

Some Characterizations of Finite Commutative Nil Rings

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Abstract. In this article, some ideal-theoretic characterizations are provided concerning when a commutative ring is nil under certain finiteness conditions of the ring. It is shown that if R is a finitely generated ring without unity, then R is nil if and only if every proper ideal of R is nil. It is also shown that if R is a finite ring with every proper ideal irreducible, then R is nil if and only if every element of R is a zero-divisor.

1 Introduction

Throughout this article, all rings are commutative and do not necessarily have unity. A ring is called *nil* if every element of the ring is nilpotent; that is, the ring R is nil if for every element x of R , there is a positive integer n for which $x^n = 0$. More strongly, the ring R is called *nilpotent* if there is a positive integer m such that $R^m = \{0\}$. The definitions for an ideal of a ring to be respectively nil or nilpotent follow the above definitions, *mutatis mutandis*. Also, for an element x in the ring R , the ideal of R generated by x will be denoted by (x) and is equal to $\{nx + rx \mid n \in \mathbb{Z}, r \in R\}$.

The study of the ideal structure in rings with a significant amount of nilpotency at the level of elements has a history spanning several decades. In fact, these studies have often focused on such rings which were either finite or finitely generated. Kruse and Price [4] examined ideals in finite nilpotent rings as did Leger [5] who determined up to isomorphism all finite nilpotent rings R whose only two-sided ideals are powers of R . Of course, much of the modern day study of nil ideals in a ring is motivated by the still-open problem of Koethe, which asks if every one-sided nil ideal of a ring is contained in a two-sided nil ideal of the ring (for recent developments along the lines of Koethe's problem, see [1], [6]). In particular, Smoktunowicz [7] has established a connection between Koethe's problem and a certain property of polynomial rings in one indeterminate over nil rings. While Koethe's problem has a trivially positive solution for commutative rings, we are nonetheless motivated to ask about the relationship between the ideal structure in a commutative ring and the quality of the ring to be nil.

To this end, we provide several characterizations of commutative nil rings under certain finiteness conditions of the ring. Our main results show that if R is a finitely generated ring without unity, then R is nil if and only if every *proper* ideal of R is nil (Theorem 2.1) and if R is a finite ring with every proper ideal irreducible, then R is nil if and only if every element of R is a zero-divisor (Theorem 2.5).

2 Results

We begin with one of our main results that showcases how the structure of proper ideals of a certain type of commutative ring impacts the behavior of the elements of the ring along the lines of nilpotency.

Theorem 2.1. *Let R be a finitely generated commutative ring without unity. Then the following are equivalent:*

- (1) R is nil;
- (2) R is nilpotent;
- (3) every proper ideal of R is nilpotent;
- (4) every proper ideal of R is nil.

Proof. (1) \Rightarrow (2): Since every element of R is nilpotent, every generator of R is nilpotent. Let $\{x_1, \dots, x_n\}$ be a set of generators for R with corresponding indices of nilpotency k_1, \dots, k_n . So, the product of an arbitrary $k = k_1 + \dots + k_n$ factors is

$$(m_{1,1}x_1 + r_{1,1}x_1 + \dots + m_{n,1}x_n + r_{n,1}x_n) \cdots (m_{1,k}x_1 + r_{1,k}x_1 + \dots + m_{n,k}x_n + r_{n,k}x_n)$$

where each $m_{i,j} \in \mathbb{Z}$ and each $r_{i,j} \in R$. Observe that this product is a sum where each term is of the form $cx_{i_1} \cdots x_{i_k}$, where c is an integer or an element of R and the x_{i_j} 's may be repeated. So, each term has at least k generator factors. Consider an arbitrary term of the sum. For each x_i , let α_i be the number of generator factors equal to x_i . So, $\alpha_1 + \dots + \alpha_k \geq k$. If $\alpha_i < k_i$ for each i , then it would be the case that $\alpha_1 + \dots + \alpha_k < k_1 + \dots + k_k = k$, a contradiction. Thus, there is some j with $\alpha_j \geq k_j$. So this term has a factor of $x_j^{k_j}$, which is equal to zero, since x_j has index of nilpotency k_j . Thus, each term of the sum is equal to zero, and so the entire sum is equal to zero. Therefore, the product of any k terms is equal to zero, whence $R^k = \{0\}$.

(2) \Rightarrow (3): This implication is clear.

(3) \Rightarrow (4): This implication is clear.

(4) \Rightarrow (1): We proceed by contradiction. Assume that every proper ideal of R is nil. Suppose that R contains an element a which is not nilpotent. Then a^2 is also not nilpotent. Note that $a^2 \in aR = \{ar \mid r \in R\}$. Thus, $aR = R$. Hence there is some $e \in R$ such that $a = ae$. Let $r \in R$. Then there is some $x \in R$ such that $r = ax$. So

$$re = (ax)e = (ae)x = ax = r.$$

Thus e is a multiplicative identity of R , a contradiction. Therefore, R is nil. \square

We now present two characterizations of finite commutative nil rings, the first (Proposition 2.2) based on a result by Frobenius [2] and the second (Proposition 2.3) based on a result [3, Proposition 1.2.2] by Kruse and Price.

Proposition 2.2. *If R is a nil ring, then $xy \neq y$ for all nonzero $x, y \in R$. If R is a finite ring, then R is nil if and only if for every nonzero $x, y \in R$, it is the case that $xy \neq y$.*

Proof. Let x and y be nonzero elements of the nil ring R . So $x^n = 0$ for some positive integer n . If $xy = y$ then

$$0 = (x^n)y = (x^{n-1})(xy) = (x^{n-1})y = (x^{n-2})(xy) = (x^{n-2})y = \dots = xy = y,$$

a contradiction. Thus, $xy \neq y$.

Now, assume that R is finite and suppose that for every nonzero $x, y \in R$, $xy \neq y$. Let $x \in R$. Let n, m be positive integers with $n < m$. So, if $x^n \neq 0$ and $x^m \neq 0$, then $x^n \neq x^n x^{m-n} = x^m$. So, each positive power of x which is not zero must be distinct. But, R is finite, so it must be the case that $x^k = 0$ for some k . Thus, every element of R is nilpotent. \square

Proposition 2.3. *If R is a finitely generated nil ring, then $IJ \subsetneq I$ for all nonzero ideals I and J of R . If R is finite, then R is nil if and only if $IJ \subsetneq I$ for all nonzero ideals I and J of R .*

Proof. Since R is a finitely generated nil ring, we that $R^k = \{0\}$ for some positive integer k . Suppose by way of contradiction that R contains nonzero ideals I and J with $IJ = I$. But then

$$\{0\} = IJ^k = (IJ)J^{k-1} = IJ^{k-1} = (IJ)J^{k-2} = \dots = IJ = I,$$

a contradiction. Therefore, $IJ \subsetneq I$.

Now assume that R is finite, and that $IJ \subsetneq I$ for all nonzero ideals I and J of R . Then by assumption, $R^k = R^{k-1}R \subsetneq R^{k-1}$ for all positive integers k . So, $R \supsetneq R^2 \supsetneq \dots$. Then since R is finite, $R^n = \{0\}$ for some positive integer n , whence R is nilpotent. \square

Since a nil ring will contain no prime ideals, it is worth considering the question of whether a nil ring can contain anything like a prime ideal. To do this, we next look at two generalizations of the concept of "prime ideal" in the context of nil rings, namely "primary ideal" and "irreducible ideal". A proper ideal P of the ring R is a *primary ideal* if $ab \in P$ implies that $a \in P$ or $b^n \in P$ for some positive integer n , where $a, b \in R$. It is easy to prove that while a nil ring will contain no prime ideals, every proper ideal of a nil ring will be primary. Proposition 2.4 establishes precisely when such rings are nil.

Proposition 2.4. *If R is a ring such that every proper ideal of R is primary, then R is nil if and only if every element of R is a zero-divisor.*

Proof. (\Rightarrow) This implication is clear.

(\Leftarrow) Suppose every element of R is a zero-divisor. Let a be a nonzero element of R . So, a is a zero-divisor. That is, there is some $b \neq 0$ in R with $ab = 0$. By assumption, $\{0\}$ is primary. It is the case that $ab \in \{0\}$ and $b \notin \{0\}$, and so $a^n \in \{0\}$ for some integer n . That is, $a^n = 0$. So, a is nilpotent. Thus, each element of R is nilpotent. \square

Another common generalization of the concept of “prime ideal” is the concept of “irreducible ideal”. A proper ideal P of the ring R is *irreducible* if it cannot be written as the intersection of ideals of R properly containing it. Unlike the situation with primary ideals, it is not the case that in a nil ring every proper ideal is irreducible. For example, $\mathbb{Z}_2 \times \mathbb{Z}_2$ with multiplication of any two elements defined to be $(0,0)$ has no prime ideals. However, the ideal $\{(0,0)\} = \{(0,0), (1,0)\} \cap \{(0,0), (0,1)\}$. Nonetheless, we provide a final characterization result that concerns when finite rings such that every proper ideal of the ring is irreducible are nil.

Theorem 2.5. *If R is a finite ring such that every proper ideal of R is irreducible, then R is nil if and only if every element of R is a zero-divisor.*

Proof. (\Rightarrow) This implication is clear.

(\Leftarrow) Suppose every element of R is a zero-divisor. We proceed by way of contradiction. Suppose that there is some element a of R with a not nilpotent. Let $\text{Nil}(R)$ and $\text{Spec}(R)$ denote the nilradical and spectrum of R , respectively. Since every proper ideal of R is irreducible and $\text{Nil}(R) = \bigcap_{P \in \text{Spec}(R)} P$, it follows that $\text{Nil}(R) \in \text{Spec}(R)$ since any two prime ideals of R must be comparable. Consider the ideal $aR = \{ar \mid r \in R\}$. If $a \cdot a \in \text{Nil}(R)$, then $a \in \text{Nil}(R)$ since $\text{Nil}(R)$ is prime, but, by assumption, $a \notin \text{Nil}(R)$. So, aR contains a non-nilpotent element. Thus, $aR \not\subseteq \text{Nil}(R)$.

Since R is finite, $\text{Nil}(R)$ is finite, say $\text{Nil}(R) = \{0, n_1, \dots, n_k\}$. Note that $a \cdot s \notin \text{Nil}(R)$ if $s \notin \text{Nil}(R)$ since $\text{Nil}(R)$ is prime. So, in order for an element n_i of $\text{Nil}(R)$ to be in aR , there must be some $n_j \in \text{Nil}(R)$ for which $an_j = n_i$. By assumption, a is a zero-divisor. So, there is some $0 \neq b \in R$ such that $ab = 0$. Then $ab \in \text{Nil}(R)$ with $a \notin \text{Nil}(R)$ which implies that $b \in \text{Nil}(R)$. Then $ab = 0$ and $a \cdot 0 = 0$. So, since $\text{Nil}(R)$ is finite, it must be the case that $a\text{Nil}(R) \subsetneq \text{Nil}(R)$. So, $\text{Nil}(R) \not\subseteq aR$. Thus, $\text{Nil}(R) \not\subseteq aR$ and $aR \not\subseteq \text{Nil}(R)$, a contradiction. Therefore, every element of R is nilpotent. \square

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