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# On Comparability Conditions and Generalized Valuation Maps

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**Abstract**. In this study, we generalize the comparability conditions, addressed in Comparability of ideals and valuation over rings [2], between certain maximal ideals and fractional ideals of D which also force D to be a quasi-local domain. Also, we introduce the notion of an almost totally ordered group and establish that: "An integral domain D is an AVD if and only if the group of divisibility of D is almost totally ordered". Further we also establish the groups of divisibility for APVD and PAVD, by which we can easily define the corresponding maps which are basically the generalization of valuation map. Finally, by a similar approach as in [2], we translate these comparability conditions into conditions on the partial ordering on the groups of divisibility of D.

## 1 Introduction

Group theory has vast applications in almost all branches of science, but our main concern is to study the abelian groups admitting a (partial) order relation compatible with the group operation. Connections between an ordered abelian group and a unitary commutative ring is the main focus throughout this study. Associated with any integral domain $D$ is a partially ordered directed group $G(D)$, known as the group of divisibility of $D$, which is the group of nonzero principal fractional ideals of $D$ with partial order defined as, $aD \leq bD$ if and only if $bD \subseteq aD$, where $a, b \in K \diagup \{0\}$, where $K$ is the quotient field of $D$.

It has proved useful on occasion to phrase a ring-theoretic problem in terms of the ordered group $G(D)$, first solve the problem there, and then pull-back the solution if possible to $D$. The main theorem in the pull-back process from the group $G(D)$ to the ring $D$ is due to Jaffard [11] and asserts that any lattice-ordered group is the group of divisibility of a Bezout domain (a domain in which every finitely generated ideal is principal).

Throughout this study, $D$ represents an integral domain with quotient field $K$. Following Hedstrom and Houston [7], $D$ is a pseudo-valuation domain (PVD) if each prime ideal $P$ of $D$ is strongly prime (that is, if $xy \in P$, where $x, y \in K$, then either $x \in P$ or $y \in P$). Equivalently, $D$ is a pseudo-valuation domain if and only if for every nonzero $x \in K$, either $x \in D$ or $ax^{-1} \in D$ for every nonunit $a \in D$ (cf. [7, Theorem 1.5]). An integral domain $D$ is said to be a valuation domain (VD) if for every nonzero element $x \in K$, either $x \in D$ or $x^{-1} \in D$. A valuation domain is a PVD, but the converse is not true. By [1, page 301], $D$ is an almost valuation domain (AVD) if for every nonzero element $x \in K$, there is an integer $n \geq 1$ such that either $x^n \in D$ or $x^{-n} \in D$. Equivalently, $D$ is an AVD if for nonzero $a, b \in D$, there is an $n = n(a, b)$ such that $a^n \mid b^n$ ($a^n$ divides $b^n$) or $b^n \mid a^n$ (see [1, Definition 5.5]). A valuation domain is an AVD, but the converse is not true. By [4, Definition 2.1], $D$ is a pseudo almost valuation domain (PAVD) if every prime ideal $P$ of $D$ is a pseudo-strongly prime (that is, if whenever $x, y \in K$ and $xyP \subseteq P$, then there is an integer $n \geq 1$ such that either $x^n \in D$ or $y^n P \subseteq P$). Equivalently, $D$ is a PAVD if and only if $D$ is quasilocal and for every nonzero $x \in K$, there is an integer $n \geq 1$ such that either $x^n \in D$ or $ax^{-n} \in D$ for every nonunit $a \in D$. By [3, Definition 3.1], $D$ is an almost pseudo valuation domain (APVD) if each prime ideal $P$ of $D$ is a strongly primary ideal, in the sense that, if $xy \in P$, $x, y \in K$, then either $x^n \in P$ for some integer $n \geq 1$ or $y \in P$. Equivalently, $D$ is an APVD if and only if $(D,\ M)$ is quasilocal such that for every nonzero element $x \in K$, either $x^n \in M$ for some integer $n \geq 1$ or $ax^{-1} \in M$ for every nonunit $a \in D$.

This study is motivated by [2], in which D. F. Anderson introduced the comparability conditions (I), (II), and (III) for a quasilocal integral domain $(D, M)$ with quotient field $K$, which are equivalent to $D$ being a valuation domain, a PVD, and a domain with valuation overring $M : M = \{r \in K : rM \subseteq M\}$, respectively.

In this study, we generalize those comparability conditions between certain maximal ideals

$M$ and fractional ideals of an integral domain $D$. These conditions force $D$ to be quasilocal and are equivalent to $D$ being an AVD, PAVD, and a domain with an almost valuation overring $M : M$, respectively. Moreover, we define a notion of an almost totally ordered group and show that the group of divisibility of an AVD is an almost totally ordered group. And by a similar approach as in [2], we establish the groups of divisibility of APVD and PAVD. With the help of these groups of divisibility we define the corresponding maps which are basically the generalization of valuation map. Finally we translate the comparability conditions (IV), (V), (VI) into conditions on the partial ordering on the group of divisibility of $D$.

## 2 Comparability conditions

Let $(D, M)$ be a quasilocal domain with quotient field $K$. By [2, pages 452, 453], the following are three comparability conditions for fractional ideals of $D$.

(I) For each $x \in K$, $xD \subseteq D$ *or* $D \subseteq xD$.

(II) For each $x \in K$, $xD \subseteq M$ *or* $M \subseteq xD$.

(III) For each $x \in K$, $xM \subseteq M$ *or* $M \subseteq xM$.

Clearly $(I) \Rightarrow (II) \Rightarrow (III)$.

(I) is an equivalent condition for $D$ to be a valuation domain, while (III) means that the overring $M : M$ of $D$ is a valuation domain, and (II) is an equivalent condition for $D$ to be a PVD.

In a similar manner, we generalize the above comparability conditions for fractional ideals of a quasilocal domain $(D, M)$ with quotient field $K$ as:

(IV) For each $x \in K$, there exists an integer $n \geq 1$ such that $x^nD \subseteq D$ or $D \subseteq x^nD$.

(V) For each $x \in K$, there exists an integer $n \geq 1$ such that $x^nD \subseteq M$ or $M \subseteq x^nD$.

(VI) For each $x \in K$, there exists an integer $n \geq 1$ such that $x^nM \subseteq M$ or $M \subseteq x^nM$.

Assume that $D$ satisfies (IV). If $x^nD \subset D$, then $x^nD \subseteq M$ since $D$ is quasilocal. If $D \subseteq x^nD$, then $M \subseteq x^nD$. Thus $D$ satisfies (V). Now suppose that $D$ satisfies (V). Then $x^nD \subseteq M$ implies $x^nM \subseteq M$, and $M \subset x^nD$ implies $x^{-n}M \subseteq M$, as $D$ is quasilocal domain. Thus $M \subseteq x^nM$. Hence $D$ satisfies (VI). Consequently, we conclude the following.

$$(IV) \quad \Rightarrow \quad (V) \quad \Rightarrow \quad (VI)$$

In the following, we establish that (IV) (resp., (V)) is an equivalent condition for $D$ to be an AVD (resp., a PAVD), while (VI) just means that the overring $M : M$ of $D$ is an AVD.

**Proposition 2.1.** *Let $(D, M)$ be a quasilocal integral domain with quotient field $K$. Then $D$ satisfies (IV) if and only if $D$ is an AVD.*

*Proof.* Suppose $D$ is an AVD. For every nonzero element $x \in K$, there is an integer $n \geq 1$ such that either $x^n \in D$ or $x^{-n} \in D$. This implies either $x^nD \subseteq D$ or $x^{-n}D \subseteq D$ (that is, $D \subseteq x^nD$), respectively. Thus (IV) holds. Conversely, let $x \in K$ be nonzero. If $x^nD \subseteq D$, then $x^n \in D$. Similarly, if $D \subseteq x^nD$, then $x^{-n}D \subseteq D$ implies $x^{-n} \in D$. Hence $D$ is an AVD. □

Let $D$ be an integral domain with quotient field $K$. As in [4], let $E(S) = \{x \in K : x^n \notin S$ for every integer $n \geq 1\}$, where $S$ is a subset of $D$.

**Proposition 2.2.** *Let $(D, M)$ be a quasilocal integral domain with quotient field $K$. Then $D$ satisfies (V) if and only if $D$ is a PAVD.*

*Proof.* Let $D$ be a PAVD and $x \in K$. If $x^n \in D$, then either $x^nD \subseteq M$ or $M \subseteq x^nD$, depending on whether $x^n$ is a nonunit or unit in $D$. If $x \in E(D)$, then $x^{-n}M \subseteq M$, by [4, Lemma 2.1]. Thus $M \subseteq x^nM \subset x^nD$. Conversely, suppose that $x^nD \subseteq M$. This implies $x^n \in M \subset D$. If $M \subseteq x^nD$, then $x^{-n}M \subseteq D$. This implies that $mx^{-n} \in D$ for each $m \in M$, which are nonunits in $D$. Hence $D$ is a PAVD. □

**Proposition 2.3.** *Let $(D, M)$ be the quasilocal integral domain with quotient field $K$. Then $D$ satisfies (VI) if and only if $M : M$ is an AVD.*

*Proof.* For nonzero $x \in K$, if $x^nM \subseteq M$, then $x^n \in M : M$, and if $M \subseteq x^nM$, then $x^{-n} \in M : M$. That is, for nonzero $x \in K$, either $x^n \in M : M$ or $x^{-n} \in M : M$. Hence $M : M$ is an AVD. Conversely, let $M : M$ be an AVD. Then for nonzero $x \in K$, either $x^n \in M : M$ or $x^{-n} \in M : M$. Hence (VI) holds. □

We conclude that: (I) $\Rightarrow$ (IV), but (IV) $\not\Rightarrow$ (I) (cf.[4, Example 2.20]). Similarly (II) $\Rightarrow$ (V), but (V) $\not\Rightarrow$ (II), and (III) $\Rightarrow$ (VI), but (VI) $\not\Rightarrow$ (III), for instance see [4, Example 3.5]. For (III) $\not\Rightarrow$ (II), see [2, Example 3.2]. However, (VI) $\Rightarrow$ (V) if the overring $V = M : M$ is an AVD with maximal ideal $Rad(MV)$ (cf. [4, Theorem 2.15]).

Consequently, the following non- reversible implications are obtained.

$$\begin{array}{ccccc} (I) & \Rightarrow & (II) & \Rightarrow & (III) \\ \Downarrow & & \Downarrow & & \Downarrow \\ (IV) & \Rightarrow & (V) & \Rightarrow & (VI) \end{array}$$

In (IV), (V), and (VI), $D$ is assumed to be a quasilocal domain with unique maximal ideal $M$. Further, we show that comparability conditions between certain maximal ideals $M$ of an integral domain $D$ and certain fractional ideals actually force $D$ to be a quasilocal domain.

**Proposition 2.4.** *Let $D$ be an integral domain with quotient field $K$. If for each $x \in K$, there exist an integer $n \geq 1$ such that $x^nD \subseteq D$ or $D \subseteq x^nD$, then $D$ is a quasilocal domain.*

*Proof.* Assume that $D$ has two distinct maximal ideals $M$ and $N$. Choose $x \in M\backslash N$ and $y \in N\backslash M$. By hypothesis, for $xy^{-1} \in K$, there exist an integer $n \geq 1$ such that either $x^ny^{-n}D \subseteq D$ or $D \subseteq x^ny^{-n}D$. That is, $x^n/y^n \in D$ or $y^n/x^n \in D$. Now, if $x^n/y^n \in D$, then $x^n = y^n(x^n/y^n) = y(y^{n-1}x^n/y^n) \in N$, and so $x \in N$ since $N$ is a prime ideal, a contradiction. A similar conclusion is obtained if $y^n/x^n \in D$. Thus $D$ is a quasilocal domain. □

**Proposition 2.5.** *Let $D$ be an integral domain with quotient field $K$. If for each $x \in K$, there is a maximal ideal $M$ of $D$ and an integer $n \geq 1$ such that either $x^nM \subseteq M$ or $M \subseteq x^nM$, then $D$ is a quasilocal domain.*

*Proof.* Assume that $D$ has two distinct maximal ideals $M$ and $N$. Choose $x \in M\backslash N$ and $y \in N\backslash M$. By hypothesis, there is a maximal ideal $P$ of $D$ such that $x^ny^{-n}P \subseteq P$ or $P \subseteq x^ny^{-n}P$. If $x^ny^{-n}P \subseteq P$, then $x^nP \subseteq y^nP \subset N$. This implies $x^np \in N$ for each $p \in P$, and therefore $p \in N$. Thus $P \subseteq N$ and hence $P = N$ since $P$ is a maximal ideal. This means $x^n = (x^ny^{-n})y^n \in N$, a contradiction. For $y$, a similar conclusion is obtained if $P \subseteq x^ny^{-n}P$. Thus $D$ is quasilocal. □

**Corollary 2.6.** *Let $D$ be an integral domain with quotient field $K$. Then the following are equivalent.*

*(1) $D$ is a quasilocal domain and satisfies (VI).*

*(2) For each $x \in K$ and maximal ideal $M$ of $D$, $x^nM$ and $M$ are comparable for some integer $n \geq 1$ (each $M : M$ is an AVD).*

*(3) For some maximal ideal $M$ of $D$, $x^nM$ and $M$ are comparable for each $x \in K$, and some integer $n \geq 1$ (some $M : M$ are AVDs).*

*(4) For each $x \in K$, there is a maximal ideal $M$ of $D$ such that $M$ and $x^nM$ are comparable for some integer $n \geq 1$.*

**Proposition 2.7.** *Let $D$ be an integral domain with quotient field $K$. If for each $x \in K$, there is a maximal ideal $M$ of domain $D$ and an integer $n \geq 1$ such that $x^nD$ and $M$ are comparable, then $D$ is a quasilocal domain.*

*Proof.* Assume that $D$ has two distinct maximal ideals $M$ and $N$. Choose $a \in M\backslash N$ and $b \in N\backslash M$. Applying the hypothesis to $a^2b^{-2}$ we conclude that, there is a maximal ideal $P$ of $D$ such that $a^{2n}b^{-2n}D$ and $P$ are comparable. If $a^{2n}b^{-2n}D \subseteq P$, then $a^{2n}D \subseteq b^{2n}P \subset N$. This implies $a^{2n} \in N$, and therefore $a \in N$, a contradiction. Thus $a^{2n}b^{-2n}D \nsubseteq P$, and hence $P \subseteq a^{2n}b^{-2n}D$. Equivalently, $b^{2n}P \subseteq a^{2n}D \subset M$, and thus, $b^{2n}c \in M$ for each $c \in P$. Since $b^{2n} \notin M$, therefore $c \in M$. This implies $P = M$, and therefore $M \subseteq a^{2n}b^{-2n}D$. Localizing at $M$, we obtain $M_M \subseteq a^{2n}D_M \subseteq aD_M \subseteq M_M$. This implies $aD_M = a^{2n}D_M = M_M$, a contradiction for $n \geq 1$. Hence $D$ must be quasilocal. □

**Corollary 2.8.** *Let $D$ be an integral domain with quotient field $K$. Then the following are equivalent.*

*(1) $D$ is a PAVD (and hence quasilocal).*

*(2) For each $x \in K$ and maximal ideal $M$ of D, $x^n D$ and $M$ are comparable for some integer $n \geq 1$.*

*(3) For some maximal ideal $M$ of D, $x^n D$ and $M$ are comparable for all $x \in K$ and some integer $n \geq 1$.*

*(4) For each $x \in K$, there is a maximal ideal $M$ of $D$ such that $x^n D$ and $M$ are comparable for some integer $n \geq 1$.*

The set of all prime ideals of an integral domain $D$ is known as the spectrum of $D$ (Spec$(D)$).

**Proposition 2.9.** *Let $(D, M)$ be a quasilocal domain with quotient field $K$. If $D$ satisfies (VI), then Spec$(D)$ is linearly ordered.*

*Proof.* Let $P$, $Q \in$ Spec$(D)$ be prime ideals of $D$ such that $P \nsubseteq Q$. Choose $x \in P \backslash Q$, and assume that $P \subsetneq M$. For each nonzero $q \in Q$, $(xq^{-1})^n M \subseteq M$ implies that $x^n M \subseteq q^n M \subseteq Q$. This gives $x^n m \in Q$, for each $m \in M$. Hence $x^n \in Q$, and therefore $x \in Q$. This is a contradiction. Next $(qx^{-1})^n M \subseteq M$ implies $q^n M \subseteq x^n M \subseteq P$. Thus $q^n \in P$, and so $q \in P$ gives $Q \subset P$. Hence Spec$(D)$ is linearly ordered. □

## 3 Group of Divisibility

Associated with any integral domain $D$, there is a partially ordered abelian group (po-group) $G(D)$, known as the group of divisibility of $D$. $G(D)$ is the group of nonzero principal fractional ideals of $D$ with $xD \leq yD$ if and only if $yD \subseteq xD$, where $0 \neq x, y \in K$, the quotient field of $D$. A necessary condition for a po-group $G$ to be a group of divisibility of an integral domain $D$ is that $G$ is directed (a po-group $G$ is directed if for each pair $a, b \in G$, there exists $c \in G$ such that $c \leq a, b$ [9, page 5]). However, this condition is not sufficient for $G$ to be the group of divisibility of a domain as shown in [13] through several examples. On the other hand, some partial orders are sufficient for the existence of a domain $D$ so that a po-group $G$ is isomorphic to $G(D)$ for some domain $D$. In fact, Krull [8] has shown that any totally ordered abelian group $G$ is the group of divisibility of a valuation ring.

Note that, if $K^*$ denotes the multiplicative group of the quotient field $K$ of an integral domain $D$ and $U$ (or $U(D)$) the group of units of $D$, then $G(D)$ is order isomorphic to $K^*/U$, where $xU \leq yU$ if and only if $y/x \in D$ (cf. [10, page 194]). The set of positive elements (positive cone) of $G(D)$ is $G(D)^+ = \{aU : aU \geqslant U\} = \{aU : a \in D^*\} = D^*/U$. Under the isomorphism $aU \longmapsto aD$, the image of $G(D)^+$ is the set of nonzero principal integral ideals of $D$ (cf. [5, page 172]).

D. F. Anderson addresses the group of divisibility of a PVD in [2, Proposition 5.1]. In this study, we extend it to the group of divisibility of an AVD, an APVD, and, a PAVD and relate them with the comparability conditions (IV), (V), and (VI).

**Almost totally ordered group**

In a valuation domain $D$, for any pair of nonzero elements $x, y$ in its quotient field $K$, either $x \mid y$ or $y \mid x$. Consequently, there exists a totally ordered group of divisibility of the valuation domain $D$ such that $xU \leq yU$ or $yU \leq xU$ (equivalently $yD \subseteq xD$ or $xD \subseteq yD$). In fact, this motivates us to think about the existence of the group of divisibility for an AVD.

**Definition 3.1.** *(a) Let $(S, \leq)$ be a partially ordered set closed under addition. The partial order $\leq$ is an almost total order if for all $s, t \in S$, there exists some integer $n \geq 1$ such that either $ns \leq nt$ or $nt \leq ns$.*

*(b) A partially ordered abelian group $(G, \leq)$ is called an almost totally ordered group if the partial order $\leq$ is an almost total order.*

**Remark 3.2.** *A totally ordered group is almost totally ordered, but the converse is not true. For example, $G = (\mathbb{Z} \oplus \mathbb{Z}_4, \leq)$ with the partial order defined as follows; $(n, \overline{k}) \leq (m, \overline{k})$, where $n, m \in \mathbb{Z}, \overline{k} \in \mathbb{Z}_4$, if and only if $n \leq m$ under the usual order on $\mathbb{Z}$, is an almost totally ordered group, in which any pair of elements are comparable through a positive integer 4 or its multiple, which is not totally ordered.*

Recall that a po-group $G$ is isolated, if for all integers $n \geq 1$ and $x \in G$, the implication $nx \geq 0 \Rightarrow x \geq 0$ holds. In a torsion free group $G$, if $a \neq 0$, then $na \neq 0$, for all integers $n \geq 1$. A group $G$ is torsion free if and only if $G$ is isomorphic to an isolated po-group (cf. [13, Proposition 7.2.2]).

The group of divisibility of an integrally closed domain is torsion free. Since an AVD is not necessarily integrally closed (IC), it is not necessary that the group of divisibility of an AVD is torsion free. Accordingly, we record an observation in the following remark.

**Remark 3.3.** *An almost totally ordered group need not be torsion free. For example, the almost totally ordered group $G = (\mathbb{Z} \oplus \mathbb{Z}_4, \leq)$ is not torsion free.*

**The Group of Divisibility of an AVD**

The following proposition defines the group of divisibility of an AVD.

**Proposition 3.4.** *An integral domain $D$ is an AVD if and only if $G(D)$ is an almost totally ordered group.*

*Proof.* Let $K$ be the quotient field of domain $D$ and $U$ be the group of unit elements in $D$. Then $G = K^*/U$ is the group of divisibility of $D$. By definition of an AVD, for $a, b \in D$, there exists an integer $n \geq 1$ such that either $a^n \mid b^n$ or $b^n \mid a^n$. This implies $b^nD \subseteq a^nD$ or $a^nD \subseteq b^nD$, and this implies $a^nU \leq b^nU$ or $b^nU \leq a^nU$ in $G$. Since in $G$, it is defined that $aU.bU = aU + bU$, therefore $a^nU = naU$. Hence $ng \leq nh$ or $nh \leq ng$, where $g = aU$ and $h = bU$. This shows that $G$ is an almost totally ordered group. Conversely, suppose that $G(D)$ is an almost totally ordered group of an integral domain $D$. Let $aU = g$, $bU = h \in G(D)^+$ such that $ng \leq nh$ for some integer $n \geq 1$. Thus $a^nU \leq b^nU$. Equivalently, $b^nD \subseteq a^nD$, which implies that $a^n \mid b^n$. Similarly, we obtain $b^n \mid a^n$ for $nh \leq ng$. Hence $D$ is an AVD. □

The following remark substantiates the existence of an almost totally ordered group of divisibility of an AVD.

**Remark 3.5.** *Let $K$ be a field with characteristic $p \neq 0$, and let $L$ be a purely inseparable extension of $K$ such that $L^p \subseteq K$. The ring $R = K + XL[X]$ is one dimensional and the localization at every height one prime is an AVD (cf [14, Example 2.13]). Consider the height one prime ideal $P = XL[X]$. In $R_P$, the only nonunits are associates of powers of $X$. So every nonzero nonunit of $R_P$ can be written as $elX^r$, where $e$ is a unit that is a fraction of the form $\frac{1+Xf(X)}{1+Xg(X)}$, $l \in L$, and $r \in \mathbb{Z}^+$. Note that for every $l \in L$ we have $l^p \in K$, and this is a unit. Now let $e_1l_1X^{r_1}$, $e_2l_2X^{r_2}$ be two nonzero nonunits in $R_P$ where we can assume that $r_1 \leq r_2$. In the first case, we have $e_1l_1X^{r_1}|e_2l_2X^{r_2}$ and consequently $e_1l_1X^{r_1}U(R_P) \leq e_2l_2X^{r_2}U(R_P)$. And in the second case, $pr_1 = pr_2$, implies $(e_1l_1X^{r_1})^p$ and $(e_2l_2X^{r_2})^p$ are associates, so $(e_1l_1X^{r_1})^p|(e_2l_2X^{r_2})^p$ holds, thus translated to $p(e_1l_1X^{r_1}U(R_P)) \leq p(e_2l_2X^{r_2}U(R_P))$ in the additive language of po-groups, that is, the two elements of the ring are related through the positive integer $p$. Hence the group of divisibility of the AVD $R_P$ is an almost totally ordered group*

A question arises whether an almost totally ordered group is a group of divisibility of an integral domain. The answer is in no. To demonstrate this fact we follow the following approach:

First Mott [12], and then Yi Chuan Yang [13] gave a principle [12, Theorem 4.4.1], and [13, Lemma7.1.4] respectively, for deciding whether a group is a group of divisibility or not, which is stated as:

"*Suppose that $v$ is a semi-valuation on a field $K$ with semi-value group $G$ and that $x, y \in K^*$ are such that $v(x)$ and $v(y)$ are not comparable under the order in $G$. If $x + y \in K^*$, then $v(x + y) \in UL\{v(x), v(y)\} \backslash (U\{v(x)\} \cup U\{v(y)\})$*", *where $UL$ $\{v(x), v(y)\}$ denotes the set of upper bounds $U$ of lower bounds $L$ of $\{v(x), v(y)\}$.*

By Mott [12] and Yang [13], any po-group that violates the above 'box condition' can not be a group of divisibility.

In [13, Example 2.2], $G = ((\mathbb{Z}_2 \oplus \mathbb{Z}), \leq)$ with partial order $(\bar{0}, 0) \leq (\bar{1}, 1), (\bar{1}, 0) \leq (\bar{0}, 1), (\bar{0}, n) \leq (\bar{0}, m)$, and $(\bar{1}, n) \leq (\bar{1}, m)$ if and only if $n \leq m$ under the usual order on $\mathbb{Z}$ is a non torsion free almost lattice ordered group (al-group), which is not a group of divisibility of any integral domain. This order $\leq$ is not just a partial order on $G$ but, in fact, it is an almost total order which says that any two elements of $G$ are comparable through an integer $c \geq 1$, where $c = 2$ or a multiple of 2. Hence, this shows that an almost totally ordered group is not necessarily a group of divisibility.

Let $D$ be an integral domain with quotient field $K$, and $I = (\{a_\alpha\})$, $J = (\{b_\beta\})$ be fractional ideals of $D$. Then the set of fractional ideals is almost totally ordered if there exists an integer $n \geq 1$ such that either $(\{a_\alpha^n\}) \subseteq (\{b_\beta^n\})$ or $(\{b_\beta^n\}) \subseteq (\{a_\alpha^n\})$.

Recall that, an integral domain $D$ is an almost principal ideal domain (API-domain) if for any non-empty subset $\{a_\alpha\} \subseteq D\backslash\{0\}$, there exists a positive integer $n = n(\{a_\alpha\})$ with $(\{a_\alpha^n\})$ principal (we call such ideals the almost principal ideal, see [1, Definiion 4.2]).

Now we may extend these notions as: "Let $D$ be an integral domain with quotient field $K$. If for any non empty subset $\{x_\alpha\} \subseteq K\backslash\{0\}$, there exist an integer $n \geq 1$ with $(\{x_\alpha^n\})$ a principal fractional ideal, then we call the fractional ideal generated by $\{x_\alpha\}$ an almost principal fractional ideal".

The following is a generalization of [5, Theorem 16.3] for an AVD.

**Theorem 3.6.** *Let $D$ be an integral domain with quotient field $K$ and $G$ be the group of divisibility of $D$. Then the followings are equivalent.*

*(1) $G$ is an almost totally ordered group.*

*(2) The set of principal fractional ideals of $D$ is almost totally ordered by inclusion.*

*(3) For each pair $(x_i)$, $i = 1, ..., k$ $(y_j)$, $j = 1, ...l$ of finitely generated fractional ideals there is a positive integer $q$ such that $(x_i^q)$ and $(y_j^q)$ are comparable.*

*(4) The set of principal integral ideals of $D$ is almost totally ordered by inclusion.*

*(5) For each pair $(x_i)$, $i = 1, ..., k$ $(y_j)$, $j = 1, ...l$ of finitely generated integral ideals there is a positive integer $q$ such that $(x_i^q)$ and $(y_j^q)$ are comparable..*

*(6) If $x \in K\backslash\{0\}$, then $x^n \in D$ or $x^{-n} \in D$ for some integer $n \geq 1$.*

*Proof.* We shall follow the scheme: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (6) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) is obvious since $G \cong P(D)$, where $P(D)$ is the set of principal fractional ideals of $D$.

(2) $\Rightarrow$ (3) Let $I = (x_1, x_2, ..., x_k)$ , $J = (y_1, y_2, ..., y_l)$ be two finitelly generated nonzero ideals. Since, by Proposition 3.4, and by (1) $\Rightarrow$ (2) the domain satisfying (2) is an AVD, there exist positive integers $m$, $n$ and elements $a, b \in K^*$ such that $(x_1^m, x_2^m, ..., x_k^m) = (a)$ and $(y_1^n, y_2^n, ..., y_l^n) = (b)$. Now it is easy to see that $(x_1^{mn}, x_2^{mn}, ..., x_k^{mn}) = (a^n)$ and $(y_1^{mn}, y_2^{mn}, ..., y_l^{mn}) = (b^m)$ . By (2) again there is a positive integer $t$ such that $(a^n)^t = (a^{nt}) \subseteq (b^{mt})$ or $(b^{mt}) \subseteq (a^{nt})$ . But since $(x_1^{mnt}, x_2^{mnt}, ..., x_k^{mnt}) = (a^{nt})$ and $(y_1^{mnt}, y_2^{mnt}, ..., y_l^{mnt}) = (b^{mt})$ we conclude that the positive integer $q$ is $mnt$.

(3) $\Rightarrow$ (5) is obvious, since the set of finitely generated integral ideals of $D$ is contained in the set of finitely generated fractional ideals of $D$.

(5) $\Rightarrow$ (4) This is again obvious, since the set of principal integral ideals of $D$ is contained in the set of finitely generated integral ideals of $D$.

(4) $\Rightarrow$ (6) Let $z = xy^{-1} \in K$, for non zero $x, y \in D$. Then for some integer $n \geq 1$, either $x^n D \subseteq y^n D$ or $y^n D \subseteq x^n D$. This implies $x^n y^{-n} D \subset D$ or $y^n x^{-n} D \subseteq D$. Hence $x^n y^{-n} \in D$ or $y^n x^{-n} \in D$, that is, $z^n \in D$ or $z^{-n} \in D$.

(6) $\Rightarrow$ (1) Suppose that $xy^{-1} \in K$, then by hypothesis, either $(xy^{-1})^n \in D$ or $(xy^{-1})^{-n} \in D$. If $(xy^{-1})^n \in D$, then $x^n D \subseteq y^n D$ which implies that $nyU \leq nxU$. If $(xy^{-1})^{-n} \in D$, then $y^n D \subset x^n D$, which implies that $nxU \leq nyU$. This shows that $G$ is an almost totally ordered group. □

Thus an integral domain that satisfies any of equivalent conditions of Theorem 3.6 is an AVD.

**Remark 3.7.** *(i) In the proof of (2) $\Rightarrow$ (3) we have used the fact that the domain described in (2) is an AVD as shown in Proposition 3.4. It is well known from [1] that an AVD is an ABD, (almost Bezout domain) (for $a, b \in D\backslash\{0\}$ there exists a positive integer $n = n(a, b)$ such that $(a^n, b^n)$ is principal see also [1, Theorem 5.6]).*

*(ii) A Noetherian AVD is an API-domain.*

## 4 Generalization of valuation map

**Semi-valuation map**

A semi-valuation map of a field $K$ is group epimorphism $w : K^* \to G$ into a po-group $G$ such that $w(a + b) \in UL\{w(a), w(b)\}$ denotes the set of upper bounds $U$ of lower bounds $L$ of $\{w(a), w(b)\}$. $G$ is called the semi-valuation group of $w$, and $R_w = \{x \in K^* \mid w(x) \geq 0\} \cup \{0\}$ is a subring of $K$ which is called the semi-valuation ring of $w$.

**Almost valuation map**

Let $w : K^* \to G$ be a semi-valuation and its corresponding domain $D_w$ be an AVD. Then by Proposition 3.4, $G$ is an almost totally ordered group. We call such a map $w$ an *almost valuation map*. That is in addition to the conditions of a semi-valuation, $w$ will satisfy

(a) For each $x, y \in K^*$ and $n \geq 1$, $nw(x) \leq nw(y)$ or $nw(y) \leq nw(x)$.

Hence $M_w = \{x \in K^* : w(x) > 0\}$ is its unique maximal ideal.

Corresponding to a valuation ring, there exists a valuation map $w$ such that for any nonzero $x, y \in D_w$, either $w(x) \leq w(y)$ or $w(y) \leq w(x)$, and hence either $y/x \in D_w$ or $x/y \in D_w$.In the same spirit for an AVD we have the following.

**Proposition 4.1.** *(1) Let $w$ be an almost valuation map. For any nonzero $x, y \in D_w$, there exist an integer $n \geq 1$ such that either $nw(x) \leq nw(y)$ or $nw(y) \leq nw(x)$, then either $y^n/x^n \in D_w$ or $x^n/y^n \in D_w$.*

*(2) Let $I$ be an ideal in an AVD $D_w$. If $x \in I$ and $y \in D_w$ with $nw(x) \leq nw(y)$, for some integers $n \geq 1$, then $y^n \in I$.*

*Proof.* (1) If $nw(x) \leq nw(y)$ for an integer $n \geq 1$, then $ng \leq nh$, where $w(x) = g$ and $w(y) = h$. This implies $0 \leq nh - ng \in G^+$, and therefore $w(y^n) + w(x^{-n}) = w(y^n x^{-n}) \in G^+$. Hence $y^n/x^n \in D_w$. Similarly, if $nw(y) \leq nw(x)$, then $x^n/y^n \in D_w$.

(2) As $x \in I$, $x^n \in D_w$, where $n \geq 1$ is an integer. So take $y^n = x^n(y^n/x^n) = x(x^{n-1}y^n/x^n)$. By (1), $y^n/x^n \in D_w$, and hence $y^n \in I$. □

It is known that the spectrum of an AVD is totally ordered (see [4, Page 1168]). Also we can verify this fact with the help of the group of divisibility of an AVD.

**Theorem 4.2.** *The spectrum of an AVD is totally ordered.*

*Proof.* Let $I$, $J$ be any pair of prime ideals of an AVD $D_w$. Suppose $J \nsubseteq I$ and take $y \in J \setminus I$. Then for every $x \in I$, $nw(x) \nleq nw(y)$, where $n \geq 1$ is an integer. By Proposition 4.1 (1), $nw(x) > nw(y)$, so $x^n/y^n \in D_w$, and by Proposition 4.1 (2) $x^n \in J$. This implies $x \in J$, and hence $I \subset J$. □

The following implication tables provide a correspondence between integral domains and their groups of divisibility.

$$\boxed{\begin{array}{ccc} VD & \Rightarrow & AVD \\ \Downarrow & & \Downarrow \\ BD(\Rightarrow GCD-domain) & \Rightarrow & ABD(\Rightarrow AGCD) \\ \Downarrow & & \\ IC & & \end{array}}$$

| | | |
|---|---|---|
| Totally ordered | ⇒ | Almost totally ordered |
| ⇓ | | ⇓ |
| Lattice ordered | ⇒ | Almost lattice ordered |
| ⇓ | | |
| Torsion free | | |

In the above table, BD stands for a Bezout Domain (a domain in which every finitely generated ideal is principal). In a GCD-domain, every nonzero pair of elements has a gcd. AGCD stands for Almost GCD-domain (a domain in which for each $a, b \in D\backslash\{0\}$, there is an integer $n = n(a, b) \geq 1$ with $a^n D \cap b^n D$ principal).

**Group of divisibility of APVD and PAVD**

We generalize the following proposition for APVD and PAVD instead of PVD.

**Proposition 4.3.** *[2, Proposition 5.1] Let $D$ be an integral domain with quotient field $K$ and group of divisibility $G$ the following are equivalent.*

*(1) $D$ is a PVD, (and hence quasilocal).*

*(2) For each $g \in G$, either $g \geq 0$ or $g < h$ for all $h \in G$ with $h > 0$.*

The following proposition defines group of divisibility of APVD.

**Proposition 4.4.** *Let $D$ be an integral domain with quotient field $K$. The following are equivalent, for the group of divisibility $G$ of $D$.*

*(1) $D$ is an APVD (and hence quasilocal).*

*(2) For each $g \in G$, there exists an integer $n \geq 1$ such that either $ng > 0$ or $g < h$ for all $h \in G$ with $h > 0$.*

*Proof.* $(1) \Rightarrow (2)$

Let $M$ be the only maximal ideal in $D$ and $g \in G$ with $g = xU$, where $x \in K^*$. So, if $x^n \in M$ for some integer $n \geq 1$, then we have $ng = x^nU > U$. If $ax^{-1} \in M$ for any nonunit $a \in D$ such that $h = aU > 0$, then $ax^{-1}U = h - g > 0$. Thus $g < h$ for $h > 0$.

$(2) \Rightarrow (1)$

First, we have to show that $D$ is quasilocal. If $D$ has two distinct maximal ideals $M$ and $N$, then choose $x \in M \backslash N$ and $y \in N \backslash M$. Let $g' = xy^{-1}U$ and $h = yU$. Then for any positive integer $n$, $ng' = x^ny^{-n}\ U$. Clearly $ng' \ngeq 0$ and $g' \nless h$, while $h > 0$ because if $g' < h$. This implies $xy^{-1}U < yU$, which implies $xU < y^2U$. Equivalently $y^2D \subseteq xD \subseteq M$, and hence $y \in M$, a contradiction to our supposition. Therefore $g' \nless h$. This contradicts the hypothesis, so $D$ must be quasilocal. Let $x \in K$ be such that $xU = g$ and $ng = x^nU$. If $ng > 0$, then $x^n \in M$ and if $g < h$, then $xU < h$. Let $a$ be a nonunit element in $D$ such that $h = aU$. Obviously $h > 0$, and hence $xU < aU$ implies $ax^{-1}U > 0$. This implies $ax^{-1} \in M$. Thus $D$ is an APVD. □

By [3, Theorem 3.4], it is clear that APVD is a class of integral domains satisfying condition (III) of [2], and hence satisfies [2, Proposition 5.2].

Following proposition defines group of divisibility of PAVD.

**Proposition 4.5.** *Let $D$ be an integral domain with quotient field $K$. The following are equivalent, for the group of divisibility $G$ of $D$.*

*(1) $D$ is a PAVD.*

*(2) For each $g \in G$, there exist an integer $n \geq 1$ such that either $ng \geq 0$ or $ng < h$ for all $h \in G$, where $h > 0$.*

*Proof.* $(1) \Rightarrow (2)$ : Let $x \in E(D)$ and $g = xU$. Then clearly $ng \ngeq 0$. In $D$ every prime ideal is a pseudo strongly prime, so by [4, Lemma 2.1] $x^{-n}M \subseteq M$. Then for each $m \in M$, $x^{-n}m \in M$. Let $xU = g$ and $mU = h > 0$. Then $(x^{-n}m)U = x^{-n}U + mU = -ng + h > 0$ implies $ng < h$ for each $h > 0$.

$(2) \Rightarrow (1)$ : If $D$ has two distinct maximal ideals $N, M$, then choose $x^n \in M \backslash N$ and $y^n \in N \backslash M$. Let $ng' = x^ny^{-n}U$ (hence $ng' \ngeq 0$) and $h = yU$. If $ng' < h$ for each $h > 0$, then $x^ny^{-n}U < yU$. This implies $x^nU < y^{n+1}U$, or equivalently, $y^{n+1}D \subseteq x^nD \subseteq M$. Hence $y^{n+1} \in M$ gives $y^n \in M$, a contradiction. Therefore $ng' \nless h$, which contradicts the hypothesis. Hence $D$ is quasilocal.

Let $M$ be the maximal ideal of $D$. To show $D$ is a PAVD, it is sufficient to show $M$ is a pseudo strongly prime ideal. For this, let $x \in E(D)$ such that $xU = g \in G$. Clearly $ng \ngeq 0$, for each integer $n \geq 1$. So, for each $m \in M$, we choose $mU = h > 0$. Then by hypothesis $ng < h$, so we have $x^nU < mU$. This implies $mx^{-n}U > 0$, and so $mx^{-n} \in M$. This implies $x^{-n}M \subseteq M$, and hence, by [4, Lemma 2.1], $M$ is pseudo strongly prime. □

**Remark 4.6.** *Let $G^*, G^{**}$, and $G^{***}$ represent the defining properties of the groups of divisibility of a PVD, APVD, and PAVD, respectively. Then, corresponding to non-reversible implications $PVD \Rightarrow APVD \Rightarrow PAVD$, we have again non reversible implications $G^* \Rightarrow G^{**} \Rightarrow G^{***}$.*

### Almost pseudo-valuation map

Let $w : K^* \to G$, where $G$ staisfies $G^{**}$, be a semi-valuation, which has the following property, if $x, y \in K^*$;

(a) $nw(x) = ng > 0$, for $n \in \mathbb{Z}^+$ or $g = w(x) < w(y) = h$, where $g, h \in G$ and $h > 0$.

This map $w$ is called almost pseudo-valuation map in which condition (a) reflects the properties in $G^{**}$. Then $D_w = \{x \in K^* : w(x) \geq 0\}$ is an APVD, it follow from the proof of 4.4.

### Pseudo almost valuation map

Let $w : K^* \to G$, where $G$ satisfies $G^{***}$ be a semi-valuation, which has the following property if $x, y \in K^*$;

(a) $nw(x) = ng \geq 0$, for $n \in \mathbb{Z}^+$ or $ng = nw(x) < w(y) = h$, where $g, h \in G$ and $h > 0$.

This map $w$ is called pseudo-almost valuation map in which condition (a) reflects the properties in $G^{***}$. Then $D_w = \{x \in K^* : w(x) \geq 0\}$ will be a PAVD, it follow from the proof of 4.5.

Consequently, the modifications in [2, Proposition 5.2] shape the following proposition.

**Proposition 4.7.** *Let $D$ be an integral domain with quotient field $K$. The following are equivalent, for the group of divisibility $G$ of $D$.*

*(1) $D$ is a quasilocal domain and satisfies (VI) (i.e. $M : M$ is an AVD).*

*(2) For each $g \in G$, there exist an integer $n \geq 1$ such that either $ng > h$ for all $h \in G$ with $h < 0$ or $ng < h$ for all $h \in G$ with $h > 0$.*

*Proof.* $(1) \Rightarrow (2)$ Let $x \in K$ such that $g = xU \in G$. If $ng \nless h$ for all $h \in G$ with $h = mU > 0$, where $m \in M$, then this implies that $mx^{-n} \notin M$. Thus by hypothesis, $x^n M \subseteq M$. Hence $x^n m \in M$ implies $ng + h > 0$ for each $h \in G$ with $h > 0$, or equivalently, $ng > -h = t$ for all $t \in G$ with $t < 0$.

$(2) \Rightarrow (1)$ To show that the domain $D$ is quasilocal, we can follow the proof of Proposition 4.5. Now, we have to show that $D$ satisfies (VI). For this, let $x \in K$ such that $xU = g$ and $yU = h$. If $ng < h$ for each $h \in G$ with $h > 0$, then $y \in M$, and therefore $h - ng > 0$. This means $yx^{-n} \in M$, and so we have $yx^{-n}M \subseteq M$. This implies $x^{-n}M \subseteq M$, and so $M \subseteq x^n M$. Similarly for $ng > h$ for all $h \in G$ with $h < 0$, we can obtain $x^n M \subseteq M$, as required. □

It is known that an AVD is a PAVD (see [4, proposition 2.12]), but, in the following, we prove it with the help of their groups of divisibility.

**Proposition 4.8.** *An AVD is a PAVD.*

*Proof.* Let $D$ be an AVD with $K$ as its quotient field and $w : K^* \to G$ be its corresponding map. If for $x \in K$, $x^n \in D$ such that $w(x^n) = ng \geq 0$, then the proof is obvious. If $x^{-n} \in D$, then $w(x^{-n}) = -ng \geq 0$. Now for every nonunit $a \in D$ such that $w(a) = h > 0$, assume that $w(ax^{-n}) < 0$. This implies $w(a) + w(x^{-n}) < 0$, that is, $h - ng < 0$. But $h > 0$ and $-ng \geq 0$, which is a contradiction. Hence $h - ng \geq 0$, that is, $ng \leq h$, and therefore $ax^{-n} \in D$. □

### 4.1 Some more on Generalization of valuation map

In [6, Page 156] a valuation monoid is defined in terms of a valuation map as follows

**Definition 4.9.** *(a)(Valuation Monoid) Let $G$ be a group. By a valuation on $G$, we mean a homomorphism $v : G \to H$, where $H$ is a totally ordered group. The set $G_v = \{x \in G \text{ s.t } v(x) \geq 0\}$ is called the value monoid of $G$ with quotient group $G$.*

*(b) In other words we may define a valuation monoid as follows: Let $G$ be a quotient group of a monoid $S$, then $S$ is called a valuation monoid if for each $x \in G$ either $x \in S$ or $-x \in S$.*

**Remark 4.10.** *$G_v$ is integrally closed in $G$.*

**Example 4.11.** *Consider $G = \langle 1/2, -1/2 \rangle^0$ and an ordered group $H = \mathbb{Z}$.*

*Define a valuation on $G$ by $v : G \to \mathbb{Z}$*

*$v(a/2 - b/2) = a - b$. then $G_v = \{a/2 - b/2 \in G : v(a/2 - b/2) \geq 0\}$ this implies that $G_v = \langle 1/2 \rangle^0$.*

**Remark 4.12.** *Every rational cyclic monoid is a valuation monoid.*

**Remark 4.13.** In a similar manner we define an almost valuation monoid as follows.

**Definition 4.14.** *(Almost valuation monoid) (a) Let $G$ be a group. By an almost valuation on $G$, we mean a homomorphism $v' : G \to G'$ where $G'$ is an almost totally ordered group. The set $G_{v'} = \{x \in G \text{ s.t } v(x) \geq 0\}$ is called the almost value monoid of $G$ with quotient group $G$.*

*(b) In other words we define an almost valuation monoid as follows: Let $G$ be a quotient group of a monoid $S$, then $S$ is called an almost valuation monoid if for each $x \in G$ there exist a positive integer $n$, either $nx \in S$ or $-nx \in S$.*

**Remark 4.15.** *$G_{v'}$ is not integrally closed in $G$.*

**Remark 4.16.** $Valuation\ Monoid \implies Almost\ Valuation\ Monoid.$

*But the converse is not true.*

**Example 4.17.** *Positive cone of any almost totally ordered group would be an almost valuation monoid which is not a valuation monoid in particular positive cone of $G = \mathbb{Z}_2 \oplus \mathbb{Z}$ is an almost valuation monoid which is not a valuation monoid.*

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