

# Transformation of 4-D dynamical systems to hyperjerk form

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**Abstract.** In this paper, we investigate the possible cases for equivalence between four-dimensional autonomous dynamical systems and hyperjerk dynamics. In fact, we show that a wide class of four-dimensional vector fields possess this property.

## 1 Introduction

A jerky dynamical system is a differential equation of the form  $x''' = J(x, x', x'')$  with an explicit time dependence (see [1] and [5]). The first three derivatives  $(x', x'', x''')$  are called *velocity*, *acceleration*, and *jerk*, respectively. Jerky dynamics form an interesting subclass of dynamical systems that can exhibit regular and chaotic behavior such as shown by several simple examples in [6-7-8-9]. Jerky dynamics can be found in several nonmechanical areas of physics, one example of which is the chaotic third-order differential equation representing an oscillator model of thermal convection given in [3]. In fact, jerky dynamics are conceptually simpler than dynamical systems, and they show all the major features of three-dimensional vector fields. In particular, they can be considered as the natural generalization of oscillator dynamics, and hence new methods to study chaos are possible.

By extension, a hyperjerk system is a dynamical system governed by an  $n^{th}$ -order ordinary differential equation with  $n > 3$  describing the time evolution of a single scalar variable of the form  $x^{(n)} = H(x, x', x'', x''', \dots)$ . The fourth derivative  $x^{(4)} = x''''$  has been called a *snap*, with successive derivatives *crackle* and *pop*, and there is no universally accepted name for the higher derivatives. Some simple forms of  $4^{th}$  and  $5^{th}$  order displaying chaos and hyperchaos were presented in [8]. The connection between externally driven nonlinear oscillators and specific uni- and bi-directionally coupled systems of two autonomous oscillators was discussed in [4] along with a reinterpretation of simple chaotic forms of hyperjerk systems with some criteria that exclude chaotic behavior in some classes of these hyperjerk systems. In fact, some 4-D quadratic continuous-time autonomous systems cannot be reduced to hyperjerk dynamics, or they possess a functionally complicated hyperjerk form.

## 2 Transformation of 4-D systems to hyperjerk form

Consider a 4-D system of the form

$$\begin{cases} x' = c_1 + b_{11}x + b_{12}y + b_{13}z + b_{14}u + n_1(x) \\ y' = c_2 + b_{21}x + b_{22}y + b_{23}z + b_{24}u + n_2(x, y, z, u) \\ z' = c_3 + b_{31}x + b_{32}y + b_{33}z + b_{34}u + n_3(x, y, z, u) \\ u' = c_4 + b_{41}x + b_{42}y + b_{43}z + b_{44}u + n_4(x, y, z, u) \end{cases} \quad (2.1)$$

where the functions  $n_1(x)$  and  $(n_i(x, y, z, u))_{2 \leq i \leq 4}$  are assumed to be at least of class  $C^2$  in their corresponding arguments. We set  $n_{11}(x) = \frac{\partial n_1(x)}{\partial x}$ ,  $n_{111} = \frac{\partial^2 n_1(x)}{\partial x^2}$ ,  $n_{i1}(x, y, z, u) = \frac{\partial n_i(x, y, z, u)}{\partial x}$ ,  $n_{i2}(x, y, z, u) = \frac{\partial n_i(x, y, z, u)}{\partial y}$ ,  $n_{i3}(x, y, z, u) = \frac{\partial n_i(x, y, z, u)}{\partial z}$ , and  $n_{i4}(x, y, z, u) = \frac{\partial n_i(x, y, z, u)}{\partial u}$ ,  $i = 2, 3, 4$ . In this paper, we derive only the hyperjerk form of system (1) with respect to  $x$ . The same logic applies for  $y, z$ , and  $u$ . For this purpose, assume that  $b_{12} \neq 0$ . Then from the first equation of (1), we have  $y(x, x', z, u) = -\frac{1}{b_{12}}(-x' + c_1 + n_1(x) + b_{11}x + b_{13}z + b_{14}u)$ . Thus  $y'(x, x', x'', z, u) = f_1z + f_2x + f_3u + f_4 + f_5$ , where

$$\left\{ \begin{array}{l} f_1 = \frac{b_{13}^2 b_{32} - b_{12} b_{13} b_{33} - b_{12} b_{14} b_{43} + b_{13} b_{14} b_{42}}{b_{12}^2} \\ f_2 = \frac{b_{11} b_{13} b_{32} - b_{12} b_{13} b_{31} + b_{11} b_{14} b_{42} - b_{12} b_{14} b_{41}}{b_{12}^2} \\ f_3 = \frac{b_{14}^2 b_{42} - b_{12} b_{13} b_{34} + b_{13} b_{14} b_{32} - b_{12} b_{14} b_{44}}{b_{12}^2} \\ f_4(x, x', z, u) = - \frac{\left( b_{13} \left( c_3 + n_3(x, y, z, u) - \frac{b_{32}(c_1 + n_1(x))}{b_{12}} \right) + b_{14} \left( c_4 + n_4(x, y, z, u) - \frac{b_{42}(c_1 + n_1(x))}{b_{12}} \right) \right)}{b_{12}} \\ f_5(x, x', x'') = - \frac{b_{11} b_{12} + b_{13} b_{32} + b_{14} b_{42} + b_{12} n_{11}(x)}{b_{12}^2} x' + \frac{1}{b_{12}} x'' \end{array} \right. \quad (2.2)$$

By equating the formula for  $y' = y'(x, x', x'', z, u)$  with the second equation of (2.1), we obtain

$$\left( f_1 - b_{23} + \frac{1}{b_{12}} b_{13} b_{22} \right) z = n_2 - f_4 - f_6 \quad (2.3)$$

where

$$f_6(x, x', x'') = \frac{f_2 b_{12} + b_{11} b_{22} - b_{12} b_{21}}{b_{12}} x - \frac{b_{22}}{b_{12}} x' - c_2 + \frac{b_{22} c_1}{b_{12}} + \frac{b_{22}}{b_{12}} n_1(x) + f_5 \quad (2.4)$$

We remark that the second member of (2.3) depends on  $z$  in the part  $n_2 - f_4$ , so if we assume that  $n_2(x, y, z, u) - f_4(x, x', z, u) = g_1(x, u)$ , i.e., this part does not depend on the variable  $z$ , with  $g_1$  being an arbitrary function of the indicated arguments, then

$$z(x, x', x'') = f_7 f_6(x, x', x'') \quad (2.5)$$

where  $f_7$  is given by

$$f_7 = \frac{b_{12}^2}{b_{12}^2 b_{23} - b_{13}^2 b_{32} - b_{12} b_{13} b_{22} + b_{12} b_{13} b_{33} + b_{12} b_{14} b_{43} - b_{13} b_{14} b_{42}} \quad (2.6)$$

with the condition

$$b_{12}^2 b_{23} - b_{13}^2 b_{32} - b_{12} b_{13} b_{22} + b_{12} b_{13} b_{33} + b_{12} b_{14} b_{43} - b_{13} b_{14} b_{42} \neq 0 \quad (2.7)$$

The function  $g_1$  does not depend on  $y$  since  $y$  itself depends on  $z$ . Hence the condition  $n_2 - f_4 = g_1(x, u)$  is equivalent to

$$(b_{12} n_2 + b_{13} n_3 + b_{14} n_4)(x, y, z, u) = b_{12} g_1(x, u) - b_{12} f_8(x) \quad (2.8)$$

where

$$f_8(x) = \frac{\left( b_{13} \left( c_3 - \frac{b_{32}(c_1 + n_1(x))}{b_{12}} \right) + b_{14} \left( c_4 - \frac{b_{42}(c_1 + n_1(x))}{b_{12}} \right) \right)}{b_{12}} \quad (2.9)$$

From (2.5), we have  $z' = f_7 \left( \frac{1}{b_{12}} x''' + f_{10}(x) x'' + f_9(x, x') x' \right)$ , where

$$\left\{ \begin{array}{l} f_9(x, x') = \left( \frac{b_{12}(f_2 - b_{21}) + b_{22} b_{11} + b_{22} n_{11}(x) - n_{11}(x) x'}{b_{12}} \right) \\ f_{10}(x) = \left( -\frac{b_{22}}{b_{12}} - \frac{(b_{11} b_{12} + b_{13} b_{32} + b_{14} b_{42} + b_{12} n_{11}(x))}{b_{12}^2} \right) \end{array} \right. \quad (2.10)$$

By equating the formula for  $z'$  with the third equation of (2.1), we obtain

$$-\frac{b_{12} b_{34} - b_{14} b_{32}}{b_{12}} u = -f_{11} - f_{12} - \frac{f_7}{b_{12}} x''' \quad (2.11)$$

where

$$\left\{ \begin{array}{l} f_{11}(x, x', x'') = \frac{b_{11} b_{32} - b_{12} b_{31}}{b_{12}} x + \frac{-b_{32} + f_7 f_9(x, x') b_{12}}{b_{12}} x' + f_7 f_{10}(x) x'' \\ f_{12}(x, x', x'', u) = -n_3(x, y, z, u) - c_3 + \frac{b_{32}(c_1 + n_1(x)) + f_6 f_7 b_{13}}{b_{12}} - f_6 f_7 b_{33} \end{array} \right. \quad (2.12)$$

We remark that the second member of (2.11) depends on  $u$  in the part  $-f_{12}$ , so if we assume that  $-f_{12} = g_2(x, z)$ , i.e., this part does not depend on the variable  $u$ , with  $g_2$  being an arbitrary function of the indicated arguments, then

$$u = \frac{f_7 x''' + b_{12} f_{11}(x, x', x'') + b_{12} f_{12}(x, x', x'')}{b_{12} b_{34} - b_{14} b_{32}} \quad (2.13)$$

with the condition

$$b_{12}b_{34} - b_{14}b_{32} \neq 0 \quad (2.14)$$

Hence the condition  $-f_{12} = g_2(x, z)$  is equivalent to

$$-n_3(x, y, z, u) - c_3 + \frac{b_{32}(c_1 + n_1(x) + f_6 f_7 b_{13})}{b_{12}} - f_6 f_7 b_{33} = g_2(x, z) \quad (2.15)$$

From (2.13), we have  $u' = f_{13}x^{(4)} + f_{14}$ , where

$$\begin{cases} f_{13} = \frac{f_7}{b_{12}b_{34} - b_{14}b_{32}} \\ f_{14}(x, x', x'', x''') = \frac{b_{12}f'_{11}(x, x', x'', x''')}{b_{12}b_{34} - b_{14}b_{32}} + \frac{b_{12}f'_{12}(x, x', x'', x''')}{b_{12}b_{34} - b_{14}b_{32}} \end{cases} \quad (2.16)$$

because  $\frac{df_{11}(x, x', x'')}{dt} = f'_{11}(x, x', x'', x''')$ , and since  $f_{12} = -g_2(x, z)$ , then  $\frac{f_{12}(x, y, z, u)}{dt} = f'_{12}(x, x', x'', x''')$ . Again, by equating the formula for  $u'$  with the fourth equation of (2.1), we obtain

$$x^{(4)} = \frac{f_{15}(x, x', x'', x''') + f_{16}(x, x', x'', x''')}{f_{13}} = H(x, x', x'', x''') \quad (2.17)$$

where

$$\begin{cases} f_{15}(x, x', x'', x''', u) = c_4 + b_{41}x + b_{43}f_7f_6 + \rho_0 \\ f_{16}(x, x', x'', x''', u) = b_{42} \left( -\frac{1}{b_{12}}(-x' + c_1 + n_1(x) + b_{11}x + \rho_1) \right) \\ \rho_0(x, x', x'', x''', u) = b_{44} \left( \frac{f_7x''' + b_{12}f_{11} + b_{12}f_{12}}{b_{12}b_{34} - b_{14}b_{32}} \right) + n_4 - f_{14} \\ \rho_1(x, x', x'', x''', u) = b_{13}f_7f_6 + b_{14} \left( \frac{f_7x''' + b_{12}f_{11} + b_{12}f_{12}}{b_{12}b_{34} - b_{14}b_{32}} \right) \end{cases} \quad (2.18)$$

Finally, the form (2.17) is the corresponding hyperjerk form of the 4-D dynamical system (2.1). Several chaotic examples of hyperjerk motion were studied in the literature where some of them are listed in [8] with many illustrations. For example, all periodically forced oscillators and some of the coupled oscillators are equivalent to a snap form as shown in [4]. This includes the frictionless forced pendulum and the periodically forced undamped oscillator with a cubic restoring force. Also, the same type of system with damping was studied in [8]. The simplest examples of chaotic hyperjerk systems have been studied in [1]. These include snap systems with one nonlinear function and the simplest dissipative chaotic case with a single quadratic nonlinearity given by  $x'''' + 6x'' = 1 - x^2$ .

### 3 The expression of the transformation

In this section, we derive a rigorous expression for the transformation between the 4-D dynamical system (2.1) and the hyperjerk form (2.17). Indeed, the above procedure defines an invertible transformation  $T = T(x, x', x'', x''')$  provided the following conditions hold:

$$\begin{cases} b_{12} \neq 0 \\ b_{12}^2b_{23} - b_{13}^2b_{32} - b_{12}b_{13}b_{22} + b_{12}b_{13}b_{33} + b_{12}b_{14}b_{43} - b_{13}b_{14}b_{42} \neq 0 \\ b_{12}b_{34} - b_{14}b_{32} \neq 0 \\ n_2 - f_4 = g_1(x, u) \\ -n_3 - c_3 + \frac{1}{b_{12}}b_{32}(c_1 + n_1(x) + f_6 f_7 b_{13}) - f_6 f_7 b_{33} = g_2(x, z) \end{cases} \quad (3.1)$$

where  $g_1$  and  $g_2$  are arbitrary functions of the indicated arguments. The transformation  $T(x, x', x'', x''')$  is defined by

$$\begin{cases} T_1(x, y, z, u) = x \\ T_2(x, y, z, u) = x' = c_1 + b_{11}x + b_{12}y + n_1(x) \\ T_3(x, y, z, u) = x'' = f_{17}(x, y) + f_{18}(x, z, u) + f_{19}(x, y, z, u) \\ T_4(x, y, z, u) = x''' = \Psi_1 + \Psi_2 \\ \Psi_1 = f_{20}x^3 + f_{21}x^2y + f_{22}x^2 + f_{23}xy + f_{24}xz + f_{25}xu \\ \Psi_2 = f_{26}x + f_{27}y^2 + f_{28}yz + f_{29}yu + f_{30}y + f_{31}z + f_{32}u + f_{33} \end{cases} \quad (3.2)$$

and its inverse is defined by

$$\left\{ \begin{array}{l} T_1^{-1}(x, x', x'', x''') = x \\ T_4^{-1}(x, x', x'', x''') = u = \frac{f_7 x''' + b_{12} f_{11}(x, x', x'') + b_{12} f_{12}(x, x', x'')}{b_{12} b_{34} - b_{14} b_{32}} \\ T_3^{-1}(x, x', x'', x''') = z = f_7 f_6(x, x', x'') \\ T_2^{-1}(x, x', x'', x''') = y = -\frac{1}{b_{12}}(-x' + c_1 + n_1(x) + b_{11}x + b_{13}z + b_{14}u) \end{array} \right. \quad (3.3)$$

where

$$\left\{ \begin{array}{l} f_{17} = (b_{11}^2 + n_{11}(x)b_{11} + b_{12}b_{21})x + (b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{11}(x))y \\ f_{18} = (b_{11}b_{13} + b_{12}b_{23} + b_{13}n_{11}(x))z + (b_{11}b_{14} + b_{12}b_{24} + b_{14}n_{11}(x))u \\ f_{19} = (b_{11}(c_1 + n_1(x)) + b_{12}(c_2 + n_2(x, y, z, u)) + n_{11}(x)(c_1 + n_1(x))) \\ f_{20} = b_{11}^2 n_{111}(x) \\ f_{21} = b_{11}b_{12}n_{111}(x) \\ f_{22} = \xi_1 + \xi_2 \\ f_{23} = \xi_3 + \xi_4 \\ f_{24} = b_{12}b_{23}n_{11}(x) + b_{13}b_{33}n_{11}(x) + b_{14}b_{43}n_{11}(x) + b_{11}b_{13}n_{111}(x) \end{array} \right. \quad (3.4)$$

and

$$\left\{ \begin{array}{l} f_{25} = b_{12}b_{24}n_{11} + b_{13}b_{34}n_{11} + b_{14}b_{44}n_{11} + b_{11}b_{14}n_{111} \\ f_{26} = \xi_5 + \xi_6 + \xi_7 + \xi_8 + \xi_9 \\ f_{27} = b_{12}^2 n_{111} \\ f_{28} = b_{12}b_{13}n_{111} \\ f_{29} = b_{12}b_{14}n_{111} \\ f_{30} = \xi_{10} + \xi_{11} + \xi_{12} \\ f_{31} = \xi_{13} + \xi_{14} \\ f_{32} = \xi_{15} + \xi_{16} \\ f_{33} = \xi_{17} + \xi_{18} + \xi_{19} + \xi_{20} \end{array} \right. \quad (3.5)$$

where

$$\left\{ \begin{array}{l} \xi_1 = b_{11}(n_{11}^2 + b_{11}n_{11} + c_1n_{111} + n_1n_{111})(x) \\ \xi_2 = b_{11}n_{111}(x)(c_1 + n_1(x)) + (b_{12}b_{21} + b_{13}b_{31} + b_{14}b_{41})n_{11}(x) \\ \xi_3 = b_{12}(n_{11}^2(x) + b_{11}n_{11}(x) + (c_1 + n_1(x))n_{111}(x)) \\ \xi_4 = (b_{12}b_{22} + b_{13}b_{32} + b_{14}b_{42})n_{11}(x) + b_{11}b_{12}n_{111}(x) \\ \xi_5 = (c_1 + n_1(x))(n_{11}^2(x) + b_{11}n_{11}(x) + (c_1 + n_1(x))n_{111}(x)) \\ \xi_6 = b_{11}(b_{11}^2 + (b_{11} + c_1 + n_1)n_{11}(x) + b_{12}b_{21} + b_{12}n_{21}(x, y, z, u)) \\ \xi_7 = b_{21}(b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22}) + b_{31}(b_{11}b_{13} + b_{12}b_{23} + b_{12}n_{23}) \\ \xi_8 = b_{41}(b_{11}b_{14} + b_{12}b_{24} + b_{12}n_{24}) + b_{12}n_{11}(x)(c_2 + n_2) \\ \xi_9 = b_{13}n_{11}(x)(c_3 + n_3(x, y, z, u)) + b_{14}n_{11}(c_4 + n_4(x, y, z, u)) \\ \xi_{10} = b_{12}(b_{11}^2 + n_{11}b_{11} + c_1n_{11} + n_1n_{11} + b_{12}b_{21} + b_{12}n_{21}) \end{array} \right. \quad (3.6)$$

and

$$\left\{ \begin{array}{l} \xi_{11} = b_{22}(b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22}) + b_{32}(b_{11}b_{13} + b_{12}b_{23} + b_{12}n_{23}) \\ \xi_{12} = b_{42}(b_{11}b_{14} + b_{12}b_{24} + b_{12}n_{24}) + b_{12}n_{111}(c_1 + n_1) \\ \xi_{13} = b_{23}(b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22}) + b_{33}(b_{11}b_{13} + b_{12}b_{23} + b_{12}n_{23}) \\ \xi_{14} = b_{43}(b_{11}b_{14} + b_{12}b_{24} + b_{12}n_{24}) + b_{13}n_{111}(c_1 + n_1) \\ \xi_{15} = b_{24}(b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22}) + b_{34}(b_{11}b_{13} + b_{12}b_{23} + b_{12}n_{23}) \\ \xi_{16} = b_{44}(b_{11}b_{14} + b_{12}b_{24} + b_{12}n_{24}) + b_{14}n_{111}(c_1 + n_1) \\ \xi_{17} = (c_1 + n_1)(b_{11}^2 + n_{11}b_{11} + c_1n_{11} + n_1n_{11} + b_{12}b_{21} + b_{12}n_{21}) \\ \xi_{18} = (c_2 + n_2(x, y, z, u))(b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22}(x, y, z, u)) \\ \xi_{19} = (c_4 + n_4(x, y, z, u))(b_{11}b_{14} + b_{12}b_{24} + b_{12}n_{24}(x, y, z, u)) \\ \xi_{20} = (c_3 + n_3(x, y, z, u))(b_{11}b_{13} + b_{12}b_{23} + b_{12}n_{23}(x, y, z, u)) \end{array} \right. \quad (3.7)$$

Note that the inverse transformation exists since the procedure described in Sec. 2 transforms the 4-D dynamical system (2.1) to the hyperjerk form in (2.17) without any singularities. Therefore, the inverse procedure defined by (3.3) can give the initial 4-D dynamical system (2.1).

#### 4 Examples of hyperjerk dynamics

Using the above method, we can prove that any dynamical system of the functional form

$$\begin{cases} x' = c_1 + b_{11}x + b_{12}y + n_1(x) \\ y' = c_2 + b_{21}x + b_{22}y + b_{23}z + n_2(x, y) \\ z' = c_3 + b_{31}x + b_{32}y + b_{33}z + b_{34}u + n_3(x, y, z) \\ u' = c_4 + b_{41}x + b_{42}y + b_{43}z + b_{44}u + n_4(x, y, z, u) \end{cases} \quad (4.1)$$

can be reduced to a hyperjerk dynamic form  $x'''' = H(x, x', x'', x''')$  if the following conditions hold:

$$b_{12} \neq 0, b_{23} \neq 0, b_{34} \neq 0 \quad (4.2)$$

That is,  $b_{13} = b_{14} = b_{24} = 0$  in system (2.1). First, we notice that if  $n_2$  and  $n_3$  do not depend on  $z$  and  $u$ , respectively, then there is no need to consider the functions  $g_1$  and  $g_2$  defined above. In this case, we have  $y(x, x') = -\frac{1}{b_{12}}(-x' + c_1 + n_1(x) + b_{11}x)$ . Thus  $y'(x, x', x'') = f_5 = -\frac{b_{11}b_{12} + b_{12}n_{11}(x)}{b_{12}^2}x' + \frac{1}{b_{12}}x''$ , and  $f_1 = 0, f_2 = 0, f_3 = 0, f_4 = 0$ . Also,  $z(x, x', x'') = f_7f_6$ , where  $f_6 = \left(\frac{b_{11}b_{22} - b_{12}b_{21}}{b_{12}}\right)x - \frac{b_{22}c_1}{b_{12}} + \frac{b_{22}}{b_{12}}n_1(x) + f_5, f_7 = \frac{1}{b_{23}}, f_8(x) = 0$ . Hence  $z' = f_7\left(\frac{1}{b_{12}}x''' + f_{10}(x)x'' + f_9x'\right)$ , where  $f_9 = \left(\frac{b_{12}(f_2 - b_{21}) + b_{22}b_{11} + b_{22}n_{11}(x) - n_{111}(x)x'}{b_{12}}\right)$  and  $f_{10}(x) = \left(-\frac{b_{22}}{b_{12}} - \frac{(b_{11} + n_{11}(x))}{b_{12}}\right)$ . By equating the formula for  $z'$  with the third equation of (26), we obtain  $u = \frac{f_7x''' + b_{12}f_{11} + b_{12}f_{12}}{b_{12}b_{34}}$ , where  $f_{11} = \frac{b_{11}b_{32} - b_{12}b_{31}}{b_{12}}x + \frac{-b_{32} + f_7f_9(x, x')b_{12}}{b_{12}}x' + f_7f_{10}(x)x''$  and  $f_{12} = -n_3 - c_3 + \frac{b_{32}(c_1 + n_1(x))}{b_{12}} - f_6f_7b_{33}$ . Hence we have  $u' = f_{13}x^{(4)} + f_{14}$ , where  $f_{13} = \frac{f_7}{b_{12}b_{34}}$  and  $f_{14} = \frac{f'_{11}}{b_{34}} + \frac{f'_{12}}{b_{34}}$ . Finally, we obtain  $x^{(4)} = \frac{f_{15} + f_{16}}{f_{13}} = H(x, x', x'', x''')$ , where  $f_{15} = c_4 + b_{41}x + b_{43}f_7f_6 + \rho_0$  and  $f_{16} = b_{42}\left(-\frac{1}{b_{12}}(-x' + c_1 + n_1(x) + b_{11}x)\right)$  with  $\rho_0 = b_{44}\left(\frac{f_7x''' + b_{12}f_{11} + b_{12}f_{12}}{b_{12}b_{34}}\right) + n_4 - f_{14}$ . In this case, the expression for the transformation  $T = T(x, x', x'', x''')$  is given by

$$\begin{cases} T_1(x, y, z, u) = x \\ T_2(x, y, z, u) = x' = c_1 + b_{11}x + b_{12}y + n_1(x) \\ T_3(x, y, z, u) = x'' = f_{17}(x, y) + f_{18}(x, z, u) + f_{19}(x, y, z, u) \\ T_4(x, y, z, u) = x''' = \Psi_1 + \Psi_2 \\ \Psi_1 = f_{20}x^3 + f_{21}x^2y + f_{22}x^2 + f_{23}xy + f_{24}xz \\ \Psi_2 = f_{26}x + f_{27}y^2 + f_{30}y + f_{31}z + f_{32}u + f_{33} \end{cases} \quad (4.3)$$

and its inverse is defined by

$$\begin{cases} T_1^{-1}(x, x', x'', x''') = x \\ T_4^{-1}(x, x', x'', x''') = u = \frac{f_7x''' + b_{12}f_{11}(x, x', x'') + b_{12}f_{12}(x, x', x'')}{b_{12}b_{34}} \\ T_3^{-1}(x, x', x'', x''') = z = f_7f_6(x, x', x'') \\ T_2^{-1}(x, x', x'', x''') = y = -\frac{1}{b_{12}}(-x' + c_1 + n_1(x) + b_{11}x) \end{cases} \quad (4.4)$$

where

$$\begin{cases} f_{17} = (b_{11}^2 + n_{11}(x)b_{11} + b_{12}b_{21})x + (b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{11}(x))y \\ f_{18} = b_{12}b_{23}z \\ f_{19} = (b_{11}(c_1 + n_1(x)) + b_{12}(c_2 + n_2(x, y)) + n_{11}(x)(c_1 + n_1(x))) \\ f_{20} = b_{11}^2n_{111}(x), f_{21} = b_{11}b_{12}n_{111}(x), f_{22} = \xi_1 + \xi_2 \\ f_{23} = \xi_3 + \xi_4, f_{24} = b_{12}b_{23}n_{11}(x), f_{25} = 0, f_{28} = 0, f_{29} = 0 \\ f_{26} = \xi_5 + \xi_6 + \xi_7 + \xi_8, f_{27} = b_{12}^2n_{111} \\ f_{30} = \xi_{10} + \xi_{11} + \xi_{12}, f_{31} = \xi_{13} + \xi_{14}, f_{32} = \xi_{15} + \xi_{16} \\ f_{33} = \xi_{17} + \xi_{18} + \xi_{19} + \xi_{20} \end{cases} \quad (4.5)$$

where

$$\left\{ \begin{array}{l} \xi_1 = b_{11} (n_{11}^2 + b_{11}n_{11} + c_1n_{111} + n_1n_{111}) (x) \\ \xi_2 = b_{11}n_{111} (x) (c_1 + n_1 (x)) + (b_{12}b_{21}) n_{11} (x) \\ \xi_3 = b_{12} (n_{11}^2 (x) + b_{11}n_{11} (x) + (c_1 + n_1 (x)) n_{111} (x)) \\ \xi_4 = (b_{12}b_{22}) n_{11} (x) + b_{11}b_{12}n_{111} (x) \\ \xi_5 = (c_1 + n_1 (x)) (n_{11}^2 (x) + b_{11}n_{11} (x) + (c_1 + n_1 (x)) n_{111} (x)) \\ \xi_6 = b_{11} (b_{11}^2 + (b_{11} + c_1 + n_1) n_{11} (x) + b_{12}b_{21} + b_{12}n_{21} (x, y)) \\ \xi_7 = b_{21} (b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22} (x, y)) + b_{31} (b_{12}b_{23} + b_{12}n_{23} (x, y)) \\ \xi_8 = b_{41} (b_{12}n_{24} (x, y)) + b_{12}n_{11} (x) (c_2 + n_2 (x, y)) \\ \xi_9 = 0 \\ \xi_{10} = b_{12} (b_{11}^2 + n_{11}b_{11} + c_1n_{11} + n_1n_{11} + b_{12}b_{21} + b_{12}n_{21}) \end{array} \right. \quad (4.6)$$

and

$$\left\{ \begin{array}{l} \xi_{11} = b_{22} (b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22}) + b_{32} (b_{12}b_{23} + b_{12}n_{23}) \\ \xi_{12} = b_{42} (b_{12}n_{24}) + b_{12}n_{111} (c_1 + n_1) \\ \xi_{13} = b_{23} (b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22}) + b_{33} (b_{12}b_{23} + b_{12}n_{23}) \\ \xi_{14} = b_{43} (b_{12}n_{24}) \\ \xi_{15} = b_{34} (b_{12}b_{23} + b_{12}n_{23}) \\ \xi_{16} = b_{44}b_{12}n_{24} \\ \xi_{17} = (c_1 + n_1) (b_{11}^2 + n_{11}b_{11} + c_1n_{11} + n_1n_{11} + b_{12}b_{21} + b_{12}n_{21}) \\ \xi_{18} = (c_2 + n_2 (x, y)) (b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22} (x, y)) \\ \xi_{19} = (c_4 + n_4 (x, y, z, u)) (b_{12}n_{24} (x, y)) \\ \xi_{20} = (c_3 + n_3 (x, y, z)) (b_{12}b_{23} + b_{12}n_{23} (x, y)) \end{array} \right. \quad (4.7)$$

We remark that if  $n_1(x) = 0$ , then the expression for the transformation  $T = T(x, x', x'', x''')$  is given by

$$\left\{ \begin{array}{l} T_1(x, y, z, u) = x \\ T_2(x, y, z, u) = x' = c_1 + b_{11}x + b_{12}y \\ T_3(x, y, z, u) = x'' = f_{17}(x, y) + f_{18}(z) + f_{19}(x, y) \\ T_4(x, y, z, u) = x''' = f_{26}x + f_{30}y + f_{31}z + f_{32}u + f_{33}(x, y, z, u) \end{array} \right. \quad (4.8)$$

and its inverse is defined by

$$\left\{ \begin{array}{l} T_1^{-1}(x, x', x'', x''') = x \\ T_4^{-1}(x, x', x'', x''') = u = \frac{f_7x''' + b_{12}f_{11}(x, x', x'') + b_{12}f_{12}(x, x', x'')}{b_{12}b_{34}} \\ T_3^{-1}(x, x', x'', x''') = z = f_7f_6(x, x', x'') \\ T_2^{-1}(x, x', x'', x''') = y = -\frac{1}{b_{12}}(-x' + c_1 + n_1(x) + b_{11}x) \end{array} \right. \quad (4.9)$$

where

$$\left\{ \begin{array}{l} f_{17} = (b_{11}^2 + b_{12}b_{21})x + (b_{11}b_{12} + b_{12}b_{22})y \\ f_{18} = b_{12}b_{23}z, f_{19} = (b_{11}(c_1) + b_{12}(c_2 + n_2(x, y))) \\ f_{20} = 0, f_{21} = 0, f_{22} = 0, f_{23} = 0, f_{24} = 0, f_{25} = 0, f_{28} = 0, f_{29} = 0, f_{27} = 0 \\ f_{26} = \xi_6 + \xi_7 + \xi_8, f_{30} = \xi_{10} + \xi_{11} + \xi_{12}, f_{31} = \xi_{13} + \xi_{14} \\ f_{32} = \xi_{15} + \xi_{16}, f_{33} = \xi_{17} + \xi_{18} + \xi_{19} + \xi_{20} \end{array} \right. \quad (4.10)$$

where

$$\left\{ \begin{array}{l} \xi_1 = 0, \xi_2 = 0, \xi_3 = 0, \xi_4 = 0, \xi_5 = 0, \xi_9 = 0 \\ \xi_6 = b_{11} (b_{11}^2 + b_{12}b_{21} + b_{12}n_{21} (x, y)) \\ \xi_7 = b_{21} (b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22} (x, y)) + b_{31} (b_{12}b_{23} + b_{12}n_{23} (x, y)) \\ \xi_8 = b_{41} (b_{12}n_{24} (x, y)), \xi_{10} = b_{12} (b_{11}^2 + b_{12}b_{21} + b_{12}n_{21} (x, y)) \\ \xi_{11} = b_{22} (b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22} (x, y)) + b_{32} (b_{12}b_{23} + b_{12}n_{23} (x, y)) \\ \xi_{12} = b_{42} (b_{12}n_{24}) \\ \xi_{13} = b_{23} (b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22} (x, y)) + b_{33} (b_{12}b_{23} + b_{12}n_{23} (x, y)) \\ \xi_{14} = b_{43} (b_{12}n_{24}) \end{array} \right. \quad (4.11)$$

and

$$\left\{ \begin{array}{l} \xi_{15} = b_{34} (b_{12}b_{23} + b_{12}n_{23} (x, y)) \\ \xi_{16} = b_{44}b_{12}n_{24} (x, y) \\ \xi_{17} = c_1 (b_{11}^2 + b_{12}b_{21} + b_{12}n_{21} (x, y)) \\ \xi_{18} = (c_2 + n_2 (x, y)) (b_{11}b_{12} + b_{12}b_{22} + b_{12}n_{22} (x, y)) \\ \xi_{19} = b_{12}n_{24} (x, y) (c_4 + n_4 (x, y, z, u)) \\ \xi_{20} = (c_3 + n_3 (x, y, z)) (b_{12}b_{23} + b_{12}n_{23} (x, y)) \end{array} \right. \quad (4.12)$$

Now we find sufficient conditions for the equivalence of a 4-D quadratic system with a hyperjerk system. According to the previous analysis, the general 4-D quadratic system has the following form

$$\left\{ \begin{array}{l} x' = c_1 + b_{11}x + b_{12}y + n_1 (x) \\ y' = c_2 + b_{21}x + b_{22}y + b_{23}z + n_2 (x, y, z, u) \\ z' = c_3 + b_{31}x + b_{32}y + b_{33}z + b_{34}u + n_3 (x, y, z, u) \\ u' = c_4 + b_{41}x + b_{42}y + b_{43}z + b_{44}u + n_4 (x, y, z, u) \end{array} \right. \quad (4.13)$$

where

$$\left\{ \begin{array}{l} n_1 (x) = a_4x^2 \\ n_2 = b_4x^2 + b_5y^2 + b_6z^2 + b_7u^2 + b_8xy + b_9xz + b_{10}yz + b_{11}xu + b_{12}zu + b_{13}yu \\ n_3 = e_4x^2 + e_5y^2 + e_6z^2 + e_7u^2 + e_8xy + e_9xz + e_{10}yz + e_{11}xu + e_{12}zu + e_{13}yu \\ n_4 = d_4x^2 + d_5y^2 + d_6z^2 + d_7u^2 + d_8xy + d_9xz + d_{10}yz + d_{11}xu + d_{12}zu + d_{13}yu \end{array} \right. \quad (4.14)$$

By using the form (4.1), it is easy to show that any 4-D quadratic system of the form

$$\left\{ \begin{array}{l} x' = c_1 + b_{11}x + b_{12}y + a_4x^2 \\ y' = c_2 + b_{21}x + b_{22}y + b_{23}z + b_4x^2 + b_5y^2 + b_8xy \\ z' = c_3 + b_{31}x + b_{32}y + b_{33}z + b_{34}u + e_4x^2 + e_5y^2 + e_6z^2 + e_8xy + e_9xz + e_{10}yz \\ u' = c_4 + b_{41}x + b_{42}y + b_{43}z + b_{44}u + n_4 (x, y, z, u) \end{array} \right. \quad (4.15)$$

can be reduced to a hyperjerk dynamic form  $x'''' = H(x, x', x'', x''')$  if the conditions (4.2) hold. We remark that the form (4.15) contains 20 nonlinearities, and the best known hyperchaotic 4-D quadratic system is the Rössler system that contains one nonlinearity  $xz$  and is given by  $x' = -y - z$ ,  $y' = x + ay + u$ ,  $z' = b + xz$ ,  $u' = cu - dz$ , but this system is not equivalent to any hyperjerk system due to the presence of singularities. In [8] the algebraically simplest hyperchaotic snap system was studied, and it is given by  $x'''' + x^4x''' + Ax'' + x' + x = 0$ . This system displays hyperchaos when  $A = 3.6$  with Lyapunov exponents  $(0.1310, 0.0358, 0, -1.2550)$ .

## 5 Conclusion

The possible cases for equivalence between four-dimensional autonomous dynamical systems and hyperjerk dynamics were studied in this paper. Some examples were also presented and discussed.

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