Palestine Journal of Mathematics

Vol. 2(1) (2013) , 50–64

Strong convergence of multi-step iterates with errors for generalized asymptotically quasi-nonexpansive mappings

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Communicated by Ayman Badawi

MSC 2010 Classifications: 47H09, 47H10.

Keywords and phrases: Generalized asymptotically quasi-nonexpansive mapping, common fixed point, multi-step iteration with errors, uniformly convex Banach space, uniformly (L, α)-Lipschitz mapping, strong convergence, weak convergence.

Abstract. In this paper, we study multi-step iteration with errors and give the necessary and sufficient condition to converge to common fixed points for a finite family of generalized asymptotically quasi-nonexpansive mappings in the framework of Banach spaces. Also we establish some strong convergence theorems to converge to common fixed points for a finite family of said mappings and scheme in a uniformly convex Banach spaces. Our results extend and improve the corresponding results of [1, 2, 5, 7, 8, 9, 11, 12, 17, 23].

1 Introduction

Let K be a subset of normed space E and $T\colon K \to K$ be a mapping. Then

(1) T is said to be an asymptotically nonexpansive mapping [3], if there exists a sequence $\{r_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$\|T^n x - T^n y\| \quad \leq \quad (1 + r_n)\,\|x - y\|\,, \tag{1.1}$$

for all $x,\, y \in K$.

(2) T is said to be (L, α)-uniformly Lipschitz [9] if there are constants $L > 0$ and $\alpha > 0$ such that

$$\|T^n x - T^n y\| \quad \leq \quad L\,\|x - y\|^{\alpha}\,, \quad \forall n \geq 1, \tag{1.2}$$

for all $x,\, y \in K$. Every asymptotically nonexpansive mapping is $(L, 1)$-uniformly Lipschitz mapping.

(3) T is said to be an asymptotically quasi-nonexpansive mapping, if $F(T) \neq \emptyset$ and there exists a sequence $\{r_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ such that

$$\|T^n x - p\| \quad \leq \quad (1 + r_n)\,\|x - p\|\,, \quad \forall x \in K \quad and \quad p \in F(T). \tag{1.3}$$

(4) T is said to be generalized asymptotically quasi-nonexpansive [18] if there exist sequences $\{r_n\}$, $\{s_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} r_n = 0 = \lim_{n\to\infty} s_n$ such that

$$\|T^n x - p\| \quad \leq \quad (1 + r_n)\,\|x - p\| + s_n, \tag{1.4}$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

If $s_n = 0$ for all $n \geq 1$, then T is known as an asymptotically quasi-nonexpansive mapping.

From the above definitions, it follows that if $F(T)$ is nonempty, then asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings are all special cases of generalized asymptotically quasi-nonexpansive mappings. But the converse does not hold in general.

In 1973, Petryshyn and Williamson [11] gave the necessary and sufficient condition for Mann iterative sequence (cf.[10]) to converge to fixed points of quasi-nonexpansive mappings. In 1997, Ghosh and Debnath [2] extended the results of Petryshyn and Williamson [11] and gave the necessary and sufficient condition for Ishikawa iterative sequence to converge to fixed points for quasi-nonexpansive mappings.

Liu [8] extended the results of [2, 11] and gave the necessary and sufficient condition for Ishikawa iterative sequence with errors to converge to fixed points of asymptotically quasi-nonexpansive mappings.

Iterative techniques for approximating fixed points of asymptotically nonexpansive and asymptotically quasi nonexpansive mappings in Banach spaces have been studied by many authors; see [3, 7, 8, 17, 19, 20, 21] and the references therein. Related work can be found in [1, 5, 12, 13, 14, 15, 23] and many others.

Recently, Tang and Peng [22] studied the following iteration scheme in Banach space:

Let $\{T_i : i = 1, 2, \ldots, k\}\colon K \to K$, where K is a nonempty subset of a Banach space E, be a finite family of uniformly quasi-Lipschitzian mappings. For a given $x_1 \in K$, then the sequence $\{x_n\}$ is defined by

$$\begin{aligned}
x_{n+1} &= a_{kn}x_n + b_{kn}T_k^n y_{(k-1)n} + c_{kn}u_{kn},\\
y_{(k-1)n} &= a_{(k-1)n}x_n + b_{(k-1)n}T_{k-1}^n y_{(k-2)n} + c_{(k-1)n}u_{(k-1)n},\\
y_{(k-2)n} &= a_{(k-2)n}x_n + b_{(k-2)n}T_{k-2}^n y_{(k-3)n} + c_{(k-2)n}u_{(k-2)n},\\
&\vdots\\
y_{2n} &= a_{2n}x_n + b_{2n}T_2^n y_{1n} + c_{2n}u_{2n}\\
y_{1n} &= a_{1n}x_n + b_{1n}T_1^n x_n + c_{1n}u_{1n}, \qquad n \geq 1,
\end{aligned} \tag{1.5}$$

where $\{a_{in}\}$, $\{b_{in}\}$, $\{c_{in}\}$ are sequences in $[0, 1]$ with $a_{in} + b_{in} + c_{in} = 1$ for all $i = 1, 2, \ldots, k$ and $n \geq 1$, $\{u_{in},\ i = 1, 2, \ldots, k,\ n \geq 1\}$ are bounded sequences in K. Also, they gave the necessary and sufficient condition to converge to common fixed points for a finite family of said class of mappings.

Remark 1.1. The iterative algorithm (1.5) is called multi-step iterative algorithm with errors. It contains well known iterations as special case. Such as, the modified Mann iteration (see, [19]), the modified Ishikawa iteration (see, [21]), the three-step iteration (see, [23]), the multi-step iteration (see, [5]).

The purpose of this paper is to study the multi-step iterative algorithm with bounded errors (1.5) for a finite family of generalized asymptotically quasi-nonexpansive mappings to converge to common fixed points in Banach spaces. The results obtained in this paper extend and improve the corresponding results of [1, 2, 5, 7, 8, 9, 11, 17, 23] and many others.

2 Preliminaries

The following lemmas will be used to prove the main results of this paper:

Lemma 2.1. (see [20]) Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \qquad n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. (Schu [19]) Let E be a uniformly convex Banach space and $0 < a \leq t_n \leq b < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E satisfying $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r$ for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Recall that the following:

A family $\{T_i : i = 1, 2, \dots, k\}$ of self-mappings of K with $\mathcal{F} = \cap_{i=1}^{k} F(T_i) \neq \emptyset$ is said to satisfy the following conditions.

(1) Condition $(\overline{A})$ [1]. If there is a nondecreasing function $f\colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0.\infty)$ such that $1/k \sum_{i=1}^{k} \|x - T_i x\|$ $\geq f(d(x, \mathcal{F}))$ for all $x \in K$, where $d(x, \mathcal{F}) = \inf\{\|x - p\| : p \in \mathcal{F}\}$.

(2) Condition $(\overline{B})$ [1]. If there is a nondecreasing function $f\colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0.\infty)$ such that $\max_{1 \leq i \leq k}\{\|x - T_i x\|\}$ $\geq f(d(x, \mathcal{F}))$ for all $x \in K$.

(3) Condition $(\overline{C})$ [1]. If there is a nondecreasing function $f\colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0.\infty)$ such that $\|x - T_l x\|\} \geq f(d(x, \mathcal{F}))$ for all $x \in K$ and for at least one $T_l,\ l = 1, 2, \dots, k$.

Note that condition $(\overline{B})$ and $(\overline{C})$ are equivalent, condition $(\overline{B})$ reduces to condition (A) [16] when all but one $T_l' s$ are identities, and in addition, it also condition $(\overline{A})$.

It is well known that every continuous and demicompact mapping must satisfy condition (A) (see [16]). Since every completely continuous mapping $T\colon K \to K$ is continuous and demicompact so that it satisfies condition (A). Thus we will use condition $(\overline{C})$ instead of the demicompactness and complete continuity of a family $\{T_i : i = 1, 2, \dots, k\}$.
Let K be a nonempty closed convex subset of a Banach space E. Then $I - T$ is demiclosed at zero if, for any sequence $\{x_n\}$ in K, condition $x_n \to x$ weakly and $\lim_{n\to\infty} \|x_n - T x_n\| = 0$ implies $(I - T)x = 0$.

3 Main Results

In this section, we prove strong convergence theorems of multi-step iterative algorithm with bounded errors for a finite family of generalized asymptotically quasi-nonexpansive mappings in a real Banach space.

Theorem 3.1. Let E be a real arbitrary Banach space, K be a nonempty closed convex subset of E. Let $\{T_i : i = 1, 2, \dots, k\}\colon K \to K$ be a finite family of generalized asymptotically quasi-nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by (1.5) with $\sum_{n=1}^{\infty} r_{in} < \infty$, $\sum_{n=1}^{\infty} s_{in} < \infty$ and $\sum_{n=1}^{\infty} c_{in} < \infty$ for all $i = 1, 2, \dots, k$. If $\mathcal{F} = \cap_{i=1}^{k} F(T_i) \neq \emptyset$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i = 1, 2, \dots, k\}$ if and only if $\liminf\limits_{n\to\infty} d(x_n, \mathcal{F}) = 0$, where $d(x, \mathcal{F})$ denotes the distance between x and the set $\mathcal{F}$.

Proof. The necessity is obvious and it is omitted. Now we prove the sufficiency. Since $\{u_{in},\ i = 1, 2, \dots, k,\ n \geq 1\}$ are bounded sequences in K, therefore there exists a $M > 0$, such that

$$M = \max\left\{\sup_{n\geq 1} \|u_{in} - p\|\,,\ i = 1, 2, \dots, k\right\}.$$

Let $p \in \mathcal{F}$, $r_n = \max\{r_{in} : i = 1, 2, \dots, k\}$ and $s_n = \max\{s_{in} : i = 1, 2, \dots, k\}$ for all n. Since $\sum_{n=1}^{\infty} r_{in} < \infty$ and $\sum_{n=1}^{\infty} s_{in} < \infty$, for all $i = 1, 2, \dots, k$, therefore $\sum_{n=1}^{\infty} r_n < \infty$ and

$\sum_{n=1}^{\infty} s_n < \infty$. For each $n \geq 1$, from (1.4) and (1.5), we note that

$$\begin{aligned}
\|y_{1n} - p\| &= \|a_{1n}x_n + b_{1n}T_1^n x_n + c_{1n}u_{1n} - p\| \\
&\leq a_{1n}\|x_n - p\| + b_{1n}\|T_1^n x_n - p\| + c_{1n}\|u_{1n} - p\| \\
&\leq a_{1n}\|x_n - p\| + b_{1n}\Big[(1 + r_{1n})\|x_n - p\| + s_{1n}\Big] \\
&\quad + c_{1n}\|u_{1n} - p\| \\
&\leq a_{1n}\|x_n - p\| + b_{1n}\Big[(1 + r_n)\|x_n - p\| + s_n\Big] \\
&\quad + c_{1n}\|u_{1n} - p\| \\
&\leq \Big(a_{1n} + b_{1n}\Big)(1 + r_n)\|x_n - p\| + b_{1n}s_n + c_{1n}M \\
&= \Big(1 - c_{1n}\Big)(1 + r_n)\|x_n - p\| + b_{1n}s_n + c_{1n}M \\
&\leq (1 + r_n)\|x_n - p\| + s_n + c_{1n}M \\
&= (1 + r_n)\|x_n - p\| + A_{1n} \qquad (3.1)
\end{aligned}$$

where $A_{1n} = s_n + c_{1n}M$, since by assumption $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} c_{1n} < \infty$, it follows that $\sum_{n=1}^{\infty} A_{1n} < \infty$.

Furthermore, from (1.5) and (3.1), we obtain

$$\begin{aligned}
\|y_{2n} - p\| &= \|a_{2n}x_n + b_{2n}T_2^n y_{1n} + c_{2n}u_{2n} - p\| \\
&\leq a_{2n}\|x_n - p\| + b_{2n}\|T_2^n y_{1n} - p\| + c_{2n}\|u_{2n} - p\| \\
&\leq a_{2n}\|x_n - p\| + b_{2n}\Big[(1 + r_{2n})\|y_{1n} - p\| + s_{2n}\Big] \\
&\quad + c_{2n}\|u_{2n} - p\| \\
&\leq a_{2n}\|x_n - p\| + b_{2n}\Big[(1 + r_n)\|y_{1n} - p\| + s_n\Big] \\
&\quad + c_{2n}\|u_{2n} - p\| \\
&\leq a_{2n}\|x_n - p\| + b_{2n}(1 + r_n)\|y_{1n} - p\| + b_{2n}s_n + c_{2n}M \\
&\leq a_{2n}\|x_n - p\| + b_{2n}(1 + r_n)\Big[(1 + r_n)\|x_n - p\| + A_{1n}\Big] \\
&\quad + b_{2n}s_n + c_{2n}M \\
&\leq \big(a_{2n} + b_{2n}\big)(1 + r_n)^2\|x_n - p\| + b_{2n}(1 + r_n)A_{1n} \\
&\quad + b_{2n}s_n + c_{2n}M \\
&= \big(1 - c_{2n}\big)(1 + r_n)^2\|x_n - p\| + b_{2n}(1 + r_n)A_{1n} \\
&\quad + b_{2n}s_n + c_{2n}M \\
&\leq (1 + r_n)^2\|x_n - p\| + (1 + r_n)A_{1n} + s_n + c_{2n}M \\
&\leq (1 + r_n)^2\|x_n - p\| + A_{2n} \qquad (3.2)
\end{aligned}$$

where $A_{2n} = (1 + r_n)A_{1n} + s_n + c_{2n}M$, since by assumption $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty} c_{2n} < \infty$ and $\sum_{n=1}^{\infty} A_{1n} < \infty$, it follows that $\sum_{n=1}^{\infty} A_{2n} < \infty$. Similarly, using (1.5) and

(3.2), we see that

$$\begin{aligned}
\|y_{3n} - p\| &= \|a_{3n}(x_n - p) + b_{3n}(T_3^n y_{2n} - p) + c_{3n}(u_{3n} - p)\| \\
&\le a_{3n}\|x_n - p\| + b_{3n}\|T_3^n y_{2n} - p\| + c_{3n}\|u_{3n} - p\| \\
&\le a_{3n}\|x_n - p\| + b_{3n}\Big[(1 + r_{3n})\|y_{2n} - p\| + s_{3n}\Big] \\
&\quad + c_{3n}\|u_{3n} - p\| \\
&\le a_{3n}\|x_n - p\| + b_{3n}\Big[(1 + r_n)\|y_{2n} - p\| + s_n\Big] \\
&\quad + c_{3n}\|u_{3n} - p\| \\
&\le a_{3n}\|x_n - p\| + b_{3n}(1 + r_n)\|y_{2n} - p\| + b_{3n}s_n + c_{3n}M \\
&\le a_{3n}\|x_n - p\| + b_{3n}(1 + r_n)\Big[(1 + r_n)^2\|x_n - p\| + A_{2n}\Big] \\
&\quad + b_{3n}s_n + c_{3n}M \\
&\le \big(a_{3n} + b_{3n}\big)(1 + r_n)^3\|x_n - p\| + b_{3n}(1 + r_n)A_{2n} \\
&\quad + b_{3n}s_n + c_{3n}M \\
&= \big(1 - c_{3n}\big)(1 + r_n)^3\|x_n - p\| + b_{3n}(1 + r_n)A_{2n} \\
&\quad + b_{3n}s_n + c_{3n}M \\
&\le (1 + r_n)^3\|x_n - p\| + (1 + r_n)A_{2n} + s_n + c_{3n}M \\
&\le (1 + r_n)^3\|x_n - p\| + A_{3n} \qquad (3.3)
\end{aligned}$$

where $A_{3n} = (1 + r_n)A_{2n} + s_n + c_{3n}M$, since by assumption $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty} c_{3n} < \infty$ and $\sum_{n=1}^{\infty} A_{2n} < \infty$, it follows that $\sum_{n=1}^{\infty} A_{3n} < \infty$. By continuing the above process, there are nonnegative real sequences $\{A_{in}\}$ in $[0, \infty)$ such that $\sum_{n=1}^{\infty} A_{in} < \infty$ and

$$\|y_{in} - p\| \le (1 + r_n)^i\|x_n - p\| + A_{in}, \ \forall\, i = 1, 2, \dots, k. \qquad (3.4)$$

For the case $i = k$, from (1.5) and (3.4), we have

$$\|x_{n+1} - p\| \le (1 + r_n)^k\|x_n - p\| + A_{kn}, \ \forall\, n \ge 1 \text{ and } p \in \mathcal{F}, \qquad (3.5)$$

where $A_{kn} = (1+r_n)A_{(k-1)n} + s_n + c_{kn}M$, since by assumption $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty} c_{kn} < \infty$ and $\sum_{n=1}^{\infty} A_{(k-1)n} < \infty$, it follows that $\sum_{n=1}^{\infty} A_{kn} < \infty$. This implies that

$$\begin{aligned}
d(x_{n+1}, \mathcal{F}) &\le (1 + r_n)^k d(x_n, \mathcal{F}) + A_{kn} \\
&= \Big(1 + \sum_{t=1}^{k} \frac{k(k-1)\dots(k-t+1)}{t!} r_n^t\Big) d(x_n, \mathcal{F}) \\
&\quad + A_{kn}. \qquad (3.6)
\end{aligned}$$

Since $\sum_{n=1}^{\infty} r_n < \infty$, it follows that $\sum_{n=1}^{\infty}\sum_{t=1}^{k}(k(k-1)\dots(k-t+1)/t!)r_n^t < \infty$ and $\sum_{n=1}^{\infty} A_{kn} < \infty$. Therefore, applying Lemma 2.1 to the inequality (3.6), we conclude that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists. Since by hypothesis $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$, so by Lemma 2.1, we have

$$\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0. \qquad (3.7)$$

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. If $x \ge 0$, then $1 + x \le e^x$ and so,

$(1+x)^k \leq e^{kx}$, for $k = 1, 2, \ldots$. Thus, from (3.5), it follows that

$$\begin{aligned}
\|x_{n+m} - p\| &\leq (1 + r_{n+m-1})^k \|x_{n+m-1} - p\| + A_{k(n+m-1)} \\
&\leq exp\{kr_{n+m-1}\} \|x_{n+m-1} - p\| + A_{k(n+m-1)} \\
&\leq \ldots \\
&\leq \ldots \\
&\leq exp\{k \sum_{i=n}^{n+m-1} r_i\} \|x_n - p\| + \sum_{i=n}^{n+m-1} A_{ki} \\
&\leq exp\{k \sum_{i=1}^{\infty} r_i\} \|x_n - p\| + \sum_{i=n}^{\infty} A_{ki} \\
&\leq Q \|x_n - p\| + \sum_{i=n}^{\infty} A_{ki} \qquad (3.8)
\end{aligned}$$

where $Q = exp\{k \sum_{i=1}^{\infty} r_i\}$, for all $p \in \mathcal{F}$ and $m, n \in \mathbb{N}$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, for each $\varepsilon > 0$, there exists a natural number n_1 such that for $n \geq n_1$,

$$d(x_n, \mathcal{F}) < \frac{\varepsilon}{4(1+Q)} \quad \text{and} \quad \sum_{i=n_1}^{n+m-1} A_{ki} < \frac{\varepsilon}{2}. \qquad (3.9)$$

Hence, there exists a point $q \in \mathcal{F}$ such that

$$\|x_{n_1} - q\| < \frac{\varepsilon}{2(1+Q)}. \qquad (3.10)$$

By (3.8), (3.9) and (3.10), for all $n \geq n_1$ and $m \geq 1$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - q\| + \|x_n - q\| \\
&\leq Q \|x_{n_1} - q\| + \sum_{i=n_1}^{\infty} A_{ki} + \|x_{n_1} - q\| \\
&\leq (1+Q) \|x_{n_1} - q\| + \sum_{i=n_1}^{\infty} A_{ki} \\
&< (1+Q).\frac{\varepsilon}{2(1+Q)} + \frac{\varepsilon}{2} = \varepsilon. \qquad (3.11)
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since E is complete, there exists a $p_1 \in E$ such that $x_n \to p_1$ as $n \to \infty$.

Now we have to prove that p_1 is a common fixed point of $\{T_i : i = 1, 2, \ldots, k\}$, that is, $p_1 \in \mathcal{F}$.

By contradiction, we assume that p_1 is not in $\mathcal{F}$. Since $\mathcal{F} = \cap_{i=1}^{k} F(T_i)$ is closed in Banach spaces, $d(p_1, \mathcal{F}) > 0$. So for all $p_2 \in \mathcal{F}$, we have

$$\|p_1 - p_2\| \leq \|p_1 - x_n\| + \|x_n - p_2\|. \qquad (3.12)$$

By the arbitrary of $p_2 \in \mathcal{F}$, we know that

$$d(p_1, \mathcal{F}) \leq \|p_1 - x_n\| + d(x_n, \mathcal{F}). \qquad (3.13)$$

By $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, above inequality and $x_n \to p_1$ as $n \to \infty$, we have

$$d(p_1, \mathcal{F}) = 0, \qquad (3.14)$$

which contradicts $d(p_1, \mathcal{F}) > 0$. Thus p_1 is a common fixed point of the mappings $\{T_i : i = 1, 2, \ldots, k\}$. This completes the proof.

Theorem 3.2. Let K be a nonempty compact convex subset of a uniformly convex Banach space E and for $i = 1, 2, \ldots, k$, let $T_i \colon K \to K$ be a finite family of uniformly (L_i, α_i)-Lipschitz and generalized asymptotically quasi-nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by (1.5) with $\sum_{n=1}^{\infty} r_{in} < \infty$, $\sum_{n=1}^{\infty} s_{in} < \infty$, $\sum_{n=1}^{\infty} c_{in} < \infty$ and $0 < \alpha \leq b_{in} \leq \beta < 1$ for all $i = 1, 2, \ldots, k$. If $\mathcal{F} = \cap_{i=1}^{k} F(T_i) \neq \emptyset$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i = 1, 2, \ldots, k\}$.

Proof. Let $p \in \mathcal{F}$, $r_n = \max\{r_{in} : i = 1, 2, \ldots, k\}$ and $s_n = \max\{s_{in} : i = 1, 2, \ldots, k\}$ for all n. From Theorem 3.1, we have that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \mathcal{F}$. Let $\lim_{n\to\infty} \|x_n - p\| = R$ for some $R > 0$. Then, from (3.1), we note that

$$\begin{aligned} \limsup_{n\to\infty} \|y_{1n} - p\| &\leq \limsup_{n\to\infty} \Big((1 + r_n) \|x_n - p\| + A_{1n} \Big) \\ &\leq \limsup_{n\to\infty} \|x_n - p\| = R, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \limsup_{n\to\infty} \|T_1^n x_n - p\| &\leq \limsup_{n\to\infty} \Big((1 + r_{1n}) \|x_n - p\| + s_{1n} \Big) \\ &\leq \limsup_{n\to\infty} \Big((1 + r_n) \|x_n - p\| + s_n \Big) \\ &\leq \limsup_{n\to\infty} \|x_n - p\| = R, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \lim_{n\to\infty} \|y_{1n} - p\| &= \lim_{n\to\infty} \|a_{1n} x_n + b_{1n} T_1^n x_n + c_{1n} u_{1n} - p\| \\ &= \lim_{n\to\infty} \|(1 - b_{1n} - c_{1n}) x_n + b_{1n} T_1^n x_n + c_{1n} u_{1n} - p\| \\ &= \lim_{n\to\infty} \|(1 - b_{1n})(x_n - p + c_{1n}(u_{1n} - x_n)) \\ &\qquad + b_{1n}(T_1^n x_n - p + c_{1n}(u_{1n} - x_n))\| \\ &= R. \end{aligned} \tag{3.17}$$

Again since $\lim_{n\to\infty} \|x_n - p\|$ exists, so $\{x_n\}$ is a bounded sequence in K. By virtue of condition $\sum_{n=1}^{\infty} c_{in} < \infty$ for all $i = 1, 2, \ldots, k$ and the boundedness of the sequence $\{x_n\}$ and $\{u_{1n}\}$, we have

$$\begin{aligned} \limsup_{n\to\infty} \|x_n - p + c_{1n}(u_{1n} - x_n)\| &\leq \limsup_{n\to\infty} \|x_n - p\| \\ &\quad + \limsup_{n\to\infty} \Big(c_{1n} \|u_{1n} - x_n\| \Big) \\ &\leq R, \ p \in \mathcal{F}. \end{aligned} \tag{3.18}$$

It follows from (3.16) that

$$\begin{aligned} \limsup_{n\to\infty} \|T_1^n x_n - p + c_{1n}(u_{1n} - x_n)\| &\leq \limsup_{n\to\infty} \|T_1^n x_n - p\| \\ &\quad + \limsup_{n\to\infty} \Big(c_{1n} \|u_{1n} - x_n\| \Big) \\ &\leq \limsup_{n\to\infty} \Big((1 + r_{1n}) \|x_n - p\| + s_{1n} \Big) \\ &\quad + \limsup_{n\to\infty} \Big(c_{1n} \|u_{1n} - x_n\| \Big) \\ &\leq \limsup_{n\to\infty} \Big((1 + r_n) \|x_n - p\| + s_n \Big) \\ &\quad + \limsup_{n\to\infty} \Big(c_{1n} \|u_{1n} - x_n\| \Big) \\ &\leq R, \ p \in \mathcal{F}. \end{aligned} \tag{3.19}$$

Therefore, from (3.17) - (3.19) and Lemma 2.2 we know that

$$\lim_{n\to\infty} \|T_1^n x_n - x_n\| = 0. \tag{3.20}$$

Again from (3.2), we note that

$$\begin{aligned}\limsup_{n\to\infty}\|y_{2n}-p\| &\le \limsup_{n\to\infty}\Big((1+r_n)^2\|x_n-p\|+A_{2n}\Big)\\ &\le \limsup_{n\to\infty}\|x_n-p\|=R, \end{aligned}\tag{3.21}$$

and from (3.15), we note that

$$\begin{aligned}\limsup_{n\to\infty}\|T_2^n y_{1n}-p\| &\le \limsup_{n\to\infty}\Big((1+r_{2n})\|y_{1n}-p\|+s_{2n}\Big)\\ &\le \limsup_{n\to\infty}\Big((1+r_n)\|y_{1n}-p\|+s_n\Big)\\ &\le \limsup_{n\to\infty}\|y_{1n}-p\|=R. \end{aligned}\tag{3.22}$$

Next, consider

$$\begin{aligned}\limsup_{n\to\infty}\|T_2^n y_{1n}-p+c_{2n}(u_{2n}-x_n)\| &\le \limsup_{n\to\infty}\|T_2^n y_{1n}-p\|\\ &\quad+\limsup_{n\to\infty}\Big(c_{2n}\|u_{2n}-x_n\|\Big)\\ &\le \limsup_{n\to\infty}\Big((1+r_{2n})\|y_{1n}-p\|+s_{2n}\Big)\\ &\quad+\limsup_{n\to\infty}\Big(c_{2n}\|u_{2n}-x_n\|\Big)\\ &\le \limsup_{n\to\infty}\Big((1+r_n)\|y_{1n}-p\|+s_n\Big)\\ &\quad+\limsup_{n\to\infty}\Big(c_{2n}\|u_{2n}-x_n\|\Big)\\ &\le R,\ p\in\mathcal{F}. \end{aligned}\tag{3.23}$$

Also,

$$\begin{aligned}\limsup_{n\to\infty}\|x_n-p+c_{2n}(u_{2n}-x_n)\| &\le \limsup_{n\to\infty}\|x_n-p\|\\ &\quad+\limsup_{n\to\infty}\Big(c_{2n}\|u_{2n}-x_n\|\Big)\\ &\le R,\ p\in\mathcal{F}, \end{aligned}\tag{3.24}$$

and

$$\begin{aligned}\lim_{n\to\infty}\|y_{2n}-p\| &= \lim_{n\to\infty}\|a_{2n}x_n+b_{2n}T_2^n y_{1n}+c_{2n}u_{2n}-p\|\\ &= \lim_{n\to\infty}\|(1-b_{2n}-c_{2n})x_n+b_{2n}T_2^n y_{1n}+c_{2n}u_{2n}-p\|\\ &= \lim_{n\to\infty}\|(1-b_{2n})(x_n-p+c_{2n}(u_{2n}-x_n))\\ &\qquad+b_{2n}(T_2^n y_{1n}-p+c_{2n}(u_{2n}-x_n))\|\\ &=R. \end{aligned}\tag{3.25}$$

Therefore, from (3.23) - (3.25) and Lemma 2.2 we know that

$$\lim_{n\to\infty}\|T_2^n y_{1n}-x_n\|=0.\tag{3.26}$$

Now, we shall show that $\lim_{n\to\infty}\|T_3^n y_{2n}-x_n\|=0$. For each $n\ge 1$,

$$\begin{aligned}\|x_n-p\| &\le \|T_2^n y_{1n}-x_n\|+\|T_2^n y_{1n}-p\|\\ &\le \|T_2^n y_{1n}-x_n\|+\Big((1+r_{2n})\|y_{1n}-p\|+s_{2n}\Big)\\ &\le \|T_2^n y_{1n}-x_n\|+\Big((1+r_n)\|y_{1n}-p\|+s_n\Big). \end{aligned}\tag{3.27}$$

Using (3.26), we have

$$\begin{aligned} R &= \lim_{n\to\infty} \|x_n - p\| \\ &\leq \liminf_{n\to\infty} \|y_{1n} - p\| . \end{aligned}$$

It follows from (3.15) that

$$\begin{aligned} R &= \lim_{n\to\infty} \|x_n - p\| \\ &\leq \liminf_{n\to\infty} \|y_{1n} - p\| \\ &\leq \limsup_{n\to\infty} \|y_{1n} - p\| \leq R. \end{aligned} \tag{3.28}$$

This implies that

$$\lim_{n\to\infty} \|y_{1n} - p\| = R. \tag{3.29}$$

On the other hand, we have

$$\|y_{2n} - p\| \leq \Big((1 + r_n)^2 \|x_n - p\| + A_{2n}\Big), \ \forall\, n \geq 1,$$

where $\sum_{n=1}^{\infty} A_{2n} < \infty$. Therefore

$$\begin{aligned} \limsup_{n\to\infty} \|y_{2n} - p\| &\leq \limsup_{n\to\infty} \Big((1 + r_n)^2 \|x_n - p\| + A_{2n}\Big), \\ &\leq R, \end{aligned} \tag{3.30}$$

and hence

$$\begin{aligned} \limsup_{n\to\infty} \|T_3^n y_{2n} - p\| &\leq \limsup_{n\to\infty} \Big((1 + r_{3n}) \|y_{2n} - p\| + s_{3n}\Big) \\ &\leq \limsup_{n\to\infty} \Big((1 + r_n) \|y_{2n} - p\| + s_n\Big) \\ &\leq \limsup_{n\to\infty} \|x_n - p\| = R. \end{aligned} \tag{3.31}$$

Next, consider

$$\begin{aligned} \limsup_{n\to\infty} \|T_3^n y_{2n} - p + c_{3n}(u_{3n} - x_n)\| &\leq \limsup_{n\to\infty} \|T_3^n y_{2n} - p\| \\ &\quad + \limsup_{n\to\infty} \Big(c_{3n} \|u_{3n} - x_n\|\Big) \\ &\leq \limsup_{n\to\infty} \Big((1 + r_{3n}) \|y_{2n} - p\| + s_{3n}\Big) \\ &\quad + \limsup_{n\to\infty} \Big(c_{3n} \|u_{3n} - x_n\|\Big) \\ &\leq \limsup_{n\to\infty} \Big((1 + r_n) \|y_{2n} - p\| + s_n\Big) \\ &\quad + \limsup_{n\to\infty} \Big(c_{3n} \|u_{3n} - x_n\|\Big) \\ &\leq R, \ p \in \mathcal{F}. \end{aligned} \tag{3.32}$$

Also,

$$\begin{aligned} \limsup_{n\to\infty} \|x_n - p + c_{3n}(u_{3n} - x_n)\| &\leq \limsup_{n\to\infty} \|x_n - p\| \\ &\quad + \limsup_{n\to\infty} \Big(c_{3n} \|u_{3n} - x_n\|\Big) \\ &\leq R, \ p \in \mathcal{F}, \end{aligned} \tag{3.33}$$

and

$$\begin{aligned}
\lim_{n\to\infty}\|y_{3n}-p\| &= \lim_{n\to\infty}\|a_{3n}x_n+b_{3n}T_3^n y_{2n}+c_{3n}u_{3n}-p\| \\
&= \lim_{n\to\infty}\|(1-b_{3n}-c_{3n})x_n+b_{3n}T_3^n y_{2n}+c_{3n}u_{3n}-p\| \\
&= \lim_{n\to\infty}\|(1-b_{3n})(x_n-p+c_{3n}(u_{3n}-x_n)) \\
&\qquad + b_{3n}(T_3^n y_{2n}-p+c_{3n}(u_{3n}-x_n))\| \\
&= R. \qquad (3.34)
\end{aligned}$$

Therefore, from (3.32) - (3.34) and Lemma 2.2 we know that

$$\lim_{n\to\infty}\|T_3^n y_{2n}-x_n\|=0. \qquad (3.35)$$

Similarly, by using the same argument as in the proof above, we have

$$\lim_{n\to\infty}\left\|T_i^n y_{(i-1)n}-x_n\right\|=0, \qquad (3.36)$$

for all $i=2,3,\ldots,k$.

Since K is compact, $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_j}\}_{j=1}^{\infty}$. Let

$$\lim_{j\to\infty}x_{n_j}=p. \qquad (3.37)$$

Then from (1.5) and (3.36), we have

$$\begin{aligned}
\left\|x_{n_j+1}-x_{n_j}\right\| &\leq b_{k_{n_j}}\left\|T_k^{n_j}y_{(k-1)n_j}-x_{n_j}\right\|+c_{k_{n_j}}\left\|u_{k_{n_j}}-x_{n_j}\right\| \\
&\to 0, \quad \text{as} \quad j\to\infty. \qquad (3.38)
\end{aligned}$$

From (1.5) and (3.20), we have

$$\begin{aligned}
\|y_{1n}-x_n\| &\leq b_{1n}\|T_1^n x_n-x_n\|+c_{1n}\|u_{1n}-x_n\| \\
&\to 0, \quad \text{as} \quad n\to\infty. \qquad (3.39)
\end{aligned}$$

Again from (3.19) and (3.37), we have

$$\lim_{j\to\infty}T_1^{n_j}x_{n_j}=p. \qquad (3.40)$$

Since $\lim_{j\to\infty}x_{n_j+1}=p$, we have

$$\lim_{j\to\infty}T_1^{n_j+1}x_{n_j+1}=p. \qquad (3.41)$$

From (3.38), (3.40) and (3.41), we have

$$\begin{aligned}
0 &\leq \|p-T_1p\| \\
&\leq \left\|p-T_1^{n_j+1}x_{n_j+1}\right\| \\
&\quad +\left\|T_1^{n_j+1}x_{n_j+1}-T_1^{n_j+1}x_{n_j}\right\| \\
&\quad +\left\|T_1^{n_j+1}x_{n_j}-T_1p\right\| \\
&\leq \left\|p-T_1^{n_j+1}x_{n_j+1}\right\|+L_1\left\|x_{n_j+1}-x_{n_j+1}\right\|^{\alpha_1} \\
&\quad +L_1\left\|T_1^{n_j}x_{n_j}-p\right\|^{\alpha_1} \\
&\quad \to 0 \quad \text{as} \quad j\to\infty. \qquad (3.42)
\end{aligned}$$

From (3.26) and (3.37), we have

$$\lim_{j\to\infty}T_2^{n_j}y_{1n_j}=p. \qquad (3.43)$$

Since $\lim_{j\to\infty} x_{n_j+1} = p$, we have

$$\lim_{j\to\infty} T_2^{n_j+1} y_{1n_j+1} = p. \tag{3.44}$$

From (3.38), (3.39), (3.43) and (3.44), we have

$$\begin{aligned}
0 &\leq \|p - T_2 p\| \\
&\leq \left\|p - T_2^{n_j+1} y_{1n_j+1}\right\| \\
&\quad + \left\|T_2^{n_j+1} y_{1n_j+1} - T_2^{n_j+1} x_{n_j+1}\right\| \\
&\quad + \left\|T_2^{n_j+1} x_{n_j+1} - T_2^{n_j+1} x_{n_j}\right\| \\
&\quad + \left\|T_2^{n_j+1} x_{n_j} - T_2^{n_j+1} y_{1n_j}\right\| \\
&\quad + \left\|T_2^{n_j+1} y_{1n_j} - T_2 p\right\| \\
&\leq \left\|p - T_2^{n_j+1} y_{1n_j+1}\right\| + L_2 \left\|y_{1n_j+1} - x_{n_j+1}\right\|^{\alpha_2} \\
&\quad + L_2 \left\|x_{n_j+1} - x_{n_j}\right\|^{\alpha_2} + L_2 \left\|x_{n_j} - y_{1n_j}\right\|^{\alpha_2} \\
&\quad + L_2 \left\|T_2^{n_j} y_{1n_j} - p\right\|^{\alpha_2} \\
&\to 0 \ \text{ as } \ j \to \infty.
\end{aligned} \tag{3.45}$$

Now, from (1.5) and (3.26), we have

$$\begin{aligned}
\|y_{2n} - x_n\| &\leq b_{2n} \|T_2^n y_{1n} - x_n\| + c_{2n} \|u_{2n} - x_n\| \\
&\to 0, \quad \text{as } \ n \to \infty.
\end{aligned} \tag{3.46}$$

Again from (3.35) and (3.37), we have

$$\lim_{j\to\infty} T_3^{n_j} y_{2n_j} = p. \tag{3.47}$$

Since $\lim_{j\to\infty} x_{n_j+1} = p$, we have

$$\lim_{j\to\infty} T_3^{n_j+1} y_{2n_j+1} = p. \tag{3.48}$$

From (3.38), (3.46), (3.47) and (3.48), we have

$$\begin{aligned}
0 &\leq \|p - T_3 p\| \\
&\leq \left\|p - T_3^{n_j+1} y_{2n_j+1}\right\| \\
&\quad + \left\|T_3^{n_j+1} y_{2n_j+1} - T_3^{n_j+1} x_{n_j+1}\right\| \\
&\quad + \left\|T_3^{n_j+1} x_{n_j+1} - T_3^{n_j+1} x_{n_j}\right\| \\
&\quad + \left\|T_3^{n_j+1} x_{n_j} - T_3^{n_j+1} y_{2n_j}\right\| \\
&\quad + \left\|T_3^{n_j+1} y_{2n_j} - T_3 p\right\| \\
&\leq \left\|p - T_3^{n_j+1} y_{2n_j+1}\right\| + L_3 \left\|y_{2n_j+1} - x_{n_j+1}\right\|^{\alpha_3} \\
&\quad + L_3 \left\|x_{n_j+1} - x_{n_j}\right\|^{\alpha_3} + L_3 \left\|x_{n_j} - y_{2n_j}\right\|^{\alpha_3} \\
&\quad + L_3 \left\|T_3^{n_j} y_{2n_j} - p\right\|^{\alpha_3} \\
&\to 0 \ \text{ as } \ j \to \infty.
\end{aligned} \tag{3.49}$$

Similarly, from (1.5) and (3.36), we have

$$\begin{aligned}
\left\|y_{(k-1)n} - x_n\right\| &\leq b_{(k-1)n} \left\|T_{k-1}^n y_{(k-2)n} - x_n\right\| + c_{(k-1)n} \left\|u_{(k-1)n} - x_n\right\| \\
&\to 0, \quad \text{as } \ n \to \infty.
\end{aligned} \tag{3.50}$$

Again from (3.36) and (3.37), we have

$$\lim_{j\to\infty} T_k^{n_j} y_{(k-1)n_j} = p. \tag{3.51}$$

Since $\lim_{j\to\infty} x_{n_j+1} = p$, we have

$$\lim_{j\to\infty} T_k^{n_j+1} y_{(k-1)n_j+1} = p. \tag{3.52}$$

From (3.38), (3.50), (3.51) and (3.52), we have

$$\begin{aligned}
0 \le\; & \|p - T_k p\| \\
\le\; & \left\|p - T_k^{n_j+1} y_{(k-1)n_j+1}\right\| \\
& + \left\|T_k^{n_j+1} y_{(k-1)n_j+1} - T_k^{n_j+1} x_{n_j+1}\right\| \\
& + \left\|T_k^{n_j+1} x_{n_j+1} - T_k^{n_j+1} x_{n_j}\right\| \\
& + \left\|T_k^{n_j+1} x_{n_j} - T_k^{n_j+1} y_{(k-1)n_j}\right\| \\
& + \left\|T_k^{n_j+1} y_{(k-1)n_j} - T_k p\right\| \\
\le\; & \left\|p - T_k^{n_j+1} y_{(k-1)n_j+1}\right\| + L_k \left\|y_{(k-1)n_j+1} - x_{n_j+1}\right\|^{\alpha_k} \\
& + L_k \left\|x_{n_j+1} - x_{n_j}\right\|^{\alpha_k} + L_k \left\|x_{n_j} - y_{(k-1)n_j}\right\|^{\alpha_k} \\
& + L_k \left\|T_k^{n_j} y_{(k-1)n_j} - p\right\|^{\alpha_k} \\
& \to 0 \quad \text{as} \quad j \to \infty.
\end{aligned} \tag{3.53}$$

Hence

$$\lim_{n\to\infty} \|p - T_i p\| = 0 \quad \forall\, i = 1, 2, \ldots, k. \tag{3.54}$$

Thus p is a common fixed point of the mappings $\{T_i : i = 1, 2, \ldots, k\}$. Since the subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ converges to p and $\lim_{n\to\infty} \|x_n - p\|$ exists, we conclude that $\lim_{n\to\infty} x_n = p$. This completes the proof.

Theorem 3.3. Let K be a nonempty closed convex subset of a uniformly convex Banach space E and for $i = 1, 2, \ldots, k$, let $T_i \colon K \to K$ be a finite family of uniformly (L_i, α_i)-Lipschitz and generalized asymptotically quasi-nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by (1.5) with $\sum_{n=1}^{\infty} r_{in} < \infty$, $\sum_{n=1}^{\infty} s_{in} < \infty$, $\sum_{n=1}^{\infty} c_{in} < \infty$ and $0 < \alpha \le b_{in} \le \beta < 1$ for all $i = 1, 2, \ldots, k$. If $\mathcal{F} = \cap_{i=1}^{k} F(T_i) \neq \emptyset$. Then $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$ for all $i = 1, 2, \ldots, k$.

Proof. From Theorem 3.2 equation (3.36), we have

$$\lim_{n\to\infty} \left\|T_i^n y_{(i-1)n} - x_n\right\| = 0, \tag{3.55}$$

for all $i = 2, 3, \ldots, k$.

In the case $i = 1$ that $\lim_{n\to\infty} \|T_1^n x_n - x_n\| = 0$, where $y_{0n} = x_n$. For $i = 2, 3, \ldots, k$, we obtain from (3.55) that

$$\begin{aligned}
\|T_i^n x_n - x_n\| \le\; & \left\|T_i^n x_n - T_i^n y_{(i-1)n}\right\| + \left\|T_i^n y_{(i-1)n} - x_n\right\| \\
\le\; & L_i \left\|x_n - y_{(i-1)n}\right\|^{\alpha_i} + \left\|T_i^n y_{(i-1)n} - x_n\right\| \\
\le\; & L_i \Big(a_{(i-1)n} \left\|T_{(i-1)}^n y_{(i-2)n} - x_n\right\| + c_{(i-1)n} \left\|u_{(i-1)n} - x_n\right\|\Big)^{\alpha_i} \\
& + \left\|T_i^n y_{(i-1)n} - x_n\right\| \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{3.56}$$

Therefore

$$\lim_{n\to\infty} \|T_i^n x_n - x_n\| = 0, \quad \forall\, i = 1, 2, \ldots, k. \tag{3.57}$$

This completes the proof.

Remark 3.1. Theorem 3.1 extend and improve the corresponding results of Khan et al. [5] and Tang and Peng [22] to the case of more general class of asymptotically quasi-nonexpansive or uniformly quasi-Lipschitzian mappings considered in this paper.

Remark 3.2. Theorem 3.1 also extend and improve the corresponding results of [1, 2, 7, 8, 11, 17]. Especially Theorem 3.1 extends and improves Theorem 1 and 2 in [8], Theorem 1 in [7] and Theorem 3.2 in [17] in the following ways:

(1) The asymptotically quasi-nonexpansive mapping in [7], [8] and [17] is replaced by finite family of generalized asymptotically quasi-nonexpansive mappings.

(2) The usual Ishikawa [4] iteration scheme in [7], the usual modified Ishikawa iteration scheme with errors in [8] and the usual modified Ishikawa iteration scheme with errors for two mappings in [17] are extended to the multi-step iteration scheme with errors for a finite family of mappings.

Remark 3.3. Theorem 3.1 also extends and improves Theorem 2.0.3 in [12] in the following aspects:

(1) Two asymptotically quasi-nonexpansive mappings in [12] is replaced by finite family of generalized asymptotically quasi-nonexpansive mappings.

(2) The usual modified Ishikawa iteration scheme with errors in the sense of Liu [6] for two mappings in [12] is extended to the multi-step iteration scheme with errors in the sense of Xu [24] for a finite family of mappings.

Remark 3.4. Theorem 3.2 extends and improves the corresponding result of [9] in the following aspects:

(1) The asymptotically quasi-nonexpansive mapping in [9] is replaced by finite family of generalized asymptotically quasi-nonexpansive mappings.

(2) The usual modified Ishikawa iteration scheme with errors in [9] is extended to the multi-step iteration scheme with errors for a finite family of mappings.

Remark 3.5. Theorem 3.1 also extends the corresponding result of [23] to the case of more general class of asymptotically nonexpansive mappings and multi-step iteration scheme with errors for a finite family of mappings considered in this paper.

4 Application

In this section we give an application of the convergence criteria established in Theorem 3.1 is given below to obtain yet another strong convergence result in our setting.

Theorem 4.1. Let K be a nonempty closed convex subset of a uniformly convex Banach space E and for $i = 1, 2, \dots, k$, let $T_i : K \to K$ be a finite family of uniformly (L_i, α_i)-Lipschitz and generalized asymptotically quasi-nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by (1.5) with $\sum_{n=1}^{\infty} r_{in} < \infty$, $\sum_{n=1}^{\infty} s_{in} < \infty$, $\sum_{n=1}^{\infty} c_{in} < \infty$ and $0 < \alpha \le b_{in} \le \beta < 1$ for all $i = 1, 2, \dots, k$. Assume that $\mathcal{F} = \cap_{i=1}^{k} F(T_i) \neq \emptyset$ and the family $\{T_i : i = 1, 2, \dots, k\}$ satisfies condition $(\overline{C})$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of the family of mappings $\{T_i : i = 1, 2, \dots, k\}$.

Proof. From Theorem 3.3 we have $\lim_{n\to\infty} \|T_i^n x_n - x_n\| = 0$ for all $i = 1, 2, \dots, k$ and the family $\{T_i : i = 1, 2, \dots, k\}$ satisfying condition $(\overline{C})$, we have that $\liminf_{n\to\infty} f(d(x_n, \mathcal{F})) = 0$. Since f is a nondecreasing function with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, it follows that $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Now by Theorem 3.1, $x_n \in \mathcal{F}$, i.e., $\{x_n\}$ converges strongly to a common fixed point of the family of mappings $\{T_i : i = 1, 2, \dots, k\}$. This completes the proof.

Example 1. Let E be the real line with the usual norm $|.|$ and $K = [0, 1]$. Define $T : K \to K$

by

$$T(x) = \begin{cases} \text{x/2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Obviously $T(0) = 0$, i.e., 0 is a fixed point of the mapping T. Thus, T is quasi-nonexpansive. It follows that T is uniformly quasi-1 Lipschitzian and asymptotically quasi-nonexpansive with constant sequence $\{k_n\} = \{1\}$ for each $n \geq 1$ and hence it is generalized asymptotically quasi-nonexpansive mapping with constant sequences $\{k_n\} = \{1\}$ and $\{s_n\} = \{0\}$ for each $n \geq 1$ but the converse is not true in general.

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Received: March 23, 2012

Accepted: July 22, 2012