On Lie ideals with symmetric bi-additive maps in rings

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Abstract. Let $R$ be a ring and $U \neq 0$ be a Lie ideal of $R$. A bi-additive symmetric map $B(\,\cdot\,): R \times R \to R$ is called symmetric bi-derivation if, for any $y \in R$, the map $x \mapsto B(x, y)$ is a derivation. A mapping $f : R \to R$ defined by $f(x) = B(x, x)$ is called the trace of $B$. In the present paper, we shall show that $U \subseteq Z(R)$ such that $R$ is a prime and semiprime ring admitting the trace $f$ satisfying the several conditions of symmetric bi-derivation.

1 Introduction

Throughout this paper, all rings will be associative. The center of a ring $R$ will be denoted by $Z(R)$. Recall that a ring $R$ is prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$ and semiprime in case $aRa = \{0\}$ implies $a = 0$. For any $x, y \in R$, the symbol $[x, y]$ will represent the commutator $xy - yx$ and the symbol $x \circ y$ stands for the anti-commutator (or skew-commutator) $xy + yx$. An additive mapping $d : R \to R$ is called derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. A derivation $d$ is inner if there exists a fixed $a \in R$ such that $d(x) = [a, x]$ holds for all $x \in R$. A mapping $A(\,\cdot\,): R \times R \to R$ is said to be symmetric if $A(x, y) = A(y, x)$ for all $x, y \in R$. A mapping $f : R \to R$ defined by $f(x) = A(x, x)$, where $A(\,\cdot\,): R \times R \to R$ is symmetric mappings, is called the trace of $A$. It is obvious that, in case $A(\,\cdot\,): R \times R \to R$ is a symmetric mapping which is also a bi-additive (i.e., additive in both arguments). The trace of $A$ satisfies the relation $f(x + y) = f(x) + f(y) + 2A(x, y)$ for all $x, y \in R$.

A symmetric bi-additive mapping $B(\,\cdot\,): R \times R \to R$ is called a symmetric bi-derivation if $B(xy, z) = B(x, z)y + xB(y, z)$ for all $x, y, z \in R$. The concept of symmetric bi-derivation was introduced by G. Maksa [7] (see also [6] where an example can be found).

A study on the theory of centralizing (commuting) maps on prime rings was initiated by the classical result of Posner [9] which stated that the existence of a nonzero centralizing derivation on a prime ring $R$ implies that $R$ is commutative. Since then, a great deal of work in this context has been done by the number of authors (see, e.g., [1], [3] and references therein). For example, as a study concerning centralizing (commuting) maps, Vukman [10],[11] investigated symmetric bi-derivations in prime and semiprime rings. In [1] Argec and Yenigul and Muthana [8] obtained the similar type of results on Lie ideals of $R$. The objective of this paper is to study the commutativity of prime and semiprime rings satisfying various identities involving the trace $f$ of the symmetric bi-derivation $B$. In fact we obtain rather more general results by considering various conditions on a subset of the ring $R$ viz. Lie ideal of $R$.

2 Preliminaries

We shall frequently use the following identities and several well known facts about the semiprime ring without specific mention.

1. $[xy, z] = x[y, z] + [x, z]y$
2. $[x, yz] = y[x, z] + [x, y]z$
3. $x \circ yz = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$
4. $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ y)z + x[y, z]$.

Remark 2.1. Let $U$ be a square closed Lie ideal of $R$. Notice that $xy + yx = (x + y)^2 - x^2 - y^2$ for all $x, y \in U$. Since $x^2 \in U$ for all $x \in U$, $xy + yx \in U$ for all $x, y \in U$. Hence we find that
2xy ∈ U for all x, y ∈ U. Therefore, for all r ∈ R, we get 2r[x, y] = 2[x, ry] = 2[x, r]y ∈ U and 2[x, y]r = 2[xy, r] = 2[y, r] ∈ U, so that 2R[U, U] ⊆ U and 2[U, U]R ⊆ U.

This remark will be freely used in the whole paper without specific reference.

**Lemma 2.1** (5, Corollary 2.1)). Let R be a 2-torsion free semiprime ring, U a Lie ideal of R such that U ⊄ Z(R) and a, b ∈ U.

(i) If aUb = {0}, then a = 0.

(ii) If aU = {0} (Ua = {0}), then a = 0.

(iii) If U is a square closed Lie ideal and aUb = {0}, then ab = 0 and ba = 0.

**Lemma 2.2** (1, Theorem 3). Let R be a prime ring with charR ≠ 2 and U be a nonzero Lie ideal of R. Let B : R × R → R be a symmetric bi-derivation and f the trace of B such that

(i) f(U) = 0, then U ⊆ Z(R) or f = 0.

(ii) f(U) ⊆ Z(R) and U be a square closed Lie ideal, then U ⊆ Z(R) or f = 0.

**Lemma 2.3** (4, Lemma 1). Let R be a 2-torsion free semiprime ring and U be a Lie ideal of R. Suppose that [U, U] ⊆ Z(R), then U ⊆ Z(R).

**Lemma 2.4** (2, Lemma 4). Let R be a prime ring of characteristic different from 2 and U ⊄ Z(R) be a Lie ideal of R and a, b ∈ R, if aUb = {0} then a = 0 or b = 0.

### 3 Results on Prime ring

We start this section with the following lemma:

**Lemma 3.1.** Let R be a prime ring with charR ≠ 2 and U be a square closed Lie ideal of R. Suppose that B : R × R → R is a symmetric bi-derivation and f the trace of B such that [f(x), y] ∈ Z(R) for all x, y ∈ U, then either U ⊆ Z(R) or f = 0.

**Proof.** Suppose on the contrary that U ⊄ Z(R). Since we have given that [f(x), y] ∈ Z(R) for all x, y ∈ U. Replacing y by 2xy and using the fact that charR ≠ 2, we get [f(x), y] + y[f(x), y] ∈ Z(R) for all x, y ∈ U. This implies that [f(x), y] + y[f(x), y] + [f(x), z] = 0 for all x, y, z ∈ U and r ∈ R i.e., [f(x), y] + y[f(x), z] = 0 for all x, y, z ∈ U and r ∈ R. Now, in particular replacing r by z, we obtain [y, z][f(x), z] = 0 for all x, y, z ∈ U. Again, replacing y by 2y and using charR ≠ 2, we get [y, z][f(x), z] = 0 for all x, y, z, t ∈ U i.e., [y, z][U[f(x), z]] = 0 for all x, y, z ∈ U. Thus in view of Lemma 2.4 we find that for each pair of x, y, z ∈ U either [y, z] = 0 or [f(x), z] = 0. For each z ∈ U, let A' = {y ∈ U|[y, z] = 0} and B' = {x ∈ U|[f(x), z] = 0}. Hence A' and B' are the additive subgroups of U whose union is U. By Brauer’s trick, we have either U = A' or U = B'. If A' = U, then [y, z] = 0 for all y, z ∈ U and have U ⊆ Z(R) a contradiction. On the other hand if U = B', then [f(x), z] = 0 for all x, z ∈ U and hence f(U) ⊆ C_R(U) = Z(R), then by Lemma 2.2, we get f = 0. This completes the proof of the lemma.

**Theorem 3.1.** Let R be a prime ring with charR ≠ 2 and U be a square closed Lie ideal of R. Suppose that B : R × R → R is a symmetric bi-derivation and f the trace of B. If [f(x), x] = 0 for all x ∈ U, then either U ⊆ Z(R) or f = 0.

**Proof.** Suppose on the contrary that U ⊄ Z(R). We have [f(x), x] = 0 for all x ∈ U. Replacing x by x + y in the above expressions, we obtain [f(x + y), x + y] = 0 for all x, y ∈ U. Thus implies that [f(x), y] + [f(y), x] + 2[B(x, y), x] = 0 for all x, y ∈ U. Replacing x by −x in the above expression, we get [f(x), y] = [f(y), x] + 2[B(x, y), x] = 0 for all x, y ∈ U. Combining above expressions and by charR ≠ 2, we find that [f(x), y] + 2[B(x, y), x] = 0 for all x, y ∈ U. Replacing y by 2y in the above expression, 2[f(x), y] + 2[B(x, y), x] = 0 for all x, y, z ∈ U. This gives 2[B(x, y) + 2B(x, y)B(x, z) = 0. In particular, x = x we get 2[y, x]B(x, x) = 0 for all x, y ∈ U. By charR ≠ 2, we get [x, y]B(x, x) = 0 for all x, y ∈ U. Replacing y by 2y and using the fact that charR ≠ 2, we get [x, y]B(x, x) = 0 for all x, y, z ∈ U. This gives [x, y]B(x, x) = 0, by Lemma 2.4, for each x ∈ U either [x, y] = 0 or B(x, x) = 0. In the first case it follows that by Lemma 2.3 that x ∈ Z(R) for all x ∈ U. Thus if x /∈ Z(R), then B(x, x) = 0. Let x, z ∈ U such that x ∈ Z(R) and z /∈ Z(R). Hence x + z /∈ Z(R) and x − z /∈ Z(R). Thus B(x + z, x + z) = 0 and B(x − z, x − z) = 0. Adding the above two relations, we find that 2B(x, x) = 0. Since charR ≠ 2, we get B(x, x) = 0. Thus for all x ∈ U, B(x, x) = 0 and from Lemma 2.2 (i), f = 0.

□
Lemma 3.1, we get the required result.

Proof. Suppose on the contrary that \( U \not\subseteq Z(R) \). We have \( f([x, y]) - f(x, y) \in Z(R) \) for all \( x, y \in U \). Replacing \( y \) by \( y + z \) in the above expression, we obtain that 
\[
[f(x, y + z)] = f([x, y + z]) - f(x, y) - f(x, z) - f(y, z) \in Z(R) \text{ for all } x, y, z \in U.
\]
This implies that \( f(x, y) + f(x, z) + 2B(x, y, z) \in Z(R) \) for all \( x, y, z \in U \). Now, using our hypothesis and \( \text{char} R \neq 2 \), we get \( B(x, y) \in Z(R) \) for all \( x, y, z \in U \). In particular, putting \( z = y \), we find that 
\[
B(x, y, y) \in Z(R) \text{ for all } x, y \in U \text{ i.e., } f([x, y]) \in Z(R) \text{ for all } x, y \in U.
\]
Combining the last expression with our hypothesis, we find that \( f([x, y]) \in Z(R) \) for all \( x, y \in U \). Thus, by Lemma 3.1, we get the required result.

\[ \square \]

Theorem 3.3. Let \( R \) be a prime ring with \( \text{char} R \neq 2 \) and \( U \) a square closed Lie ideal of \( R \). Suppose that \( B : R \to R \) is a symmetric bi-derivation and \( f \) is the trace of \( B \) such that 
\[
f(x) = f([x, y]) \in Z(R) \text{ for all } x, y \in U.
\]
Then either \( U \subseteq Z(R) \) or \( f = 0 \).

Proof. Suppose on the contrary that \( U \not\subseteq Z(R) \). Given that \( f(x) - f([x, y]) \in Z(R) \) for all \( x, y \in U \). This implies that \( 2yf(x) \in Z(R) \) for all \( x, y \in U \). \( \text{char} R \neq 2 \) implies that \( yf(x) \in Z(R) \) for all \( x, y \in U \). Hence \( yf(x) = 0 \) for all \( x, y \in U \) and \( r \in R \) i.e.,
\[
y[f(x), r] + [y, r]f(x) = 0 \text{ for all } x, y \in U \text{ and } r \in R.
\]
(3.1)
Replacing \( y \) by \( 2y \) and using \( \text{char} R \neq 2 \), we obtain \( t[yf(x), r] + [y, r]f(x) + [t, r]yf(x) = 0 \) for all \( x, y, t \in U \) and \( r \in R \). Using (3.1), we get \( t, r) yf(x) = 0 \) for all \( x, y, t \in U \) and \( r \in R \). This implies that \( [t, r] yf(x) = 0 \) for all \( x, t \in U \) and \( r \in R \). By Lemma 2.4, we get either \( [t, r] = 0 \) or \( f(x) = 0 \) for all \( x, t \in U \) and \( r \in R \). If \( [t, r] = 0 \), then \( U \subseteq Z(R) \) a contradiction. Hence if \( f(x) = 0 \) for all \( x, t \in U \), then by Lemma 2.2 (i), we get \( f = 0 \).

\[ \square \]

Theorem 3.4. Let \( R \) be a prime ring with \( \text{char} R \neq 2 \) and \( U \) a square closed Lie ideal of \( R \). Suppose that \( B : R \to R \) is a symmetric bi-derivation and \( f \) is the trace of \( B \) such that \( f(x) = f([x, y]) \in Z(R) \text{ for all } x, y \in U \). Then either \( U \subseteq Z(R) \) or \( f = 0 \).

Proof. Suppose on the contrary that \( U \not\subseteq Z(R) \). Since \( f(x) - f([x, y]) \in Z(R) \) for all \( x, y \in U \). Replacing \( x \) by \( x + z \) in the above expression, we obtain that 
\[
f(x + z, y) - f(x, y) \in Z(R) \text{ for all } x, y, z \in U.
\]
This implies that \( f(x, y) + f(x, z) + 2B(x, z, y) \in Z(R) \) for all \( x, y, z \in U \). Now, using our hypothesis and \( \text{char} R \neq 2 \), we get \( B(x, z, y) \in Z(R) \) for all \( x, y, z \in U \). In particular, putting \( z = x \), we find that \( B(x, x, y) \in Z(R) \) for all \( x, y \in U \). Hence by Lemma 3.1, we get the required result.

\[ \square \]

Theorem 3.5. Let \( R \) be a prime ring with \( \text{char} R \neq 2 \) and \( U \) a square closed Lie ideal of \( R \). Suppose that \( B : R \to R \) is a symmetric bi-derivation and \( f \) is the trace of \( B \) such that 
\[
f(x) = f([x, y]) \in Z(R) \text{ for all } x, y \in U.
\]
Then either \( U \subseteq Z(R) \) or \( f = 0 \).

Proof. Suppose on the contrary that \( U \not\subseteq Z(R) \). We have \( f(x) - f([x, y]) \in Z(R) \) for all \( x, y \in U \). Replacing \( y \) by \( y + z \) in the above expression, we obtain that 
\[
f(x, y + z) = f(x, y) + f(x, z) + 2f(x) + B(y, z) \in Z(R) \text{ for all } x, y, z \in U.
\]
Now, using our hypothesis and \( \text{char} R \neq 2 \), we find that \( f(x) \in B(y, z) \in Z(R) \) for all \( x, y, z \in U \). In particular, putting \( z = y \), we get \( f(x) \in Z(R) \) for all \( x, y \in U \). Combining the last step with our hypothesis, we find that \( f(x) \in Z(R) \) for all \( x, y \in U \). Thus, by Lemma 3.1, we get the required result.

\[ \square \]
Theorem 3.7. Let $R$ be a prime ring with $\text{char} R \neq 2$ and $U$ be a square closed Lie ideal of $R$. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and $f$ is the trace of $B$ and $g : R \to R$ be any mapping such that $f(xy) - f(x)g(y) \in Z(R)$ for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $f = 0$.

Proof. Suppose on the contrary that $U \not\subseteq Z(R)$. We have $f(xy) - f(x)g(y) \in Z(R)$ for all $x, y \in U$. Replacing $x$ by $x + z$ in the above expression, we obtain $f(x + z)y - (x + z)g(y) \in Z(R)$ for all $x, y, z \in U$. This implies that $f(xy) + f(z)y + 2B(x, z)y - zg(y) \in Z(R)$ for all $x, y, z \in U$. Using our hypothesis and $\text{char} R \neq 2$, we find that $B(x, z)y \in Z(R)$ for all $x, y, z \in U$. In particular $z = x$, we get $B(x, y) \in Z(R)$ for all $x, y \in U$ i.e., $f(xy) \in Z(R)$ for all $x, y \in U$. This implies that $[f(xy), r] = 0$ for all $x, y \in U$ and $r \in R$ i.e., $f(x)[y, r] + [f(x), r]y = 0$ for all $x, y \in U$ and $r \in R$. Replacing $y$ by $2y$ and using the fact that $\text{char} R \neq 2$, we get $f(x)y[t, r] + [f(x), r]y = 0$ for all $x, y, t \in U$ and $r \in R$. Therefore we obtain, $f(x)y[t, r] = 0$ for all $x, y, t \in U$ and $r \in R$. Hence using the same arguments as used in the last paragraph of proof of Theorem 3.4, we get the required result.

Theorem 3.8. Let $R$ be a prime ring with $\text{char} R \neq 2$ and $U$ is a square closed Lie ideal of $R$. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and $f$ the trace of $B$ such that $f(xy) - f(x)f(y) \in Z(R)$ holds for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $f = 0$.

Proof. Suppose on the contrary that $U \not\subseteq Z(R)$. Given that $f(xy) - f(x)f(y) \in Z(R)$ holds for all $x, y \in U$. Replacing $x$ by $x + z$ in the above relation, we obtain $f(xy) + f(z)f(y) + 2B(x, z)y - zf(y) \in Z(R)$ for all $x, y, z \in U$. Again using relation (3.2), we have $f(xy) \in Z(R)$ for all $x, y \in U$. In particular $z = x$, we get $f(xy) \in Z(R)$ for all $x, y \in U$. This means that $[f(xy), r] = 0$ for all $x, y \in U$ and $r \in R$. This can be re-written as $f(xy)[y, r] + [f(xy), r]y = 0$ for all $x, y \in U$ and $r \in R$. In particular, putting $r = f(x)$, we get $f(xy)[f(x), y] = 0$ for all $x, y \in U$ and $r \in R$. Replacing $y$ by $2yz$ and using that $\text{char} R \neq 2$, we conclude that $f(xy)[f(x), z] = 0$ for all $x, y, z \in U$.

Multiplying the above equation left by $z$, we get $zf(xy)[f(x), z] = 0$ for all $x, y, z \in U$. Replacing $y$ by $2zy$ in relation (3.3) and using the fact that $\text{char} R \neq 2$, we get $f(xy)[f(z), x] = 0$ for all $x, y, z \in U$. Now combining the last two expressions, we find that $[f(xy), z]y[5]f(x), z] = 0$ for all $x, y, z \in U$ that is $[f(xy), z]U[f(x), y] = \{0\}$. Using Lemma 2.1, we get $[f(xy), z] = 0$ for all $x, y, z \in U$ and hence by Lemma 3.1, we get $f = 0$.

Theorem 3.9. Let $R$ be a prime ring with $\text{char} R \neq 2$ and $U$ be a square closed Lie ideal of $R$. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and $f$ the trace of $B$ such that $f(xy) - yf(x) - x \in Z(R)$ holds for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $f = 0$.

Proof. Suppose on the contrary that $U \not\subseteq Z(R)$. Given that $f(xy) - yf(x) - f(y)x \in Z(R)$ holds for all $x, y \in U$. Replacing $x$ by $x + z$ in the above relation, we obtain $f(xy) + 2B(xy, xz) - yf(x) - zf(y)x \in Z(R)$ for all $x, y, z \in U$. Then using our hypothesis and $\text{char} R \neq 2$, we get $B(xy, xz) \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = x$, we find that $f(xy) - yf(x) \in Z(R)$ holds for all $x, y \in U$. Combining this with our hypothesis, we obtain $f(y)x \in Z(R)$ for all $x, y \in U$. This gives $[f(xy), r] = 0$ for all $x, y \in U$ and $r \in R$. This yields that $f(y)[x, r] + [f(y), r]x = 0$ holds for all $x, y \in U$ and $r \in R$.

Replacing $x$ by $2xz$ and using $\text{char} R \neq 2$, we find that $[f(xy), r] = 0$ for all $x, y, z \in U$ and $r \in R$. Using relation (3.4), we get $f(xy)[y, r] = 0$ for all $x, y, z \in U$ and $r \in R$. Using the same technique as we have used in Theorem 3.4, we get the result.
Theorem 3.10. Let $R$ be a prime ring with $\text{char}R \neq 2$ and $U$ be a square closed Lie ideal of $R$. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and $f$ the trace of $B$ such that $f(xy) - xf(y) - yf(x) \in Z(R)$ holds for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $f = 0$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. Given that $f(xy) - xf(y) - yf(x) \in Z(R)$ holds for all $x, y \in U$. Replacing $x$ by $x + z$ in the above relation, we get $f(xy) + f(zy) + 2B(xy, zy) - x(f(y)) - z(f(y)) - yf(z) + yf(x) \in Z(R)$ for all $x, y, z \in U$. Combining this with our hypothesis, we obtain $2B(xy, zy) - 2yB(x, z) \in Z(R)$ for all $x, y, z \in U$. Combining this with our hypothesis, we get $B(xy, zy) - yB(x, z) \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = x$, we get $f(xy) - yf(x) \in Z(R)$ for all $x, y \in U$. Using the last expression with our hypothesis, we find that $xf(z) \in Z(R)$ holds for all $x, y \in U$. This gives that $[xf(z), x] = 0$ holds for all $x, y \in U$ and $r \in R$. Now, using the similar argument as used in the last paragraph of the proof of Theorem 3.4, we get required result.

Theorem 3.11. Let $R$ be a prime ring with $\text{char}R \neq 2$ and $U$ be a square closed Lie ideal of $R$. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and $f$ the trace of $B$ such that $f(xy) - [f(x), y] - [x, f(y)] \in Z(R)$ holds for all $x, y \in U$. Then either $U \subseteq Z(R)$ or $f = 0$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have given that $f(xy) - [f(x), y] - [x, f(y)] \in Z(R)$ holds for all $x, y \in U$. Replacing $x$ by $x + z$ in the above relation, we find that $f(xy) + f(zy) + 2B(xy, zy) - [f(x), y] - [f(z), y] - 2B(x, y) - yf(x) - [z, f(y)] \in Z(R)$ for all $x, y, z \in U$. Combining our hypothesis with above relation, we get $2B(xy, zy) - 2B(x, y) \in Z(R)$ for all $x, y, z \in U$. Since $\text{char}R \neq 2$, we obtain $B(xy, zy) - 2B(x, y) \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = x$, we get $f(xy) \in Z(R)$ for all $x, y \in U$. Using Lemma 3.1, we get required result.

4 Results on Semiprime ring

Theorem 4.1. Let $R$ be a $2$-torsion free semiprime ring and $U$ be a Lie ideal of $R$. Suppose that $A : R \times R \to R$ is a symmetric bi-additive mapping and $f$ is the trace of $A$ such that $f(xy) - xA(y) - A(x)y \in Z(R)$ for all $x, y \in U$. Then $U \subseteq Z(R)$.

Proof. We have $f(xy) - [x, y] \in Z(R)$ for all $x, y \in U$. Replacing $x$ by $x + z$ in the above expression, we obtain that $f(xy) + f(zy) + 2A(xy, zy) - [x, y] - [z, y] \in Z(R)$ for all $x, y, z \in U$. Now, using our hypothesis and the fact that $R$ is $2$-torsion free, we get $A(xy, zy) \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = x$, we find that $A(xy, x) \in Z(R)$ for all $x, y \in U$ i.e., $f(xy) \in Z(R)$. Combining the last step with our hypothesis, we find that $[x, y] \in Z(R)$ for all $x, y \in U$. By Lemma 2.3, we get the required result.

Theorem 4.2. Let $R$ be a $2$-torsion free semiprime ring and $U$ be a square closed Lie ideal of $R$. Suppose that $A : R \times R \to R$ is a symmetric bi-additive mapping and $f$ is the trace of $A$ such that $f(xy) - (x \circ y) \in Z(R)$ for all $x, y \in U$. Then $U \subseteq Z(R)$.

Proof. Suppose on the contrary that $U \nsubseteq Z(R)$. We have $f(xy) - x \circ y \in Z(R)$ for all $x, y \in U$. Replacing $x$ by $x + z$ in above expression, we obtain that $f(xy) + f(zy) + 2A(x \circ y, z \circ y) - x \circ y - z \circ y \in Z(R)$ for all $x, y, z \in U$. Now, using our hypothesis and the fact that $R$ is $2$-torsion free, we get $A(x \circ y, z \circ y) \in Z(R)$ for all $x, y, z \in U$. In particular, putting $z = x$, we find that $A(x \circ y, x \circ y) \in Z(R)$ for all $x, y \in U$ i.e., $f(xy) \in Z(R)$. Combining the last step with our hypothesis, we find that $x \circ y \in Z(R)$ for all $x, y \in U$. Replacing $x$ by $2yx$, we get $2y(x \circ y) \in Z(R)$ for all $x, y \in U$. This implies that $[y, z] = 0$ for all $x, y, z \in U$. On solving and using the fact that $R$ is $2$-torsion free, we conclude that $[y, z](x \circ y) = 0$ for all $x, y, z \in U$. Again replacing $x$ by $2xz$ and using the fact that $R$ is $2$-torsion free, we get $[y, z]x \circ z = 0$ for all $x, y, z \in U$. By Lemma 2.1, we get $U \subseteq Z(R)$, a contradiction.
Theorem 4.3. Let \( R \) be a 2-torsion free semiprime ring and \( U \) be a square closed Lie ideal of \( R \). Suppose that \( A : R \times R \to R \) is a symmetric bi-additive mapping and \( f \) is the trace of \( A \) such that \( f([x, y]) = [x, y] \in Z(R) \) for all \( x, y \in U \). Then \( U \subseteq Z(R) \).

Proof. Suppose on the contrary that \( U \nsubseteq Z(R) \). Given that \( f([x, y]) = [x, y] \in Z(R) \) for all \( x, y \in U \). Replacing \( x \) by \( x+z \) in the above expression, we obtain that \( f((x+z), y) = [x+z, y] \in Z(R) \) for all \( x, y, z \in U \). This implies that \( f(x, y) + f([z, y]) + 2A(x, y, z) - [x, y] - [z, y] \in Z(R) \) for all \( x, y, z \in U \). Now, using our hypothesis and the fact that \( R \) is 2-torsion free, we get \( A([x, y], [z, y]) \in Z(R) \) for all \( x, y, z \in U \). In particular, putting \( z = x \), we find that \( A([x, y], [x, y]) \in Z(R) \) for all \( x, y \in U \) i.e., \( f([x, y]) \in Z(R) \). Combining the last step with our hypothesis, we find that \( x \circ y \in Z(R) \) for all \( x, y \in U \). Now, the same steps as we have used in Theorem 4.2 we get the required result.

Theorem 4.4. Let \( R \) be a 2-torsion free semiprime ring and \( U \) be a Lie ideal of \( R \). Suppose that \( A : R \times R \to R \) is a symmetric bi-additive mapping and \( f \) is the trace of \( A \) such that \( f([x, y]) = [x, y] \in Z(R) \) for all \( x, y \in U \). Then \( U \subseteq Z(R) \).

Proof. We have given that \( f(x \circ y) = [x, y] \in Z(R) \) for all \( x, y \in U \). Replacing \( x \) by \( x+z \) in the above expression, we obtain that \( f((x+z), y) = [x+z, y] \in Z(R) \) for all \( x, y, z \in U \). This implies that \( f(x \circ y) + f([z, y]) + 2A(x, y, z) - [x, y] - [z, y] \in Z(R) \) for all \( x, y, z \in U \). Now, using our hypothesis and the fact that \( R \) is 2-torsion free, we get \( A([x, y], [z, y]) \in Z(R) \) for all \( x, y, z \in U \). In particular, putting \( z = x \), we find that \( A([x, y], [x, y]) \in Z(R) \) for all \( x, y \in U \) i.e., \( f(x \circ y) \in Z(R) \). Combining the last step with our hypothesis, we find that \( x \circ y \in Z(R) \) for all \( x, y \in U \). Then, by Lemma 2.3, we get the required result.

Theorem 4.5. Let \( R \) be a 2-torsion free semiprime ring and \( U \) be a square closed Lie ideal of \( R \). Suppose that \( A : R \times R \to R \) is a symmetric bi-additive mapping and \( f \) the trace of \( A \) such that \( 2(x \circ y) = f(x) - f(y) \) for all \( x, y \in U \). Then \( U \subseteq Z(R) \).

Proof. Suppose on the contrary that \( U \nsubseteq Z(R) \). Since we have \( 2(x \circ y) = f(x) - f(y) \) for all \( x, y \in U \). Replacing \( x \) by \( x+y \) in the above expression, we obtain \( 4y^2 = 2A(x, y) + f(y) \) for all \( x, y \in U \). Replacing \( x \) by \( -z \) in the above expression, we get \( 4y^2 = -2A(x, y) + f(y) \) for all \( x, y \in U \). Now, combining the last two expression, we obtain \( 4y^2 = f(y) \) for all \( x, y \in U \). Putting \( y = x \) in our hypothesis, we find that \( 4y^2 = 0 \). This implies that \( f(y) = 0 \) for all \( y \in U \). Hence \( 2(x \circ y) = 0 \) for all \( x, y \in U \). Since \( R \) is 2-torsion free, we get \( x \circ y = 0 \) for all \( x, y \in U \). Using the same argument as used in the proof of the Theorem 4.2, we get the required result.

Theorem 4.6. Let \( R \) be a 2-torsion free semiprime ring and \( U \) be a square closed Lie ideal of \( R \). Suppose that \( A : R \times R \to R \) is a symmetric bi-additive mapping and \( f \) is the trace of \( A \) such that \( f(x) \circ f(y) - x \circ y \in Z(R) \) for all \( x, y \in U \). Then \( U \subseteq Z(R) \).

Proof. Suppose on the contrary that \( U \nsubseteq Z(R) \). We have \( f(x) \circ f(y) - x \circ y \in Z(R) \) for all \( x, y \in U \). Replacing \( y \) by \( y+z \) in the above expression, we obtain that \( f(x) \circ f(y) + f(x) \circ f(z) + 2f(x) \circ A(y, z) - [x, y] - x \circ z \in Z(R) \) for all \( x, y, z \in U \). Now, using our assumption and the fact that \( R \) is 2-torsion free, we find that \( f(x) \circ A(y, z) \in Z(R) \) for all \( x, y, z \in U \). In particular, putting \( z = y \), we get \( f(x) \circ f(y) \in Z(R) \) for all \( x, y \in U \). Combining the last step with our hypothesis, we find that \( x \circ y \in Z(R) \) for all \( x, y \in U \). Then using the similar technique as used in Theorem 4.2, we get the required result.

Theorem 4.7. Let \( R \) be a 2-torsion free semiprime ring and \( U \) be a Lie ideal of \( R \). Suppose that \( A : R \times R \to R \) is a symmetric bi-additive mapping and \( f \) is the trace of \( A \) such that \( f(x) \circ f(y) - [x, y] \in Z(R) \) for all \( x, y \in U \). Then \( U \subseteq Z(R) \).

Proof. Given that \( f(x) \circ f(y) - [x, y] \in Z(R) \) for all \( x, y \in U \). Replacing \( y \) by \( y+z \) in the above expression, we obtain that \( f(x) \circ f(y) + f(x) \circ f(z) + 2f(x) \circ A(y, z) - [x, y] - [x, z] \in Z(R) \) for all \( x, y, z \in U \). Now, using our assumption and fact that \( R \) is 2-torsion free, we find that \( f(x) \circ A(y, z) \in Z(R) \) for all \( x, y, z \in U \). In particular, putting \( z = y \), we get \( f(x) \circ f(y) \in Z(R) \) for all \( x, y \in U \). Combining the last step with our assumption, we find that \( [x, y] \in Z(R) \) for all \( x, y \in U \). Thus, by Lemma 2.3, we get the required result.

Theorem 4.8. Let \( R \) be a 2-torsion free semiprime ring and \( U \) be a Lie ideal of \( R \). Suppose that \( A : R \times R \to R \) is a symmetric bi-additive mapping and \( f \) is the trace of \( A \) such that \( xy - f(x) = yx - f(y) \) holds for all \( x, y \in U \). Then \( U \subseteq Z(R) \).
Proof. We have \( xy - f(x) = yx - f(y) \) for all \( x, y \in U \). This can be re-written as \( [x, y] = f(x) - f(y) \) for all \( x, y \in U \). Replacing \( x \) by \( x + y \) in the above relation, we obtained, \( [x, y] = f(x) - 2A(x, y) \) for all \( x, y \in U \). Now, substituting \( -x \) in place of \( x \) and combining the above relation, we get \( 2f(x) = 0 \) for all \( x, y \in U \). Since \( R \) is 2-torsion free, we find that \( f(x) = 0 \) for all \( x \in U \). Now, combining it with our hypothesis, we arrive at \( [x, y] = 0 \) for all \( x, y \in U \). Hence, by Lemma 2.3, we get \( U \subseteq Z(R) \).

Theorem 4.9. Let \( R \) be a 2-torsion free semiprime ring and \( U \) be a square closed Lie ideal of \( R \). Suppose that \( B : R \times R \rightarrow R \) is a symmetric bi-derivation and \( f \) the trace of \( B \) such that \([x, y] = f(xy) - f(yx) \) holds for all \( x, y \in U \). Then \( U \subseteq Z(R) \).

Proof. Given that \([x, y] = f(xy) - f(yx) \) holds for all \( x, y \in U \). This can be re-written as
\[
[x, y] = [x^2, f(y)] + [f(x), y^2] + 2xB(x, y)y - 2yB(x, y)x \quad \text{for all } x, y \in U. \tag{4.1}
\]
Now, replacing \( x \) by \( x + y \) in (4.1), we obtained
\[
[x, y] = [x^2, f(y)] + [xy, f(y)] + [yx, f(y)] + [f(x), y^2] + 2[B(x, y), y^2] + 2xB(x, y)y + 2xf(y)y - 2yB(x, y)x - 2yf(y)x \quad \text{for all } x, y \in U. \tag{4.2}
\]
Thus in view of expression of (4.1) yields that
\[
0 = [xy, f(y)] + [yx, f(y)] + 2[B(x, y), y^2] + 2xf(y)y - 2yf(y)x \quad \text{for all } x, y \in U. \tag{4.3}
\]
Replacing \( x \) by \( x + y \) in (4.2) and using (4.2), we obtained
\[
2([x^2, f(y)] + [f(x), y^2] + 2xB(x, y)y - 2yB(x, y)x) = 0 \quad \text{for all } x, y \in U. \tag{4.4}
\]
Since \( R \) is 2-torsion free, the last expression implies that \([x, y] = [x^2, f(y)] + [f(x), y^2] + 2xB(x, y)y - 2yB(x, y)x = 0 \) for all \( x, y \in U \). This yields that \( U \subseteq Z(R) \).

Theorem 4.10. Let \( R \) be a 2-torsion free semiprime ring and \( U \) be a square closed Lie ideal of \( R \). Suppose that \( B : R \times R \rightarrow R \) is a symmetric bi-derivation and \( f \) the trace of \( B \) such that \([x, y] = f(xy) + f(yx) \) holds for all \( x, y \in U \). Then \( U \subseteq Z(R) \).

Proof. We have \([x, y] = f(xy) + f(yx) \in Z(R) \) for all \( x, y \in U \). This can be re-written as
\[
[x, y] = [x^2, f(y)] + [f(x), y^2] - 2xB(x, y)y + 2yB(x, y)x \in Z(R) \quad \text{for all } x, y \in U. \tag{4.5}
\]
Now using the similar argument as we have used form (4.1) to (4.3), we get
\[
0 = [xy, f(y)] + [yx, f(x)] + 2[B(x, y), y^2] + 2xf(y)y - 2yf(y)x \in Z(R) \quad \text{for all } x, y \in U. \tag{4.6}
\]
Further replacing \( y \) by \( x + y \) in the last expression and using the fact that \( R \) is 2-torsion free, we find that \([f(xy) - f(yx)] \in Z(R) \) for all \( x, y \in U \). Combining this our hypothesis, we get \([x, y] \in Z(R) \) for all \( x, y \in U \). Hence using Lemma 2.3, we get the required result.

References


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