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The generalized sequence space $BV_\sigma(\mathcal{M}, p, q, u, s)$ defined by a sequence of Orlicz functions

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Abstract. In this paper we introduce the sequence space $BV_\sigma(\mathcal{M}, p, q, u, s)$ defined by a sequence of Orlicz functions over a seminormed sequence space. We establish some inclusion relations on this space under some conditions and examine some properties of this space.

1. INTRODUCTION

Let ℓ_∞ and ω denote the set of bounded and all sequences $x = (x_k)$ with complex terms.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Lindenstrauss and Tzafriri [3] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space of ℓ_M is a Banach space, with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}$$

which is called an Orlicz space.

If M is a convex function and $M(0) = 0$, then $M(\lambda x) \le \lambda M(x)$ for all with $0 \le \lambda \le 1$.

An Orlicz function M is said to satisfy the Δ_2- condition for small x or at 0 if for each $k > 0$ there exist $R_k > 0$ and $x_k > 0$ such that $M(kx) \le R_k M(x)$, $\forall x \in (0, x_k]$ [2].

Let σ be a one to one mapping of the set of positive integers into itself such that $\sigma^k(n) = \sigma(\sigma^{k-1}(n))$, $k = 1, 2, ...$ A continuous linear functional φ on ℓ_∞ is said to be an invariant mean or a $\sigma-$ mean if and only if

(i) $\varphi(x) \ge 0$ when the sequence $x = (x_k)$ has $x_k \ge 0$ for all k,

(ii) $\varphi(e) = 1$, where $e = (1, 1, 1, ...)$ and

(iii) $\varphi\left(\{x_{\sigma(k)}\}\right) = \varphi\left(\{x_k\}\right)$ for all $x \in \ell_\infty$.

If σ is the translation mapping $n \to n + 1$, a $\sigma-$ mean is often called a Banach limit, and V_σ, the set of $\sigma-$ convergent sequences is

$$V_\sigma = \left\{ x = (x_n) : \lim_k t_{kn}(x) = Le \text{ uniformly in } n, L = \sigma - \lim x \right\}$$

where

$$t_{kn}(x) = \frac{1}{k+1} \sum_{j=0}^{k} T^j x_n.$$

Mursaleen [5] defined

$$BV_\sigma = \left\{ x \in \ell_\infty : \sum_k |\phi_{k,n}(x)| < \infty, \text{ uniformly in } n \right\}$$

where

$$\phi_{k,n}(x) = t_{kn}(x) - t_{k-1,n}(x)$$

assuming that $t_{kn}(x) = 0$, for $k = -1$.

For any sequences x, y and scalar λ, we have

$$\phi_{k,n}(x + y) = \phi_{k,n}(x) + \phi_{k,n}(y) \text{ and } \phi_{k,n}(\lambda x) = \lambda \phi_{k,n}(x).$$

Definitition 1.1. A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ [2].

Remark. It is well known that a sequence space E is normal implies that E is monotone [2].

Definition 1.2. Let q_1 and q_2 be seminorms on a linear space X. Then q_1 is said to be stronger than q_2 if and only if there exists a constant T such that $q_2(x) \leq T q_1(x)$ for all $x \in X$ [7].

The following inequality and $p = (p_k)$ sequence will be used frequently throughout this paper

$$|a_k + b_k|^{p_k} \leq D\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\}$$

where $a_k, b_k \in \mathbb{C}$, $0 < p_k \leq \sup_k p_k = G$ and $D = \max\left(1, 2^{G-1}\right)$ [4].

2. MAIN RESULTS

Definition 2.1. Let X be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q, $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a sequence of strictly positive real numbers, $u = (u_k)$ be a sequence of positive real numbers and $s \geq 0$ be a fixed real number. Then we define the sequence space $BV_\sigma(\mathcal{M}, p, q, u, s)$ as follows:

$$BV_\sigma(\mathcal{M}, p, q, u, s) = \left\{x = (x_k) \in X : \sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k \phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k} < \infty\right\}$$

where for some $\rho > 0$ and uniformly in n.

Theorem 2.2. The sequence space $BV_\sigma(\mathcal{M}, p, q, u, s)$ is a linear space over the field $\mathbb{C}$ of complex numbers.

Proof. Let $x, y \in BV_\sigma(\mathcal{M}, p, q, u, s)$ and $\alpha, \beta \in \mathbb{C}$. There exist some positive numbers ρ_1 and ρ_2 such that

$$\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k \phi_{k,n}(x)}{\rho_1}\right)\right)\right]^{p_k} < \infty, \text{ uniformly in } n$$

and

$$\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k \phi_{k,n}(y)}{\rho_2}\right)\right)\right]^{p_k} < \infty, \text{ uniformly in } n.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M_k is non decreasing and convex, q is a seminorm, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k \phi_{k,n}(\alpha x + \beta y)}{\rho_3}\right)\right)\right]^{p_k} \\
\leq\ & \sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{\alpha u_k \phi_{k,n}(x)}{\rho_3}\right) + q\left(\frac{\beta u_k \phi_{k,n}(y)}{\rho_3}\right)\right)\right]^{p_k} \\
\leq\ & \sum_{k=1}^{\infty} \frac{1}{2^{p_k}} k^{-s}\left[M_k\left(q\left(\frac{u_k \phi_{k,n}(x)}{\rho_1}\right) + M_k\left(q\left(\frac{u_k \phi_{k,n}(y)}{\rho_2}\right)\right)\right)\right]^{p_k} \qquad (1) \\
\leq\ & D\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k \phi_{k,n}(x)}{\rho_1}\right)\right)\right]^{p_k} + D\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k \phi_{k,n}(y)}{\rho_2}\right)\right)\right]^{p_k} \\
<\ & \infty, \text{ uniformly in } n
\end{aligned}$$

where $D = \max(1, 2^{H-1})$. This proves that $BV_\sigma(\mathcal{M}, p, q, u, s)$ is a linear space. □

Theorem 2.3. The sequence space $BV_\sigma(\mathcal{M}, p, q, u)$ is a paranormed (not necessarily total paranormed) space with

$$g(x) = \inf\left\{\rho^{\frac{p_m}{H}} : \left(\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k \phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1,\ m, n = 1, 2, 3, ...\right\}$$

where $H = \max(1, \sup_k p_k)$.

Proof. Since $q(\theta) = 0$ and $M_k(0) = 0$, we get $\inf\{\rho^{\frac{p_m}{H}}\} = 0$ for $x = \theta$. Clearly $g(x) = g(-x)$. The subaddivity of g follows from (1), on taking $\alpha = 1$ and $\beta = 1$. Finally, we prove that the scalar multiplication is continuous. Let λ be any number. By definition,

$$g(\lambda x) = \inf\left\{\rho^{\frac{p_m}{H}} : \left(\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{\lambda u_k \phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1,\ m,n = 1,2,3,...\right\}.$$

Then

$$g(\lambda x) = \inf\left\{(\lambda r)^{\frac{p_m}{H}} : \left(\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k \phi_{k,n}(x)}{r}\right)\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1,\ m,n = 1,2,3,...\right\}$$

where $r = \frac{\rho}{\lambda}$. Since $|\lambda|^{p_k} \leq \max(1, |\lambda|^H)$, it follows that $|\lambda|^{\frac{p_k}{H}} \leq \left(\max(1, |\lambda|^H)\right)^{\frac{1}{H}}$. Hence

$$g(\lambda x) = \left(\max(1, |\lambda|^H)\right)^{\frac{1}{H}} . \inf\left\{r^{\frac{p_m}{H}} : \left(\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k \phi_{k,n}(x)}{r}\right)\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1,\ m,n = 1,2,3,...\right\}$$

and therefore $g(\lambda x)$ converges to zero when $g(x)$ converges to zero in $BV_\sigma(\mathcal{M}, p, q, u, s)$. Now suppose that $\lambda_n \to 0$ and x is in $BV_\sigma(\mathcal{M}, p, q, u, s)$. For arbitrary $\varepsilon > 0$, let N be a positive integer such that

$$\sum_{k=N+1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k \phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k} < \left(\frac{\varepsilon}{2}\right)^H$$

for some $\rho > 0$ and all n. This implies that

$$\left(\sum_{k=N+1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k \phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k}\right)^{\frac{1}{H}} < \frac{\varepsilon}{2}$$

for some $\rho > 0$ and all n.

Let $0 < |\lambda| < 1$, using convexity of M_k , we get for all n

$$\begin{aligned}\sum_{k=N+1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{\lambda u_k \phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k} &\leq \sum_{k=N+1}^{\infty} k^{-s}\left[|\lambda|\, M_k\left(q\left(\frac{u_k \phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k} \\ &< \left(\frac{\varepsilon}{2}\right)^H.\end{aligned}$$

Since M_k is continuous everywhere in $[0, \infty)$, then

$$f(t) = \sum_{k=1}^{N} k^{-s}\left[M_k\left(q\left(\frac{t u_k \phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k}$$

is continuous at 0. So there is $0 < \delta < 1$ such that $|f(t)| < \frac{\varepsilon}{2}$ for $0 < t < \delta$. Let K be such that $|\lambda_i| < \delta$ for $i > K$. Then for $i > K$ and all n, we have

$$\left(\sum_{k=1}^{N} k^{-s}\left[M_k\left(q\left(\frac{\lambda_i u_k \phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k}\right)^{\frac{1}{H}} < \frac{\varepsilon}{2}.$$

Since $0 < \varepsilon < 1$ we have

$$\left(\sum_{k=1}^{N} k^{-s}\left[M_k\left(q\left(\frac{\lambda_i u_k \phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k}\right)^{\frac{1}{H}} < 1$$

for $i > K$ and all n. If we take limit on $\inf\{\rho^{\frac{p_m}{H}}\}$ we get $g(\lambda x) \to 0$. □

Theorem 2.4. Let $\mathcal{M} = (M_k)$ and $\mathcal{T} = (T_k)$ be sequences of Orlicz functions, q, q_1, q_2 be seminorms and $s, s_1, s_2 \geq 0$. Then

(i) $BV_\sigma(\mathcal{M}, p, q, u, s) \cap BV_\sigma(\mathcal{T}, p, q, u, s) \subseteq BV_\sigma(\mathcal{M} + \mathcal{T}, p, q, u, s)$,

(ii) $BV_\sigma(\mathcal{M}, p, q_1, u, s) \cap BV_\sigma(\mathcal{M}, p, q_2, u, s) \subseteq BV_\sigma(\mathcal{M}, p, q_1 + q_2, u, s)$,

(iii) If $s_1 \leq s_2$ then $BV_\sigma(\mathcal{M}, p, q, u, s_1) \subseteq BV_\sigma(\mathcal{M}, p, q, u, s_2)$

(iv) If q_1 is stronger than q_2 and M_k are Orlicz functions that satisfy Δ_2- condition, then $BV_\sigma(\mathcal{M}, p, q_1, u, s) \subset BV_\sigma(\mathcal{M}, p, q_2, u, s)$.

Proof. (i) Let $x \in BV_\sigma(\mathcal{M}, p, q, u, s) \cap BV_\sigma(\mathcal{T}, p, q, u, s)$.

$$
\begin{aligned}
& k^{-s}\left[(M_k + T_k)\left(q\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k} \\
= \;& k^{-s}\left[M_k\left(q\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right) + T_k\left(q\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k} \\
\leq \;& Dk^{-s}\left[M_k\left(q\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k} + Dk^{-s}\left[T_k\left(q\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k}.
\end{aligned}
$$

If we take summation from $k = 1$ to ∞ on above inequality, we get $x \in BV_\sigma(\mathcal{M} + \mathcal{T}, p, q, u, s)$.

(ii) Let $x \in BV_\sigma(\mathcal{M}, p, q_1, u, s) \cap BV_\sigma(\mathcal{M}, p, q_2, u, s)$. If we take $\rho = \max\{2\rho_1, 2\rho_2\}$, then we have

$$
\begin{aligned}
& \sum_{k=1}^{\infty} k^{-s}\left[M_k\left((q_1 + q_2)\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k} \\
= \;& \sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q_1\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right) + q_2\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k} \\
\leq \;& D\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q_1\left(\frac{u_k\phi_{k,n}(x)}{\rho_1}\right)\right)\right]^{p_k} + D\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q_2\left(\frac{u_k\phi_{k,n}(x)}{\rho_2}\right)\right)\right]^{p_k} \\
< \;& \infty
\end{aligned}
$$

where uniformly in n. Hence we get $x \in BV_\sigma(\mathcal{M}, p, q_1 + q_2, u, s)$.

(iii) For $s_1 \leq s_2$ let $x \in BV_\sigma(\mathcal{M}, p, q, u, s_1)$. Then, we write

$$
\sum_{k=1}^{\infty} k^{-s_1}\left[M_k\left(q\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k} < \infty
$$

for some $\rho > 0$, uniformly in n. Then

$$
\begin{aligned}
\sum_{k=1}^{\infty} k^{-s_2}\left[M_k\left(q\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k} \;&\leq\; \sum_{k=1}^{\infty} k^{-s_1}\left[M_k\left(q\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k} \\
&<\; \infty, \text{ uniformly in } n.
\end{aligned}
$$

(iv) Let $x \in BV_\sigma(\mathcal{M}, p, q_1, u, s)$. Then

$$
\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q_1\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k} < \infty.
$$

Since q_1 is stronger than q_2 and M_k is Orlicz function that satisfy Δ_2- condition, we have

$$
\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q_2\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k} \leq R_k\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q_1\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k}
$$

Hence $BV_\sigma(\mathcal{M}, p, q_1, u, s) \subset BV_\sigma(\mathcal{M}, p, q_2, u, s)$. □

Theorem 2.5. Let $0 < r_k \leq t_k$ and (t_k/r_k) be bounded. Then $BV_\sigma(\mathcal{M}, r, q, u, s) \subseteq BV_\sigma(\mathcal{M}, t, q, u, s)$ where $r = (r_k)$ and $t = (t_k)$ sequences of positive real numbers.

Proof. Let $x \in BV_\sigma(\mathcal{M}, r, q, u, s)$. Then there exists some $\rho > 0$ such that

$$
\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{r_k} < \infty, \text{ uniformly in } n.
$$

This implies that $k^{-s}\left[M_k\left(q\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right] \leq 1$ for suffeciently large values of k, say $k \geq k_0$ for some fixed $k_0 \in \mathbb{N}$. Since $r_k \leq t_k$ for each $k \in \mathbb{N}$, we get

$$
\left[M_k\left(q\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{t_k} \leq \left[M_k\left(q\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{r_k}
$$

for all $k \geq k_0$, and therefore

$$\sum_{k\geq k_0}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{t_k} \leq \sum_{k\geq k_0}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{r_k}.$$

Hence we have

$$\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{t_k} < \infty, \text{ uniformly in } n.$$

Hence we have $x \in BV_\sigma(\mathcal{M}, t, q, u, s)$. □

Theorem 2.6. (i) If $0 < p_k \leq 1$ for all $k \in \mathbb{N}$, then $BV_\sigma(\mathcal{M}, p, q, u, s) \subseteq BV_\sigma(\mathcal{M}, q, u, s)$.
(ii) If $p_k \geq 1$ for all $k \in \mathbb{N}$, then $BV_\sigma(\mathcal{M}, q, u, s) \subseteq BV_\sigma(\mathcal{M}, p, q, u, s)$.

Proof. (i) If we take $t_k = 1$ for all $k \in \mathbb{N}$ in Theorem 2.5, we have the result.
(ii) If we take $t_k = p_k$ and $r_k = 1$ or all $k \in \mathbb{N}$ in Theorem 2.5, we have the result. □

Theorem 2.7. The sequence space $BV_\sigma(\mathcal{M}, p, q, u, s)$ is solid.

Proof. Let $x \in BV_\sigma(\mathcal{M}, p, q, u, s)$, i.e,

$$\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k} < \infty, \text{ uniformly in } n.$$

Let $\alpha = (\alpha_k)$ be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then, since ϕ is lineer, q is seminorm and M_k is Orlicz function for each $k \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{\alpha_k u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k} &\leq \sum_{k=1}^{\infty} k^{-s}\left[M_k\left(q\left(\frac{u_k\phi_{k,n}(x)}{\rho}\right)\right)\right]^{p_k}\\ &< \infty, \text{ uniformly in } n.\end{aligned}$$

Hence $\alpha x \in BV_\sigma(\mathcal{M}, p, q, u, s)$ when $x \in BV_\sigma(\mathcal{M}, p, q, u, s)$ under the above restrictions. Therefore the space $BV_\sigma(\mathcal{M}, p, q, u, s)$ is a solid sequence space. □

Corollary 2.8. The sequence space $BV_\sigma(\mathcal{M}, p, q, u, s)$ is monotone.

Proof. Proof is seen from Remark. □

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