Palestine Journal of Mathematics

Vol. 2(1) (2013), 100–103

Related fixed point on two metric spaces

Taieb Hamaizia and Abdelkrim Aliouche

Communicated by Ayman Badawi

MSC 2010 Classifications: 54E50; 58J20

Keywords and phrases: Related fixed point; metric space

Abstract. In this paper, A new fixed point theorem for two pairs of mappings on two metric spaces is proved. This result generalizes the main theorem from [2].

1 Introduction

The following related fixed point theorem was proved in [2], See also [1].

Theorem 1.1. *Let (X, d) and (Y, ρ) be complete metric spaces, let T be a mappings of X into Y, and let S be a mappings of Y into X satisfying the inequalities*

$$\begin{aligned} d(Sy, Sy')d(STx, STx') &\leq c\max\{d(Sy, Sy')\rho(Tx, Tx'), d(x', Sy)\rho(y', Tx), \\ &\quad d(x, x')d(Sy, Sy'), d(Sy, STx)d(Sy', STx')\}, \end{aligned}$$

$$\begin{aligned} \rho(Tx, Tx')\rho(TSy, TSy') &\leq c\max\{d(Sy, Sy')\rho(Tx, Tx'), d(x', Sy)\rho(y', Tx), \\ &\quad \rho(y, y')\rho(Tx, Tx'), \rho(Tx, TSy)\rho(Tx', TSy')\}, \end{aligned}$$

for all x, x' in X and y, y' in Y, where $0 \leq c \leq 1$. If either the mappings T or S is continuous then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

2 Main results

We now prove the following related fixed point theorem.

Theorem 2.1. *Let (X, d) and (Y, ρ) be complete metric spaces, let A, B be mappings of X into Y, and let S, T be mappings of Y into X satisfying the inequalities*

$$\begin{aligned} d(Sy, Ty')d(SAx, TBx') &\leq c\max\{d(Sy, Ty')\rho(Ax, Bx'), d(x', Sy)\rho(y', Ax), \\ &\quad d(x, x')d(Sy, Ty'), d(Sy, SAx)d(Ty', TBx')\}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \rho(Ax, Bx')\rho(BSy, ATy') &\leq c\max\{d(Sy, Ty')\rho(Ax, Bx'), d(x', Sy)\rho(y', Ax), \\ &\quad \rho(y, y')\rho(Ax, Bx'), \rho(Ax, BSy)\rho(Bx', ATy')\}, \end{aligned} \tag{2.2}$$

for all x, x' in X and y, y' in Y, where $0 \leq c < 1$. If one of the mappings A, B, S and T is continuous then SA and TB have a common fixed point z in X and BS and AT have a common fixed point w in Y. Further, $Az = Bz = w$ and $Sw = Tw = z$.

Proof. Let x be an arbitrary point in X, we define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by $Sy_{2n-1} = x_{2n-1}, Bx_{2n-1} = y_{2n}, Ty_{2n} = x_{2n}, Ax_{2n} = y_{2n+1}$

Applying inequality (2.1)), we get

$$\begin{aligned} d(Sy_{2n-1}, Ty_{2n})d(SAx_{2n}, TBx_{2n-1}) &\leq c\max\{d(Sy_{2n-1}, Ty_{2n})\rho(Ax_{2n}, Bx_{2n-1}), \\ &\quad d(x_{2n-1}, Sy_{2n})\rho(y_{2n}, Ax_{2n}), d(x_{2n}, x_{2n-1})d(Sy_{2n-1}, Ty_{2n}), \\ &\quad d(Sy_{2n-1}, SAx_{2n})d(Ty_{2n}, TBx_{2n-1})\}, \end{aligned} \tag{2.3}$$

or

$$d(x_{2n-1},x_{2n})d(y_{2n+1},x_{2n}) \leq c\max\{d(x_{2n-1},x_{2n})\rho\left(y_{2n+1},y_{2n}\right),d(x_{2n},x_{2n-1})d(x_{2n-1},x_{2n})\},$$

from which it follows that

$$d(x_{2n+1},x_{2n}) \leq c\max\{\rho\left(y_{2n},y_{2n+1}\right),d(x_{2n},x_{2n-1})\}. \tag{2.4}$$

Applying inequality (2.2) , we get

$$\begin{aligned}\rho(y_{2n},y_{2n+1})\rho(y_{2n},y_{2n+1}) \leq\ & c\max\{d(x_{2n-1},x_{2n})\rho\left(y_{2n},y_{2n+1}\right), \\ & d(x_{2n},x_{2n-1})\rho\left(y_{2n},y_{2n}\right),\rho(y_{2n-1},y_{2n})\rho(y_{2n},y_{2n+1}), \\ & \rho(y_{2n},y_{2n})\rho(y_{2n+1},y_{2n+1})\},\end{aligned} \tag{2.5}$$

from which it follows that

$$\rho(y_{2n},y_{2n+1}) \leq c\max\{d(x_{2n-1},x_{2n}),\rho(y_{2n-1},y_{2n})\}. \tag{2.6}$$

It now follows from inequalities (2.3), (2.4), (2.5) and (2.6) that, for some n

$$d(x_{n+1},x_n) \leq c\max\{\rho\left(y_n,y_{n+1}\right),d(x_n,x_{n-1})\},$$

$$\rho(y_n,y_{n+1}) \leq c\max\{d(x_{n-1},x_n),\rho(y_{n-1},y_n)\},$$

and easily by induction that

$$d(x_{n+1},x_n) \leq c^n\max\{\rho\left(y_1,y_2\right),d(x_1,x_2)\},$$

similarly,

$$\rho(y_{n+1},y_n) \leq c^n\max\{d(x_1,x_2),\rho(y_1,y_2)\},$$

for n=1,2,3.... Since $0 \leq c < 1$, it follows that $\{x_n\}$ and $\{y_n\}$ are the cauchy sequences with the limits z in X and w in Y.

Now suppose that A is continuous. Then

$$\lim Ax_{2n} = Az = \lim y_{2n+1} = w$$

and so $Az = w$.

Using inequality (2.1), we are successively obtained

$$\begin{aligned}d(Sy_{2n-1},Ty_{2n})d(SAx_{2n},TBx_{2n-1}) \leq\ & c\max\{d(Sy_{2n-1},Ty_{2n})\rho\left(Ax_{2n},Bx_{2n-1}\right), \\ & d(x_{2n-1},Sy_{2n})\rho\left(y_{2n},Ax_{2n}\right), \\ & d(x_{2n},x_{2n-1})d(Sy_{2n-1},Ty_{2n}), \\ & d(Sy_{2n-1},SAx_{2n})d(Ty_{2n},TBx_{2n-1})\},\end{aligned}$$

which implies

$$d(SAz,TBx_{2n-1}) \leq c\max\{\rho\left(Az,Bx_{2n-1}\right),\rho\left(y_{2n},Az\right),d(x_{2n-1},x_{2n})\},$$

Letting n approches to infinity, we have

$$d(Sw,z) \leq c\max\{\rho\left(Az,w\right),\rho\left(w,Az\right),0\}.$$

Then $Sw = z = SAz$.

Further, Applying inequality (2.2) we obtain

$$\begin{aligned}\rho(Ax_{2n},Bx_{2n-1})\rho(BSy_{2n-1},ATy_{2n}) \leq\ & c\max\{d(Sy_{2n-1},Ty_{2n})\rho\left(Ax_{2n},Bx_{2n-1}\right), \\ & d(x_{2n-1},Sy_{2n-1})\rho\left(y_{2n},Ax_{2n}\right), \\ & \rho(y_{2n},y_{2n-1})\rho(Ax_{2n},Bx_{2n-1}), \\ & \rho(Ax_{2n},BSy_{2n-1})\rho(Bx_{2n-1},ATy_{2n})\},\end{aligned}$$

thus,

$$\begin{aligned}\rho(BSy_{2n-1}, ATy_{2n}) &\leq c\max\{d(Sy_{2n-1}, Ty_{2n}), d(x_{2n-1}, Sy_{2n-1}),\\ &\quad \rho(y_{2n}, y_{2n-1}), \rho(Az, BSy_{2n-1})\},\end{aligned}$$

Letting n approaches to infinity, we have

$$\rho(w, Az) \leq c\max\{d(z, Tw), d(z, Sw), 0, \rho(Az, w)\},$$

Then $Tw = z = TBz$.

By the symmetry, the same results again hold if one of the mappings B, S, T is continuous instead of A.

To prove the uniqueness, suppose that TB and SA have a second distinct common fixed point z'. Then, using inequality (2.1), we get

$$\begin{aligned}d(Sy, Ty')d(SAz, TBz') &\leq c\max\{d(Sy, Ty')\rho(Az, Bz'), d(z', Sy)\rho(y', Az),\\ &\quad d(z, z')d(Sy, Ty'), d(Sy, SAz)d(Ty', TBz')\},\end{aligned}$$

that is,

$$\begin{aligned}d(z, z')d(SAz, TBz') &= [d(z, z')]^2 \leq c\max\{d(z, z')\rho(Az, Bz'), d(z', z)\rho(Bz', Az),\\ &\quad d(z, z')d(z, z'), d(z, z)d(z', z')\},\end{aligned}$$

and hence

$$\begin{aligned}d(z, z') &\leq c\max\{\rho(Az, Bz'), \rho(Bz', Az),\\ &\quad d(z, z')d(z, z'), d(z, z)d(z', z')\}.\end{aligned}$$

Therefore,

$$d(z, z') \leq c\rho(Az, Bz'). \tag{2.7}$$

Further, applying inequality (2.2), we obtain

$$\begin{aligned}\rho(Az, Bz')\rho(BSy, ATy') &\leq c\max\{d(Sy, Ty')\rho(Az, Bz'), d(z', Sy)\rho(y', Az),\\ &\quad \rho(y, y')\rho(Az, Bz'), \rho(Az, BSy)\rho(Bz', ATy')\},\end{aligned}$$

that is,

$$\begin{aligned}\rho(Az, Bz')\rho(BSy, ATy') &= [\rho(Az, Bz')]^2 \leq c\max\{d(Sy, Ty')\rho(Az, Bz'), d(z', Sy)\rho(Bz', Az),\\ &\quad \rho(Az, Bz')\rho(Az, Bz'), \rho(Az, BSy)\rho(Bz', ATy')\}.\end{aligned}$$

Therefore,

$$\rho(Az, Bz') \leq cd(z, z'). \tag{2.8}$$

It now follows from inequalities (2.7) and (2.8) that

$$d(z, z') \leq c\rho(Az, Bz') \leq c^2 d(z, z').$$

So $z = z'$. The uniqueness of w is proved similary. This complete the proof of the theorem. □

References

[1] A. Aliouche, B. Fisher, *A related fixed point theorem for two pairs of mappings on two complete metric spaces*, Hacettepe Journal of Mathematics and Statistics, Volume 34, 39-45 (2005).

[2] R. K. Namdeo and N. K. Tiwari, B.Fisher and Kenan Tas, *Related fixed point theorems on two complete and compact metric spaces*, Internat. J. Math. & Math. Sci. Vol. 21 No. 3, 559-564 (1998).

Author information

Taieb Hamaizia and Abdelkrim Aliouche, University of Larbi Ben M'Hidi, Oum-El-Bouaghi, 04000, Algeria.
E-mail: tayeb042000@yahoo.fr

Received: April 2, 2012

Accepted: July 12, 2012