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Direct limits of finite products of fields

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Abstract. Let R be a von Neumann regular ring. The main results of this paper assert whether a von Neumann regular ring is expressible as a directed union of finite products of fields.

1 Introduction

All rings considered in this paper are commutative with unit and all ring-homomorphisms are unital. If R is a subring of a ring S, we assume that R and S have the same unity. We let Idem(R), $\mathcal{C}(R)$ and $\mathcal{A}(R)$ respectively be the set of idempotents of R, the set $\{char(R/M) : M$ is a maximal ideal of $R\}$ and the set of Artinian subrings of R.
Recall that R is von Neumann regular (vNr for short) if R is reduced and zero-dimensional. If R is vNr with Idem(R) finite then R is a finite product of fields, and hence Artinian [13, Lemma 1].
During the last ten years, many papers, that are source of motivation for this work, investigated vNr rings, Artinian rings and directed unions of finite products of fields (see e.g. [4, 5, 10, 13]). Particularly, it was shown that a hereditarily zero-dimensional ring is a directed union of finite products of fields, and that for a ring R, $\mathcal{A}(R)$ needs not be directed. On the other hand, [4, Theorem 6.7] gave necessary and sufficient conditions for a product $\prod_{\alpha\in A} R_\alpha$ of non zero rings to be directed union of Artinian subrings.

In this paper, we deal with the problem of when a vNr is expressible as a directed union of finite products of fields, raised by Gilmer and Heinzer in 1992 ([2, Problem 42]). Of particular interest is [4, Corollary 5.5], which shows that any zero–dimensional ring R with a finite spectrum is a directed union of finite products of fields. The result we give in Theorem 3.1 determines necessary and sufficient conditions under which a vNr ring is a directed union of finite products of fields. We also investigate this class of rings in connection with their families of residue fields $\mathcal{F}(R) = \{R/M : M$ a maximal ideal of $R\}$. On the other hand, let $\{R_\alpha\}_{\alpha\in A}$ be a nonempty family of nonzero rings and $\prod_{\alpha\in A} R_\alpha$ their direct product. We frequently consider $\prod_{\alpha\in A} R_\alpha$ as the set of all functions $f : A \longrightarrow \bigcup_{\alpha\in A} R_\alpha$, such that $f(\alpha) \in R_\alpha$ for each $\alpha \in A$, with addition and multiplication defined pointwise. In this perspective, the direct sum ideal of $\prod_{\alpha\in A} R_\alpha$, denoted $\bigoplus_{\alpha\in A} R_\alpha$, is the set of all functions $f \in \prod_{\alpha\in A} R_\alpha$ that are finitely nonzero (i.e. $\{\alpha \in A : f(\alpha) \neq 0$ in $R_\alpha\}$ is finite).
The paper is organized as follows. In Section 2, we consider conditions under which S is a directed union of finite products of fields. Firstly, we show that if R is a von Neumann regular ring such that $R \subset S \subset \prod_{\lambda\in\Lambda} \frac{R}{M_\lambda}$, where $\{M_\lambda\}_{\lambda\in\Lambda} = Max(R)$ and S is a directed union of finite products of fields, then R and S have the same set of residue fields. In Section 3, we investigate some conditions under which a von Neumann regular ring is a directed union of finite products of fields.

2 General results and counterexample

Let (R_j, f_{jk}) be a directed system of rings, indexed by a directed set $(I, \leq)$. Let $R = \bigcup_{j\in I} R_j$, together with the canonical maps $f_j : R_j \longrightarrow R$. The ring R is said to be a directed union of the R_j's if the f_{jk}'s are inclusion maps. Thus, directed unions can be treated by assuming all f_{jk} to be monomorphisms. Notice that R needs not be Artinian even if each R_j is Artinian.
There are several well–known results concerning rings which can be written as a directed union of finite products of fields. For instance, every ring with only finitely many idempotent elements is a directed union of Artinian subrings (see [4, Corollary 5.5]). Now, let $\mathcal{F} = \{F_i\}_{i\in I}$ and $\mathcal{G} = \{K_j\}_{j\in J}$ be two indexed families of fields. We say that $\mathcal{F} = \mathcal{G}$ if there exists a bijection $f : I \longrightarrow J$ such that F_i and $K_{f(i)}$ are isomorphic fields for each $i \in I$. We say that $\mathcal{F} \subseteq \mathcal{G}$ if there exists an injection map $f : I \longrightarrow J$ such that $F_i \simeq K_{f(i)}$ for each i.

Proposition 2.1. *Let R be a Von Neumann regular ring and $R \subset T \subset \prod_{\lambda\in\Lambda} \frac{R}{M_\lambda}$ such that $T = \bigcup_{i\in I} T_i$ is a directed union of finite products of fields. Then $\mathcal{F}(R) = \mathcal{F}(T)$.*

Proof. Let $K \in \mathcal{F}(R)$ then there exists $M_{\lambda_o} \in Max(R)$ such that $K \simeq R/M_{\lambda_o}$. If $Q = \prod_{\lambda\in\Lambda} A_\lambda$ such that $A_{\lambda_o} \equiv \bar{0}$ modulo M_{λ_o} and $A_\lambda = R/M_\lambda$ for $\lambda \neq \lambda_o$, then $Q \cap R = M_{\lambda_o}$. Let $P = Q \cap T$, then $R/M_{\lambda_o} \subseteq T/P$ and, up to isomorphism, $T/P \subseteq (\prod_{\lambda\in\Lambda} R/M_\lambda)/Q \simeq R/M_{\lambda_o}$. Therefore $R/M_{\lambda_o} \simeq T/P$. Given $L \in \mathcal{F}(T)$, there exists $P \in Max(T)$ such that $L \simeq T/P$. Since $T = \bigcup_{i\in I} T_i$, we have $P = \bigcup_{i\in I} P_i$, where $P_i = P \cap T_i \in Spec(T_i)$. For each i, T_i is a subring of $\prod_{\lambda\in\Lambda} R/M_\lambda$ and T_i is isomorphic to a finite product of fields. Since $\{T_i\}_{i\in I}$ is directed, so is $\{T_i/P_i\}_{i\in I}$. It follows that $\bigcup_{i\in I} T_i/P_i$ is a field. Hence, there exists $M_{\lambda_o} \in Max(R)$ such that $\bigcup_{i\in I} T_i/P_i \subseteq R/M_{\lambda_o}$. It is shown in [9, Proposition 6.1.2, page 128] that $T/P = \bigcup_{i\in I} T_i/P_i$. Let $Q = \prod_{\lambda\in\Lambda} K_\lambda$ such that $K_\lambda = R/M_\lambda$ for each $\lambda \in \Lambda\diagdown\{\lambda_o\}$ and $K_{\lambda_o} \equiv \bar{0}$ modulo M_{λ_o}. Therefore, $P \subseteq Q\cap T$, and since P is maximal in T, we have $P = Q\cap T$. Since $P \cap R = Q \cap T \cap R = Q \cap R = M_{\lambda_o}$, $R/M_{\lambda_o} \subseteq T/P$, i.e., $T/P = R/M_{\lambda_o}$ and hence $\mathcal{F}(T) \subseteq \mathcal{F}(R)$. Thus, $\mathcal{F}(T) = \mathcal{F}(R)$. □

From Proposition 2.1, we deduce that if R is a directed union of finite products of fields then $\mathcal{F}(R) = \mathcal{F}(T)$, where $R \subset T \subset \prod_{\lambda\in\Lambda} \frac{R}{M_\lambda}$, but the converse fails as shown in the following example.

Counterexample 2.2. Let p be a positive prime integer and $\{q_i\}_{i\in\mathbb{N}^*}$ be an infinite family of positive prime integers. Let $\mathcal{F} = \{GF(p)\} \bigcup \{GF(p^{q_i})\}_{i=1}^{\infty}$ be a family of finite Galois fields. We denote by φ_i the natural imbedding of $GF(p)$ into $GF(p^{q_i})$ for each $i \in \mathbb{N}^*$. Let $\varphi = \prod_{i=1}^{\infty} \varphi_i$, $T = \prod_{i=1}^{\infty} GF(p^{q_i})$ and $I = \bigoplus_{i=1}^{\infty} GF(p^{q_i})$ be the direct sum ideal of T. We denote $R_o = GF(p)^* = \varphi(GF(p))$ the diagonal imbedding of $GF(p)$ in T. Let $V = R_o + I$, since V is a subring of T and dim(V)=0 [6, Proposition 2.7], then V is a Von Neumann regular ring. We claim that $\mathcal{F}(V) = \{GF(p)\} \bigcup \{GF(p^{q_i})\}_{i=1}^{\infty}$. Let $p_i : T \to GF(p^{q_i})$ be the canonical projection and $p_{i|V}$ its restriction on V, which is a surjective homomorphism. We have $Kerp_{i|V} = (1 - e_i)T \bigcap V = (1 - e_i)V = M_i$, with e_i the primitive idempotent with support $\{i\}$, and $V/M_i \simeq GF(p^{q_i})$ for each $i \in \mathbb{N}^*$. Also, I is a maximal ideal of V and $V/I \simeq (GF(p))^* \simeq GF(p)$. Thus $\{I\} \cup \{M_i\}_{i=1}^{\infty} \subseteq Max(V)$. Let $P \in Spec(V)$, if $I \subseteq P$, then $I = P$. If $I \nsubseteq P$ then $e_i \notin P$, for some $i \in \mathbb{N}^*$, and hence $1 - e_i \in P$. Therefore, $M_i \subseteq P$ and $P = M_i$. As consequence, $Max(V) = \{I\} \bigcup \{M_i\}_{i=1}^{\infty}$. Thus $\mathcal{F}(V) = \{GF(p)\} \bigcup \{GF(p^{q_i})\}$. From [11, Theorem 5.5 page 247], $GF(p^{q_i}) = GF(p)(a_i)$, where a_i is a p^{q_i}-th primitive root of unity, for each $i \in \mathbb{N}^*$. Let R^o be the minimal zero-dimensional subring of $\prod_{i=1}^{\infty} GF(p^{q_i})$ containing $V[a]$, where $a = \{a_i\}_{i=1}^{\infty}$. Since $V[a] = R_o[a] + I[a]$, we have $V[a]/I[a] \simeq R_o[a] \simeq GF(p)[X]$ because a is a transcendental element over R_o. Therefore, $qf(V[a]/I[a]) \simeq GF(p)(X)$. By [7, Theorem 3.3], $\mathcal{F}(R^o) = \{qf(R/P) : P \in Spec(R)$ is contracted from $T\} = \{GF(p^{q_i})\}_{i=1}^{\infty} \bigcup \{GF(p)(X)\}$. Since $a \in R^o$ and a is in no finite product of fields, then R^o is not a directed union of finite products of fields. By [4, Proposition 5.3 (2)], $R = R^o \oplus GF(p)$ is a Von Neumann regular ring which is not a directed union of finite products of fields. Let Ω be a field containing each $GF(p^{q_i})$ and $GF(p)(X)$. Given $y = \{y_i\}_{i\in\mathbb{Z}^+} \in \prod_{i=1}^{\infty} GF(p^{q_i}) \times GF(p)(X) \times GF(p)$, let $||y|| = \{y_i : i \in \mathbb{Z}^+\} \subseteq \Omega$. Finally, put $S = \{y \in \Omega^{\mathbb{Z}^+} : ||y||$ is finite$\}$. Let $\varphi : \Omega^{\mathbb{Z}^+} \to \Omega^{\mathbb{Z}^+}$ a homomorphism defined by $\varphi(X) = a$ and let $S_1 = \varphi(S)$. Therefore, S_1 is a directed union of finite products of fields and $\mathcal{F}(S_1) = \mathcal{F}(R)$.

Let R be a ring and $\{R_\alpha\}_{\alpha\in A}$ an infinite family of nonzero rings such that R is, up to isomorphism, a subring of each R_α. We use R^* to denote the diagonal imbedding of R in $\prod_{\alpha\in A} R_\alpha$, that is $R^* = \varphi(R)$, where $\varphi : R \hookrightarrow \prod_{\alpha\in A} R_\alpha$ is the monomorphism defined by $\varphi(x) = \{x_\alpha\}_{\alpha\in A}$ such that $x_\alpha = x$ for each $\alpha \in A$.

Proposition 2.3. *Let R be a von Neumann regular ring with $\mathcal{F}(R) = \{L_\alpha\}_{\alpha\in A}$, $\mathcal{C}(R) = \{p\}$, and $\mathcal{S} = \{\ \{r_\alpha\}_{\alpha\in A} \in \prod_{\alpha\in A} L_\alpha :\ \ \{r_\alpha\}_{\alpha\in A}\ \ has\ \ only\ finitely\ \ many\ distinct\ coordinates\}$. Assume that there exists a field Ω that contains all but finitely many L_α's. Then $\mathcal{S}$ is a directed union of finite products of fields.*

Proof. To show that $\mathcal{S}$ is a directed union of finite products of fields, it suffices to prove that $\mathcal{S}$ is covered by a directed union of finite products of fields. Let $f \in \mathcal{S}$, then $\{f(\alpha) : \alpha \in A\} = \{f_1, ..., f_t\}$ a finite set. Let $A_i = \{\alpha \in A : f(\alpha) = f_i\}$ and denote $f_i^* = (f_i, f_i, ..., f_i, ...) \in \prod_{\alpha\in A_i} L_\alpha$. Then $\{f(\alpha)\}_{\alpha\in A} = (f_1^*, ..., f_t^*)$. Since $A = \cup_{i=1}^{t} A_i$, and all the fields L_α, $\alpha \in A_i$, have the same characteristic, then, up to isomorphism, $\bigcap_{\alpha\in A_i} L_\alpha = K_i$ is a field with $f_i^* \in K_i^*$, the diagonal imbedding of K_i in $\prod_{\alpha\in A_i} L_\alpha$. It follows that $f \in K_1^* \times ... \times K_t^* \simeq K_1 \times ... \times K_t$. Therefore, $\mathcal{S}$ is covered by a directed union of finite products of fields. □

Theorem 2.4. *Let R be a von Neumann regular ring with $\mathcal{F}(R) = \{L_\alpha\}_{\alpha\in A}$ and $\mathcal{S} = \{\ \{r_\alpha\}_{\alpha\in A} \in \prod_{\alpha\in A} L_\alpha :\ \ \{r_\alpha\}_{\alpha\in A}\ \ has\ \ only\ \ finitely\ \ many\ distinct\ coordinates\}$ Then, $\mathcal{S}$ is a directed union of finite products of fields if and only if $\mathcal{C}(R)$ is finite.*

Proof. Assume that $\mathcal{S}$ is a directed union of finite products of fields and let $f \in \mathcal{S}$. Then f has only finitely many distinct components f_1,..., f_t. Now, let $A_i = \{\alpha \in A : f(\alpha) = f_i\}$, for each $i = 1, ..., t$. So $A = \bigcup_{i=1}^t A_i$ is a partition of A. We set $f_i^* = (f_i, ..., f_i, ...) \in \prod_{\alpha\in A_i} L_\alpha$. Then, up to isomorphism, $\{f(\alpha)\}_{\alpha\in A} = (f_1^*, ..., f_t^*)$ belongs to a finite product of fields. In fact $(f_1^*, ..., f_t^*) \in \prod_{i=1}^t K_i^* \simeq \prod_{i=1}^t K_i$, where K_i^* is the diagonal imbedding of K_i into $\prod_{\alpha\in A_i} L_\alpha$ for each $i = 1, ..., t$, with $K_i^* \simeq K_i = \bigcap_{\alpha\in A_i} L_\alpha$. Since $\mathcal{S}$ is a subring of $\prod_{\alpha\in A} L_\alpha$, this shows that $\prod_{i=1}^t K_i$ is isomorphic to a subring of $\prod_{\alpha\in A} L_\alpha$. It follows that $\mathcal{C}(R)$ is finite. Conversely, let $\mathcal{C}(R) = \{p_1, ..., p_n\}$. We can write $\prod_{\alpha\in A} L_\alpha = \oplus_{i=1}^n T_i$, where $T_i = \prod_{\alpha\in\Lambda_i} L_\alpha$ and $\Lambda_i = \{\alpha \in A : char(L_\alpha) = p_i\}$. Let e_j be the idempotent element of $\prod_{\alpha\in A} L_\alpha$ associated with {j}. Hence $\mathcal{S} = \mathcal{S}e_1 \oplus\ ...\ \oplus \mathcal{S}e_n$. Form Proposition 2.3, each $\mathcal{S}e_j$ is a subring of T_j which is a directed union of finite products of fields. By [4, Proposition 5.3], $\mathcal{S}$ is a directed union of finite products of fields. □

3 Behavior with respect to residue fields

Let R be a von Neumann regular ring with maximal ideals $\mathfrak{m}_i$, $i \in I$ and corresponding residue fields K_i. We assume that there exists a field Ω containing each K_i. (We can always make this assumption if the K_i have the same characteristic.) Assuming $\mathcal{C}(R)$ finite, we have the partition $\mathcal{F}(R) = \mathcal{F}_1 \bigcup\ ...\ \bigcup \mathcal{F}_n$ of $\mathcal{F}(R)$ with respect to the characteristic. We assume that for each $i \in \{1, ...,\ n\}$ there exists F_i such that $F_i \in \mathcal{F}_i$ and each element of $\mathcal{F}_i$ is an algebraic extension of F_i. We use F^* to denote the diagonal imbedding of F into $\prod_{i\in I} F_i$.

Theorem 3.1. *The ring R is a directed union of finite products of fields if and only if for each $f \in R$, f is integral over $F_1^* \times\ ...\ \times F_n^*$.*

In order to prove this result, we need the following Lemma.

Lemma 3.2. *Let R be a Von Neumann regular ring and $\mathcal{F}(R) = \{F_i\}_{i\in I}$. Assume that each F_i is an algebraic extension of F. Then R is a directed union of finite products of fields if and only if for each $f \in R$, f is integral over F^*.*

Proof. Suppose that R is a directed union of finite products of fields and let $f \in R$. Then f belongs to a finite product of fields, in other words f is in only finitely many fields F_i. By [1, Proposition 3, page 9], f is an integral over F^*. Conversely, let $f \in R$ to be integral over F^*. Then there exists a monic polynomial $H(X)$ in $F^*[X]$ that vanishes at f. The polynomial $H(X)$ has only finitely many roots of $H(X)$. We note also that $\{f(i)\}_{i\in I}$ is the unique solution of $H(X)$. It follows that f has only finitely many distinct components. We conclude that $R \subset \mathcal{S}$ and Theorem 2.4 completes the proof. □

Proof of Theorem 3.1. We can write $\prod_{i\in I} F_i = \bigoplus_{j=1}^n T_j$, where $T_j = \prod_{i\in I_j} F_i$ and $I_j = \{i \in I\ :\ \ char(F_i) = p_j\}$. From Lemma 3.2, to show that R is a directed union of finite products of fields it suffices to show that $R[e_1, ...,\ e_n]$ has the same property, where e_j is the idempotent associated with j, for $j = 1,\ ...,\ n$. Moreover, since $R[e_1, ...,\ e_n] = Re_1 \oplus\ ...\ \oplus Re_n$, to prove that the condition of Theorem 3.1 is satisfied for $R[e_1, ...,\ e_n]$ it suffices to show that it is satisfied for each Re_j. Let $f \in R$ be integral over $\prod_{i=1}^n F_i^*$. Let $H(X)$ be a monic polynomial of $\prod_{i=1}^n F_i^*[X]$ such that $H(f) = 0$. Let $H_j = He_j$, for each $j = 1,\ ...\ n$. The polynomial H_j is monic in $F_i^*[X]$ and satisfy $H_j(\{f(i)\}_{i\in I_j}) = 0$, for each $j = 1,\ ...\ n$. Therefore, $\{f(i)\}_{i\in I_j}$ is integral over F_j^*. From Lemma 3.2, Re_j is a directed union of finite products of fields, for each $j = 1\ ...\ n$. Conversely, assume that R is a directed union of Finite products of fields. We know that fe_j is integral over F_j^* for each $j = 1,\ ...,\ n$. Let $H_j(X) \in F_j^*[X]$ be a monic polynomial that vanishes at fe_j, for $j = 1,\ ...,\ n$. If we set $H = \prod_{i=1}^n H_j$, then H is a monic polynomial $\prod_{j=1}^n F_j^*$ that vanishes at f (cf. [1, Proposition 3, page 9]). □

Example 3.3. Let p be a positive prime integer and $\{q_i\}_{i\in\mathbb{N}^*}$ be an infinite family of distinct prime integers. Let $\mathcal{F} = \{\mathbb{Q}\} \bigcup \{\mathbb{Q}(\zeta_i)\}_{i=1}^\infty$ be an infinite family of fields, where ζ_i is a $p^{q_i}-$primitive root of unity. We denote by φ_i the imbedding of $\mathbb{Q}$ into $\mathbb{Q}(\zeta_i)$ for each $i \in \mathbb{Z}^+$. Let $\varphi = \prod_{i=1}^\infty \varphi_i$, $T = \prod_{i=1}^\infty \mathbb{Q}(\zeta_i)$, $I = \bigoplus_{i=1}^\infty \mathbb{Q}(\zeta_i)$ the direct sum ideal of T. We denote $\mathbb{Q}^* = \varphi(\mathbb{Q}) = R_o \simeq \mathbb{Q}$ the diagonal imbedding of $\mathbb{Q}$ in T. Let $R_1 = R_o + I$. Then $\mathcal{F}(R_1) =$

$\{F_i\}_{i=1}^{\infty} \bigcup \{\mathbb{Q}\}$. Let $f \in \prod_{i=1}^{\infty} F_i$ such that $f(i) = i$ for each $i \in \mathbb{Z}^+$. If R is the minimal zero-dimensional subring of T containing $R_1[f]$, then $\mathcal{F}(R) = \{F_i\}_{i=1}^{\infty} \bigcup \{\mathbb{Q}(t)\}$, where $F_i = \mathbb{Q}(\zeta_i)$ for each $i \in \mathbb{Z}^+$. On the other hand, since $\mathcal{S} = R_o + J$, where $J = \mathbb{Q}(t) + I$, we have $\mathcal{F}(\mathcal{S}) = \{F_i\}_{i=1}^{\infty} \bigcup \{\mathbb{Q}(t)\} \bigcup \{\mathbb{Q}\}$. Now, by [4, Proposition 5.3 (2)], the ring $R = R_1 \oplus \mathbb{Q}$ is not a directed union of finite products of fields, even if $\mathcal{F}(R) = \mathcal{F}(\mathcal{S})$. Notice that $F_i \cap F_j = \mathbb{Q}$ for $i \neq j$ and $F_i \cap \mathbb{Q}(t) = \mathbb{Q}$ with $\mathbb{Q} \in \mathcal{F}(R)$. Even though, $\mathbb{Q} \in \mathcal{F}(R)$ R is not a directed union of finite products of fields.

Let R be a von Neumann regular ring and $\{M_i\}_{i \in I}$ its spectrum. Since R is a reduced ring, we have $\bigcap_{i \in I} M_i = (0)$ and hence the homomorphism $\varphi : R \to \prod_{i \in I} \frac{R}{M_i}$, defined by $\varphi(x) = x + M_i$, is injective. This allows us to view R as a subring of $\prod_{i \in I} \frac{R}{M_i}$. We identify x with its image $\{x_i\}_{i \in I} \in \prod_{i \in I} \frac{R}{M_i}$. Finally, we denote $F_i = \frac{R}{M_i}$ for each $i \in I$.

Corollary 3.4. *With the notation and assumptions above, we assume that there is a field Ω containing each F_i and F is the prime subfield of Ω. If each distinct pair of fields F_j and F_k in $\mathcal{F}_i$ satisfying $F_j \bigcap F_k = F \notin \mathcal{F}(R)$, then R is not a directed union of finite products of fields.*

Proof. Suppose that $F \notin \mathcal{F}(R)$. Let S be the subring of $\prod_{i \in I} F_i$ consisting of eventually constant sequences. Thus $S = F^* + I$, the F-subalgebra of $\prod_{i \in I} F_i$ generated by the direct sum ideal $I = \oplus_{i \in I} F_i$, where F^* is the diagonal imbedding of F into $\prod_{i \in I} F_i$. First claim that S is the maximal subring of $\prod_{i \in I} F_i$ with respect to being a directed union of finite product of fields. Let $T = \bigcup_{j \in J} T_j$ be a subring of $\prod_{i \in I} F_i$ which is a directed union of finite product of fields. Let $t = \{t_i\}_{i \in I} \in T$ then there exists $j_o \in J$ such that $t \in T_{j_o}$ which is a finite products of fields, then t has only finitely many distinct coordinates, i.e., $t \in S$. If R is a directed union of finite products of fields, then $R \subseteq S$ and hence $\mathcal{F}(R) = \mathcal{F}(S)$ (see Proposition 2.1), a contradiction with $F \in \mathcal{F}(S) \diagdown \mathcal{F}(R)$. □

Example 3.5. Let $\mathbb{Q}$ be the field of rational numbers, α an element such that $\alpha^2 = d$ (d without squire factor in $\mathbb{Q}$) and $\mathcal{P}$ be an infinite family of distinct prime integers. Let $\Omega = \mathbb{Q}(\alpha)$ be a simple algebraic extension of $\mathbb{Q}$ and $R = \mathbb{Q}(\alpha) + I$, the $\Omega-$subalgebra of $T = \prod_{i=1}^{\infty} \Omega(\zeta_i)$ generated by the direct sum ideal $I = \oplus_{i=1}^{\infty} \Omega(\zeta_i)$, where ζ_i is a p^q-primitive root of unity and p is a prime integer with $q \in \mathcal{P}$. For each $i \in \mathbb{Z}^+$, let $\phi_i : \Omega \to \Omega(\zeta_i)$ be the field-homomorphism taking α to ζ_i. Let $\phi = \{\phi_i\}_{i=1}^{\infty} : T \to T$, a ring-homomorphism. Let $R_o = \phi(R)$. Being isomorphic to R, R_o is a directed union of finite products fields. We remark that the element $\{\zeta_i\}_{i=1}^{\infty} \in R_o$ which is not in $\mathcal{S}$ (the maximum among all subrings of T that are directed union of finite products of fields in proof of Corollary 3.4).

Proposition 3.6. *Let R be a zero-dimensional ring and $N(R)$ be the nilradical of R. Then the following conditions are equivalent:*

(i) R is a directed union of zero-dimensional subrings with finite spectra;

(ii) $R/N(R)$ is a directed union of finite products of fields.

Proof. (i)⇒(ii). Suppose that $R = \bigcup_{i \in I} R_i$ is a directed union of zero-dimensional subrings with finite spectra, then by [15, Proposition 6.1.2, page 128], $R/N(R) \simeq \bigcup_{i \in I} R/N(R) \cap R_i = \bigcup_{i \in I} R_i/N(R_i)$ is a directed union of $R_i/N(R_i)$, where $N(R_i)$ is the nilradical of R_i. The ring $R_i/N(R_i)$ is Von Neumann regular with finite spectra, then $R_i/N(R_i)$ is Artinian.
(ii)⇒ (i). Suppose $R/N(R) = \bigcup_{i \in I} S_i$ is a directed union of Finite products of fields and let $\varphi :\ R \to R/N(R)$ be the canonical epimorphism. We denote $R_i = \varphi^{-1}(S_i)$ the inverse image of S_i by φ, for each $i \in I$. Since $\{S_i\}_{i \in I}$ is directed, the family $\{R_i\}_{i \in I}$ is also directed. We have $R_i/N(R_i) \simeq S_i$, as $Spec(S_i)$ is finite the ring R_i has only finitely many prime ideals, for each $i \in I$. Because S_i is zero-dimensional, R_i is also zero-dimensional but need not be Artinian. It follows that $R = \bigcup_{i \in I} R_i$ is a directed union of zero-dimensional quasilocal subrings. □

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