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# Direct limits of finite products of fields

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**Abstract**. Let $R$ be a von Neumann regular ring. The main results of this paper assert whether a von Neumann regular ring is expressible as a directed union of finite products of fields.

## 1 Introduction

All rings considered in this paper are commutative with unit and all ring-homomorphisms are unital. If $R$ is a subring of a ring $S$, we assume that $R$ and $S$ have the same unity. We let Idem$(R)$, $\mathcal{C}(R)$ and $\mathcal{A}(R)$ respectively be the set of idempotents of $R$, the set $\{char(R/M) : M$ is a maximal ideal of $R\}$ and the set of Artinian subrings of $R$.
Recall that $R$ is von Neumann regular (vNr for short) if $R$ is reduced and zero-dimensional. If $R$ is vNr with Idem$(R)$ finite then $R$ is a finite product of fields, and hence Artinian [13, Lemma 1].
During the last ten years, many papers, that are source of motivation for this work, investigated vNr rings, Artinian rings and directed unions of finite products of fields (see e.g. [4, 5, 10, 13]). Particularly, it was shown that a hereditarily zero-dimensional ring is a directed union of finite products of fields, and that for a ring $R$, $\mathcal{A}(R)$ needs not be directed. On the other hand, [4, Theorem 6.7] gave necessary and sufficient conditions for a product $\prod_{\alpha\in A} R_\alpha$ of non zero rings to be directed union of Artinian subrings.

In this paper, we deal with the problem of when a vNr is expressible as a directed union of finite products of fields, raised by Gilmer and Heinzer in 1992 ([2, Problem 42]). Of particular interest is [4, Corollary 5.5], which shows that any zero–dimensional ring $R$ with a finite spectrum is a directed union of finite products of fields. The result we give in Theorem 3.1 determines necessary and sufficient conditions under which a vNr ring is a directed union of finite products of fields. We also investigate this class of rings in connection with their families of residue fields $\mathcal{F}(R) = \{R/M : M$ a maximal ideal of $R\}$. On the other hand, let $\{R_\alpha\}_{\alpha\in A}$ be a nonempty family of nonzero rings and $\prod_{\alpha\in A} R_\alpha$ their direct product. We frequently consider $\prod_{\alpha\in A} R_\alpha$ as the set of all functions $f : A \longrightarrow \bigcup_{\alpha\in A} R_\alpha$, such that $f(\alpha) \in R_\alpha$ for each $\alpha \in A$, with addition and multiplication defined pointwise. In this perspective, the direct sum ideal of $\prod_{\alpha\in A} R_\alpha$, denoted $\bigoplus_{\alpha\in A} R_\alpha$, is the set of all functions $f \in \prod_{\alpha\in A} R_\alpha$ that are finitely nonzero (i.e. $\{\alpha \in A : f(\alpha) \neq 0$ in $R_\alpha\}$ is finite).
The paper is organized as follows. In Section 2, we consider conditions under which $S$ is a directed union of finite products of fields. Firstly, we show that if $R$ is a von Neumann regular ring such that $R \subset S \subset \prod_{\lambda\in\Lambda} \frac{R}{M_\lambda}$, where $\{M_\lambda\}_{\lambda\in\Lambda} = Max(R)$ and $S$ is a directed union of finite products of fields, then $R$ and $S$ have the same set of residue fields. In Section 3, we investigate some conditions under which a von Neumann regular ring is a directed union of finite products of fields.

## 2 General results and counterexample

Let $(R_j, f_{jk})$ be a directed system of rings, indexed by a directed set $(I, \leq)$. Let $R = \bigcup_{j\in I} R_j$, together with the canonical maps $f_j : R_j \longrightarrow R$. The ring $R$ is said to be a directed union of the $R_j$'s if the $f_{jk}$'s are inclusion maps. Thus, directed unions can be treated by assuming all $f_{jk}$ to be monomorphisms. Notice that $R$ needs not be Artinian even if each $R_j$ is Artinian.
There are several well–known results concerning rings which can be written as a directed union of finite products of fields. For instance, every ring with only finitely many idempotent elements is a directed union of Artinian subrings (see [4, Corollary 5.5 ]). Now, let $\mathcal{F} = \{F_i\}_{i\in I}$ and $\mathcal{G} = \{K_j\}_{j\in J}$ be two indexed families of fields. We say that $\mathcal{F} = \mathcal{G}$ if there exists a bijection $f : I \longrightarrow J$ such that $F_i$ and $K_{f(i)}$ are isomorphic fields for each $i \in I$. We say that $\mathcal{F} \subseteq \mathcal{G}$ if there exists an injection map $f : I \longrightarrow J$ such that $F_i \simeq K_{f(i)}$ for each $i$.

**Proposition 2.1.** *Let $R$ be a Von Neumann regular ring and $R \subset T \subset \prod_{\lambda\in\Lambda} \frac{R}{M_\lambda}$ such that $T = \bigcup_{i\in I} T_i$ is a directed union of finite products of fields. Then $\mathcal{F}(R) = \mathcal{F}(T)$.*

**Proof.** Let $K \in \mathcal{F}(R)$ then there exists $M_{\lambda_o} \in Max(R)$ such that $K \simeq R/M_{\lambda_o}$. If $Q = \prod_{\lambda\in\Lambda} A_\lambda$ such that $A_{\lambda_o} \equiv \bar{0}$ modulo $M_{\lambda_o}$ and $A_\lambda = R/M_\lambda$ for $\lambda \neq \lambda_o$, then $Q \cap R = M_{\lambda_o}$. Let $P = Q \cap T$, then $R/M_{\lambda_o} \subseteq T/P$ and, up to isomorphism, $T/P \subseteq (\prod_{\lambda\in\Lambda} R/M_\lambda)/Q \simeq R/M_{\lambda_o}$. Therefore $R/M_{\lambda_o} \simeq T/P$. Given $L \in \mathcal{F}(T)$, there exists $P \in Max(T)$ such that $L \simeq T/P$. Since $T = \bigcup_{i\in I} T_i$, we have $P = \bigcup_{i\in I} P_i$, where $P_i = P \cap T_i \in Spec(T_i)$. For each $i$, $T_i$ is a subring of $\prod_{\lambda\in\Lambda} R/M_\lambda$ and $T_i$ is isomorphic to a finite product of fields. Since $\{T_i\}_{i\in I}$ is directed, so is $\{T_i/P_i\}_{i\in I}$. It follows that $\bigcup_{i\in I} T_i/P_i$ is a field. Hence, there exists $M_{\lambda_o} \in Max(R)$ such that $\bigcup_{i\in I} T_i/P_i \subseteq R/M_{\lambda_o}$. It is shown in [9, Proposition 6.1.2, page 128] that $T/P = \bigcup_{i\in I} T_i/P_i$. Let $Q = \prod_{\lambda\in\Lambda} K_\lambda$ such that $K_\lambda = R/M_\lambda$ for each $\lambda \in \Lambda \diagdown \{\lambda_o\}$ and $K_{\lambda_o} \equiv \bar{0}$ modulo $M_{\lambda_o}$. Therefore, $P \subseteq Q \cap T$, and since $P$ is maximal in $T$, we have $P = Q \cap T$. Since $P \cap R = Q \cap T \cap R = Q \cap R = M_{\lambda_o}$, $R/M_{\lambda_o} \subseteq T/P$, i.e., $T/P = R/M_{\lambda_o}$ and hence $\mathcal{F}(T) \subseteq \mathcal{F}(R)$. Thus, $\mathcal{F}(T) = \mathcal{F}(R)$. □

From Proposition 2.1, we deduce that if $R$ is a directed union of finite products of fields then $\mathcal{F}(R) = \mathcal{F}(T)$, where $R \subset T \subset \prod_{\lambda\in\Lambda} \frac{R}{M_\lambda}$, but the converse fails as shown in the following example.

**Counterexample 2.2.** Let $p$ be a positive prime integer and $\{q_i\}_{i\in\mathbb{N}^*}$ be an infinite family of positive prime integers. Let $\mathcal{F} = \{GF(p)\} \bigcup \{GF(p^{q_i})\}_{i=1}^{\infty}$ be a family of finite Galois fields. We denote by $\varphi_i$ the natural imbedding of $GF(p)$ into $GF(p^{q_i})$ for each $i \in \mathbb{N}^*$. Let $\varphi = \prod_{i=1}^{\infty} \varphi_i$, $T = \prod_{i=1}^{\infty} GF(p^{q_i})$ and $I = \bigoplus_{i=1}^{\infty} GF(p^{q_i})$ be the direct sum ideal of $T$. We denote $R_o = GF(p)^* = \varphi(GF(p))$ the diagonal imbedding of $GF(p)$ in $T$. Let $V = R_o + I$, since $V$ is a subring of $T$ and dim($V$)=0 [6, Proposition 2.7], then $V$ is a Von Neumann regular ring. We claim that $\mathcal{F}(V) = \{GF(p)\} \bigcup \{GF(p^{q_i})\}_{i=1}^{\infty}$. Let $p_i : T \to GF(p^{q_i})$ be the canonical projection and $p_{i|V}$ its restriction on $V$, which is a surjective homomorphism. We have $Kerp_{i|V} = (1 - e_i)T \bigcap V = (1 - e_i)V = M_i$, with $e_i$ the primitive idempotent with support $\{i\}$, and $V/M_i \simeq GF(p^{q_i})$ for each $i \in \mathbb{N}^*$. Also, $I$ is a maximal ideal of $V$ and $V/I \simeq (GF(p))^* \simeq GF(p)$. Thus $\{I\} \cup \{M_i\}_{i=1}^{\infty} \subseteq Max(V)$. Let $P \in Spec(V)$, if $I \subseteq P$, then $I = P$. If $I \nsubseteq P$ then $e_i \notin P$, for some $i \in \mathbb{N}^*$, and hence $1 - e_i \in P$. Therefore, $M_i \subseteq P$ and $P = M_i$. As consequence, $Max(V) = \{I\} \bigcup \{M_i\}_{i=1}^{\infty}$. Thus $\mathcal{F}(V) = \{GF(p)\} \bigcup \{GF(p^{q_i})\}$. From [11, Theorem 5.5 page 247], $GF(p^{q_i}) = GF(p)(a_i)$, where $a_i$ is a $p^{q_i}$-th primitive root of unity, for each $i \in \mathbb{N}^*$. Let $R^o$ be the minimal zero-dimensional subring of $\prod_{i=1}^{\infty} GF(p^{q_i})$ containing $V[a]$, where $a = \{a_i\}_{i=1}^{\infty}$. Since $V[a] = R_o[a] + I[a]$, we have $V[a]/I[a] \simeq R_o[a] \simeq GF(p)[X]$ because $a$ is a transcendental element over $R_o$. Therefore, $qf(V[a]/I[a]) \simeq GF(p)(X)$. By [7, Theorem 3.3], $\mathcal{F}(R^o) = \{qf(R/P) : P \in Spec(R)$ is contracted from $T\} = \{GF(p^{q_i})\}_{i=1}^{\infty} \bigcup \{GF(p)(X)\}$. Since $a \in R^o$ and $a$ is in no finite product of fields, then $R^o$ is not a directed union of finite products of fields. By [4, Proposition 5.3 (2)], $R = R^o \oplus GF(p)$ is a Von Neumann regular ring which is not a directed union of finite products of fields. Let $\Omega$ be a field containing each $GF(p^{q_i})$ and $GF(p)(X)$. Given $y = \{y_i\}_{i\in\mathbb{Z}^+} \in \prod_{i=1}^{\infty} GF(p^{q_i}) \times GF(p)(X) \times GF(p)$, let $||y|| = \{y_i : i \in \mathbb{Z}^+\} \subseteq \Omega$. Finally, put $S = \{y \in \Omega^{\mathbb{Z}^+} : ||y||$ is finite$\}$. Let $\varphi : \Omega^{\mathbb{Z}^+} \to \Omega^{\mathbb{Z}^+}$ a homomorphism defined by $\varphi(X) = a$ and let $S_1 = \varphi(S)$. Therefore, $S_1$ is a directed union of finite products of fields and $\mathcal{F}(S_1) = \mathcal{F}(R)$.

Let $R$ be a ring and $\{R_\alpha\}_{\alpha\in A}$ an infinite family of nonzero rings such that $R$ is, up to isomorphism, a subring of each $R_\alpha$. We use $R^*$ to denote the diagonal imbedding of $R$ in $\prod_{\alpha\in A} R_\alpha$, that is $R^* = \varphi(R)$, where $\varphi : R \hookrightarrow \prod_{\alpha\in A} R_\alpha$ is the monomorphism defined by $\varphi(x) = \{x_\alpha\}_{\alpha\in A}$ such that $x_\alpha = x$ for each $\alpha \in A$.

**Proposition 2.3.** *Let $R$ be a von Neumann regular ring with $\mathcal{F}(R) = \{L_\alpha\}_{\alpha\in A}$, $\mathcal{C}(R) = \{p\}$, and $\mathcal{S} = \{\ \{r_\alpha\}_{\alpha\in A} \in \prod_{\alpha\in A} L_\alpha :\ \ \{r_\alpha\}_{\alpha\in A}\ \ has\ \ only\ finitely\ \ many\ distinct\ coordinates\}$. Assume that there exists a field $\Omega$ that contains all but finitely many $L_\alpha$'s. Then $\mathcal{S}$ is a directed union of finite products of fields.*

**Proof.** To show that $\mathcal{S}$ is a directed union of finite products of fields, it suffices to prove that $\mathcal{S}$ is covered by a directed union of finite products of fields. Let $f \in \mathcal{S}$, then $\{f(\alpha) : \alpha \in A\} = \{f_1, ..., f_t\}$ a finite set. Let $A_i = \{\alpha \in A : f(\alpha) = f_i\}$ and denote $f_i^* = (f_i, f_i, ..., f_i, ...) \in \prod_{\alpha\in A_i} L_\alpha$. Then $\{f(\alpha)\}_{\alpha\in A} = (f_1^*, ..., f_t^*)$. Since $A = \cup_{i=1}^{t} A_i$, and all the fields $L_\alpha$, $\alpha \in A_i$, have the same characteristic, then, up to isomorphism, $\bigcap_{\alpha\in A_i} L_\alpha = K_i$ is a field with $f_i^* \in K_i^*$, the diagonal imbedding of $K_i$ in $\prod_{\alpha\in A_i} L_\alpha$. It follows that $f \in K_1^* \times ... \times K_t^* \simeq K_1 \times ... \times K_t$. Therefore, $\mathcal{S}$ is covered by a directed union of finite products of fields. □

**Theorem 2.4.** *Let $R$ be a von Neumann regular ring with $\mathcal{F}(R) = \{L_\alpha\}_{\alpha\in A}$ and $\mathcal{S} = \{\ \{r_\alpha\}_{\alpha\in A} \in \prod_{\alpha\in A} L_\alpha :\ \ \{r_\alpha\}_{\alpha\in A}\ \ has\ \ only\ \ finitely\ \ many\ distinct\ coordinates\}$ Then, $\mathcal{S}$ is a directed union of finite products of fields if and only if $\mathcal{C}(R)$ is finite.*

**Proof.** Assume that $\mathcal{S}$ is a directed union of finite products of fields and let $f \in \mathcal{S}$. Then $f$ has only finitely many distinct components $f_1$,..., $f_t$. Now, let $A_i = \{\alpha \in A : f(\alpha) = f_i\}$, for each $i = 1, ..., t$. So $A = \bigcup_{i=1}^{t} A_i$ is a partition of $A$. We set $f_i^* = (f_i, ..., f_i, ...) \in \prod_{\alpha\in A_i} L_\alpha$. Then, up to isomorphism, $\{f(\alpha)\}_{\alpha\in A} = (f_1^*, ..., f_t^*)$ belongs to a finite product of fields. In fact $(f_1^*, ..., f_t^*) \in \prod_{i=1}^{t} K_i^* \simeq \prod_{i=1}^{t} K_i$, where $K_i^*$ is the diagonal imbedding of $K_i$ into $\prod_{\alpha\in A_i} L_\alpha$ for each $i = 1, ..., t$, with $K_i^* \simeq K_i = \bigcap_{\alpha\in A_i} L_\alpha$. Since $\mathcal{S}$ is a subring of $\prod_{\alpha\in A} L_\alpha$, this shows that $\prod_{i=1}^{t} K_i$ is isomorphic to a subring of $\prod_{\alpha\in A} L_\alpha$. It follows that $\mathcal{C}(R)$ is finite. Conversely, let $\mathcal{C}(R) = \{p_1, ..., p_n\}$. We can write $\prod_{\alpha\in A} L_\alpha = \oplus_{i=1}^{n} T_i$, where $T_i = \prod_{\alpha\in\Lambda_i} L_\alpha$ and $\Lambda_i = \{\alpha \in A : char(L_\alpha) = p_i\}$. Let $e_j$ be the idempotent element of $\prod_{\alpha\in A} L_\alpha$ associated with {j}. Hence $\mathcal{S} = \mathcal{S}e_1 \oplus\ ...\ \oplus \mathcal{S}e_n$. Form Proposition 2.3, each $\mathcal{S}e_j$ is a subring of $T_j$ which is a directed union of finite products of fields. By [4, Proposition 5.3], $\mathcal{S}$ is a directed union of finite products of fields. □

## 3 Behavior with respect to residue fields

Let $R$ be a von Neumann regular ring with maximal ideals $\mathfrak{m}_i$, $i \in I$ and corresponding residue fields $K_i$. We assume that there exists a field $\Omega$ containing each $K_i$. (We can always make this assumption if the $K_i$ have the same characteristic.) Assuming $\mathcal{C}(R)$ finite, we have the partition $\mathcal{F}(R) = \mathcal{F}_1 \bigcup\ ...\ \bigcup \mathcal{F}_n$ of $\mathcal{F}(R)$ with respect to the characteristic. We assume that for each $i \in \{1, ..., n\}$ there exists $F_i$ such that $F_i \in \mathcal{F}_i$ and each element of $\mathcal{F}_i$ is an algebraic extension of $F_i$. We use $F^*$ to denote the diagonal imbedding of $F$ into $\prod_{i\in I} F_i$.

**Theorem 3.1.** *The ring $R$ is a directed union of finite products of fields if and only if for each $f \in R$, $f$ is integral over $F_1^* \times\ ...\ \times F_n^*$.*

In order to prove this result, we need the following Lemma.

**Lemma 3.2.** *Let $R$ be a Von Neumann regular ring and $\mathcal{F}(R) = \{F_i\}_{i\in I}$. Assume that each $F_i$ is an algebraic extension of $F$. Then $R$ is a directed union of finite products of fields if and only if for each $f \in R$, $f$ is integral over $F^*$.*

**Proof.** Suppose that $R$ is a directed union of finite products of fields and let $f \in R$. Then $f$ belongs to a finite product of fields, in other words $f$ is in only finitely many fields $F_i$. By [1, Proposition 3, page 9], $f$ is an integral over $F^*$. Conversely, let $f \in R$ to be integral over $F^*$. Then there exists a monic polynomial $H(X)$ in $F^*[X]$ that vanishes at $f$. The polynomial $H(X)$ has only finitely many roots of $H(X)$. We note also that $\{f(i)\}_{i\in I}$ is the unique solution of $H(X)$. It follows that $f$ has only finitely many distinct components. We conclude that $R \subset \mathcal{S}$ and Theorem 2.4 completes the proof. □

**Proof of Theorem 3.1.** We can write $\prod_{i\in I} F_i = \bigoplus_{j=1}^{n} T_j$, where $T_j = \prod_{i\in I_j} F_i$ and $I_j = \{i \in I\ :\ \ char(F_i) = p_j\}$. From Lemma 3.2, to show that $R$ is a directed union of finite products of fields it suffices to show that $R[e_1, ...,\ e_n]$ has the same property, where $e_j$ is the idempotent associated with $j$, for $j = 1,\ ...,\ n$. Moreover, since $R[e_1, ...,\ e_n] = Re_1 \oplus\ ...\ \oplus Re_n$, to prove that the condition of Theorem 3.1 is satisfied for $R[e_1, ...,\ e_n]$ it suffices to show that it is satisfied for each $Re_j$. Let $f \in R$ be integral over $\prod_{i=1}^{n} F_i^*$. Let $H(X)$ be a monic polynomial of $\prod_{i=1}^{n} F_i^*[X]$ such that $H(f) = 0$. Let $H_j = He_j$, for each $j = 1,\ ...\ n$. The polynomial $H_j$ is monic in $F_i^*[X]$ and satisfy $H_j(\{f(i)\}_{i\in I_j}) = 0$, for each $j = 1,\ ...\ n$. Therefore, $\{f(i)\}_{i\in I_j}$ is integral over $F_j^*$. From Lemma 3.2, $Re_j$ is a directed union of finite products of fields, for each $j = 1\ ...\ n$. Conversely, assume that $R$ is a directed union of Finite products of fields. We know that $fe_j$ is integral over $F_j^*$ for each $j = 1,\ ...,\ n$. Let $H_j(X) \in F_j^*[X]$ be a monic polynomial that vanishes at $fe_j$, for $j = 1,\ ...,\ n$. If we set $H = \prod_{i=1}^{n} H_j$, then $H$ is a monic polynomial $\prod_{j=1}^{n} F_j^*$ that vanishes at $f$ (cf. [1, Proposition 3, page 9]). □

**Example 3.3.** Let $p$ be a positive prime integer and $\{q_i\}_{i\in\mathbb{N}^*}$ be an infinite family of distinct prime integers. Let $\mathcal{F} = \{\mathbb{Q}\} \bigcup \{\mathbb{Q}(\zeta_i)\}_{i=1}^{\infty}$ be an infinite family of fields, where $\zeta_i$ is a $p^{q_i}-$ primitive root of unity. We denote by $\varphi_i$ the imbedding of $\mathbb{Q}$ into $\mathbb{Q}(\zeta_i)$ for each $i \in \mathbb{Z}^+$. Let $\varphi = \prod_{i=1}^{\infty} \varphi_i$, $T = \prod_{i=1}^{\infty} \mathbb{Q}(\zeta_i)$, $I = \bigoplus_{i=1}^{\infty} \mathbb{Q}(\zeta_i)$ the direct sum ideal of $T$. We denote $\mathbb{Q}^* = \varphi(\mathbb{Q}) = R_o \simeq \mathbb{Q}$ the diagonal imbedding of $\mathbb{Q}$ in $T$. Let $R_1 = R_o + I$. Then $\mathcal{F}(R_1) =$

$\{F_i\}_{i=1}^{\infty} \bigcup \{\mathbb{Q}\}$. Let $f \in \prod_{i=1}^{\infty} F_i$ such that $f(i) = i$ for each $i \in \mathbb{Z}^+$. If $R$ is the minimal zero-dimensional subring of $T$ containing $R_1[f]$, then $\mathcal{F}(R) = \{F_i\}_{i=1}^{\infty} \bigcup \{\mathbb{Q}(t)\}$, where $F_i = \mathbb{Q}(\zeta_i)$ for each $i \in \mathbb{Z}^+$. On the other hand, since $\mathcal{S} = R_o + J$, where $J = \mathbb{Q}(t) + I$, we have $\mathcal{F}(\mathcal{S}) = \{F_i\}_{i=1}^{\infty} \bigcup \{\mathbb{Q}(t)\} \bigcup \{\mathbb{Q}\}$. Now, by [4, Proposition 5.3 (2)], the ring $R = R_1 \oplus \mathbb{Q}$ is not a directed union of finite products of fields, even if $\mathcal{F}(R) = \mathcal{F}(\mathcal{S})$. Notice that $F_i \cap F_j = \mathbb{Q}$ for $i \neq j$ and $F_i \cap \mathbb{Q}(t) = \mathbb{Q}$ with $\mathbb{Q} \in \mathcal{F}(R)$. Even though, $\mathbb{Q} \in \mathcal{F}(R)$ $R$ is not a directed union of finite products of fields.

Let $R$ be a von Neumann regular ring and $\{M_i\}_{i\in I}$ its spectrum. Since $R$ is a reduced ring, we have $\bigcap_{i\in I} M_i = (0)$ and hence the homomorphism $\varphi : R \to \prod_{i\in I} \frac{R}{M_i}$, defined by $\varphi(x) = x + M_i$, is injective. This allows us to view $R$ as a subring of $\prod_{i\in I} \frac{R}{M_i}$. We identify $x$ with its image $\{x_i\}_{i\in I} \in \prod_{i\in I} \frac{R}{M_i}$. Finally, we denote $F_i = \frac{R}{M_i}$ for each $i \in I$.

**Corollary 3.4.** *With the notation and assumptions above, we assume that there is a field $\Omega$ containing each $F_i$ and $F$ is the prime subfield of $\Omega$. If each distinct pair of fields $F_j$ and $F_k$ in $\mathcal{F}_i$ satisfying $F_j \bigcap F_k = F \notin \mathcal{F}(R)$, then $R$ is not a directed union of finite products of fields.*

**Proof.** Suppose that $F \notin \mathcal{F}(R)$. Let $S$ be the subring of $\prod_{i\in I} F_i$ consisting of eventually constant sequences. Thus $S = F^* + I$, the $F$-subalgebra of $\prod_{i\in I} F_i$ generated by the direct sum ideal $I = \oplus_{i\in I} F_i$, where $F^*$ is the diagonal imbedding of $F$ into $\prod_{i\in I} F_i$. First claim that $S$ is the maximal subring of $\prod_{i\in I} F_i$ with respect to being a directed union of finite product of fields. Let $T = \bigcup_{j\in J} T_j$ be a subring of $\prod_{i\in I} F_i$ which is a directed union of finite product of fields. Let $t = \{t_i\}_{i\in I} \in T$ then there exists $j_o \in J$ such that $t \in T_{j_o}$ which is a finite products of fields, then $t$ has only finitely many distinct coordinates, i.e., $t \in S$. If $R$ is a directed union of finite products of fields, then $R \subseteq S$ and hence $\mathcal{F}(R) = \mathcal{F}(S)$ (see Proposition 2.1), a contradiction with $F \in \mathcal{F}(S) \diagdown \mathcal{F}(R)$. □

**Example 3.5.** Let $\mathbb{Q}$ be the field of rational numbers, $\alpha$ an element such that $\alpha^2 = d$ ($d$ without squire factor in $\mathbb{Q}$) and $\mathcal{P}$ be an infinite family of distinct prime integers. Let $\Omega = \mathbb{Q}(\alpha)$ be a simple algebraic extension of $\mathbb{Q}$ and $R = \mathbb{Q}(\alpha) + I$, the $\Omega-$subalgebra of $T = \prod_{i=1}^{\infty} \Omega(\zeta_i)$ generated by the direct sum ideal $I = \oplus_{i=1}^{\infty} \Omega(\zeta_i)$, where $\zeta_i$ is a $p^q$-primitive root of unity and $p$ is a prime integer with $q \in \mathcal{P}$. For each $i \in \mathbb{Z}^+$, let $\phi_i : \Omega \to \Omega(\zeta_i)$ be the field-homomorphism taking $\alpha$ to $\zeta_i$. Let $\phi = \{\phi_i\}_{i=1}^{\infty} : T \to T$, a ring-homomorphism. Let $R_o = \phi(R)$. Being isomorphic to $R$, $R_o$ is a directed union of finite products fields. We remark that the element $\{\zeta_i\}_{i=1}^{\infty} \in R_o$ which is not in $\mathcal{S}$ (the maximum among all subrings of $T$ that are directed union of finite products of fields in proof of Corollary 3.4).

**Proposition 3.6.** *Let $R$ be a zero-dimensional ring and $N(R)$ be the nilradical of $R$. Then the following conditions are equivalent:*

*(i) $R$ is a directed union of zero-dimensional subrings with finite spectra;*

*(ii) $R/N(R)$ is a directed union of finite products of fields.*

**Proof.** (i)⇒(ii). Suppose that $R = \bigcup_{i\in I} R_i$ is a directed union of zero-dimensional subrings with finite spectra, then by [15, Proposition 6.1.2, page 128], $R/N(R) \simeq \bigcup_{i\in I} R/N(R) \cap R_i = \bigcup_{i\in I} R_i/N(R_i)$ is a directed union of $R_i/N(R_i)$, where $N(R_i)$ is the nilradical of $R_i$. The ring $R_i/N(R_i)$ is Von Neumann regular with finite spectra, then $R_i/N(R_i)$ is Artinian.
(ii)⇒ (i). Suppose $R/N(R) = \bigcup_{i\in I} S_i$ is a directed union of Finite products of fields and let $\varphi :\ R \to R/N(R)$ be the canonical epimorphism. We denote $R_i = \varphi^{-1}(S_i)$ the inverse image of $S_i$ by $\varphi$, for each $i \in I$. Since $\{S_i\}_{i\in I}$ is directed, the family $\{R_i\}_{i\in I}$ is also directed. We have $R_i/N(R_i) \simeq S_i$, as $Spec(S_i)$ is finite the ring $R_i$ has only finitely many prime ideals, for each $i \in I$. Because $S_i$ is zero-dimensional, $R_i$ is also zero-dimensional but need not be Artinian. It follows that $R = \bigcup_{i\in I} R_i$ is a directed union of zero-dimensional quasilocal subrings. □

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