Palestine Journal of Mathematics

Vol. 1 (2012) , 1–10

Exponential decay for low and higher energies in the third order linear Moore-Gibson-Thompson equation with variable viscosity

Barbara Kaltenbacher and Irena Lasiecka

Communicated by Jawad Abuhlail

MSC2010 classification: 35B33,35B40,35L05,35M10

Keywords: third order abstract differential equations, Moore- Gibson -Thompson equation, variable coefficients, exponential decays rates, higher energies, evolution semigroups.

1 Abstract

We consider the Moore-Gibson-Thompson equation which arises, e.g., as a linearization of a model for wave propagation in viscous thermally relaxing fluids. This third order in time equation displays, even in the linear version, a variety of dynamical behaviors for their solutions that depend on the physical parameters in the equation. These range from non-existence [3] and instability to exponential stability (in time) [11]. By neglecting diffusivity of the sound coefficient there is a lack of generation of a semigroup associated with the linear dynamics. When diffusivity of the sound is positive, the linear dynamics is described by a strongly continuous evolution. We shall show that this evolution is exponentially stable provided sufficiently large viscous damping is accounted for in the model. The viscosity considered is *time and space dependent* which then leads to evolution rather then semigroup generators. Decay rates for both natural and higher level energies are derived.

2 Introduction

Investigations on nonlinear propagation of sound in the situation of high amplitude waves have put forth extensive literature on physically well-based partial differential models see, e.g., [2, 1, 4, 14, 12, 18, 20]. This still highly active field of reasearch is driven by a wide range of applications such as the medical and industrial use of high intensity ultrasound in lithotripsy, thermotherapy, ultrasound cleaning and sonochemistry.

The classical models of nonlinear acoustics are Kuznetsov's equation, the Westervelt equation, and the KZK (Kokhlov-Zabolotskaya-Kuznetsov) equation. For a mathematical well-posedness analyis of several types of initial boundary value problems for these nonlinear second order in time PDEs we refer to, e.g., [7, 8, 9, 10, 19]. Motivated mainly by the fact that the use of classical Fourier's law leads to an infinite signal speed paradoxon, the use of several other constitutive relations for the heat flux within the derivation of nonlinear acoustic wave equations have been considered Jordan [6]. Among these is the Maxwell-Cattaneo law, whose combination with the usual balance equations (conservation of mass, momentum and energy) as well as the equation of state, leads to a third order in time PDE model. Since this linearized version appears in a slightly different setting in Moore & Gibson [15] and Thompson [17] (see eq. (11.84) on p 556 there), we here call it Moore-Gibson-Thompson equation, where the fully nonlinear version will be referred to as the Jordan-Moore-Gibson-Thompson equation.

It should be noted that the analysis of the third order equations is very different from that of the second order, where a positive diffusivity coefficient provides a regularizing parabolic effect. This is no longer true in the third order equations which are of *hyperbolic type*, thus requiring a very different type of analysis than the related second order equations. The case of constant coefficient equations and finite energy solutions has been studied in [11]. The aim of this paper is to provide: (i) analysis of *time-space* dependent coefficients in the equations and (ii) *stability estimates for higher energy solutions* . This is a critical perquisite for studying nonlinear dynamics.

We remark that while in the constant coefficient case, stability analysis for linear dynamics can be carried out via spectral analysis, this is not the case in the non-autonomous case where the location of the spectrum for each temporary point in the generator provides only limited information on the overall stability. (counterexamples are known for simple ODE's). For this reason a different -energy based method - will be developped. We shall show that under suitable assumptions on the variations of the coefficients both low and high energy dynamics are well-posed and exponentially stable. The results presented provide first and necessary step for the analysis of the corresponding nonlinear model. The latter subject will be pursued in a forthcoming publication by the authors.

2.1 Third Order abstract model

Let $\mathcal{H} \equiv L_2(\Omega)$ where $\Omega \in R^n$, $n \leq 3$ is a bounded and smooth domain. Let $\mathcal{A} \equiv -\Delta$, defined on $\mathcal{D}(\mathcal{A}) = H_0^1(\Omega) \cap H^2(\Omega)$. It is well known that $\mathcal{A}$ is a selfadjoint positive operator defined on $\mathcal{H}$ with a dense domain $\mathcal{D}(\mathcal{A})$.

Moreover fractional powers $\mathcal{A}^\alpha$ are well defined. In particular, $\mathcal{D}(\mathcal{A}^{1/2}) \sim H_0^1(\Omega)$ [16].

We shall consider the linearized version of Kuzntesov Moore Gibson Equation written in a general abstract form.

$$\tau u_{ttt} + \alpha(t,x)u_{tt} + c^2 \mathcal{A}u + b\mathcal{A}u_t = 0 \tag{2.1}$$

where $\alpha(t,x)$ is a variable "damping" coefficient depending on a viscosity of the fluid. It is interesting to notice that the third order in time model has very different characteristics from a familiar second order equation ($\tau = 0, \alpha > 0$). The wellposedness of solutions fails even in the simplest case when $b = 0$ [3]. Thus, structural damping is essential for the well-behaved systems. As we know, for the second order equations the presence of the structural damping is immaterial for the wellposedness, it does however play a role in asymptotic behavior and regularity of solutions. Instead, for the third order equations structural damping ($b > 0$) is critical for the wellposedness [11]. More specifically, it does affect both the wellposedness and stability. Our main goal is to provide a complete analysis and classification of parameters leading to both wellposedness and stability of the abstract model under consideration.

Exponential stability of the trajectories depends on the critical parameter $\frac{sound\ speed \times relaxation\ parameter}{sound\ diffusivity}$ which is required to be small enough with respect to a natural damping α in the system. More specifically, in the constant coefficient case the exponential stability of natural energy function requires $\gamma \equiv \alpha - \frac{\tau c^2}{b} > 0$. In the complementary region of the parameters the system is unstable ($\gamma < 0$) or marginally stable ($\gamma = 0$). In the present paper we shall extend this result to non-constant coefficient case and we shall also incoroporate the analysis of higher energy stability. To this end we shall provide semigroup formulation for the model.

2.2 Semigroup formulation

With $\mathcal{A}$ defined in the previous section we consider the following third order in time abstract ODE defined on $\mathcal{H} = L_2(\Omega)$

$$\tau u_{ttt} + \alpha(x,t)u_{tt} + c^2\mathcal{A}u + b\mathcal{A}u_t = 0, \;\; t > 0 \tag{2.2}$$

with the initial conditions given by

$$u(0) = u_0, u_t(0) = u_1, u_{tt}(0) = u_2. \tag{2.3}$$

The coefficient α can be thought of as control parameter inducing viscous damping in the equation. In fact, this coefficient has important implications on controlling the size of the potential well corresponding to global existence of nonlinear solutions. In what follows we shall assume that the damping coefficient α is both space and time dependent with the imposed $L_\infty(\Omega \times R^+)$ bound.

The above system given in (2.2) can be written as a first order system of the following form:

$$U_t(t) = A(t)U(t), \; t > 0\,, U(0) = U_0 \in H \tag{2.4}$$

where $H \equiv \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{H}$ and

$$U \equiv \begin{pmatrix} u \\ u_t \\ u_{tt} \end{pmatrix}; \;\; A(t) = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -c^2\mathcal{A} & -b\mathcal{A} & -\alpha(\cdot,t)I \end{pmatrix}.$$

The natural domain of $A(t)$ is given by

$$\mathcal{D}(A(t)) = \mathcal{D}(A) = \{U = (u_1,u_2,u_3) \in H, u_3 \in \mathcal{D}(\mathcal{A}^{1/2}), u_i \in \mathcal{D}(\mathcal{A}), i = 1,2\}$$

The goal of this paper is the analysis of wellposedness and of asymptotic stability of the model defined in (2.4). As noticed in [11] when $\tau = 0$ the model reduces to a classical wave equation with ($b > 0$) structural damping -which corresponds to an analytic semigroup.. However, $\tau > 0$ makes the model of hyperbolic type and the wellposedness is no longer valid unless the parameters are appropriately selected. As shown in [3] the problem is not well-posed when the diffusivity constant $b = 0$. In what follows we shall assume that $b > 0$. It will be shown that for these parameters system (2.4) is always wellposed. However, wellposedness of the corresponding nonlinear equation depends on long time characteristics of the model. These are affected by the damping coefficient $\alpha(t,x)$. Thus, the interaction between the damping α and diffusivity parameter b lies in the heart of the problem. Of particular interest to this paper is not only long time behavior of the natural energy associated with A, but also the higher energy that is critical for the construction of potential well corresponding to nonlinear problem.

Notation: $(u,v) \equiv (u,v)_\mathcal{H}$. Similarly $|u|^2 \equiv |u|^2_\mathcal{H}$. $\mathcal{A}^\theta$, $\theta \in [0,1]$ denotes fractional powers of $\mathcal{A}$ [16].

2.3 Main results

There are two energy functions that are of interest to this work. The first one is the basic *finite energy* corresponding to the operator A

$$\mathcal{E}_0(t) \equiv |\mathcal{A}^{1/2}u(t)|^2 + |\mathcal{A}^{1/2}u_t(t)|^2 + |u_{tt}|^2 \tag{2.5}$$

The *higher energy* is given by
$$\mathcal{E}_1(t) \equiv |\mathcal{A}u(t)|^2 + |\mathcal{A}^{1/2}u_t(t)|^2 + |u_{tt}|^2 \tag{2.6}$$
We note that the *higher* energy is not just an energy corresponding to strong (regular) solutions. It represents a different measure of regularity of solutions that is somewhat reminiscent to parabolic effects. Though the system is not parabolic. We believe this being an interesting aspect of the model studied.

We shall be using the following Assumption imposed on variable coefficient $\alpha(x,t)$.

Assumption 2.1.

(i) $\alpha(x,t) \in [\alpha_0, \alpha_1], \forall x, t \in \bar{\Omega} \times R^+$ where $\alpha_0 \geq 0, \alpha_1 < \infty$.

(ii) The map $u \to \alpha(t)u$ is continuous in $t > 0$ from $\mathcal{D}(\mathcal{A}^{1/2}) \to \mathcal{H}$

(iii) The map $u \to \dot{\alpha}u$ is bounded from $\mathcal{D}(\mathcal{A}^{1/2}) \to L_\infty(R, [\mathcal{D}(\mathcal{A}^{1/2})]')$.

Theorem 2.2. *Let $b > 0$, α satisfies Assumption 2.1 with $\alpha_0 \geq 0$. The system given in (2.2) generates a strongly continuous evolution $U(t,s) : H \to H$ with $0 \leq s \leq t$. In addition $U(t,s)$ is also a strongly continuous evolution when acting $U(t,s) : H_1 \to H_1$ with*
$$H_1 \equiv \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{H}$$
Both evolutions are time reversible.

Our second result describes exponential decays for the two energies functions. It turns out that the latter depends on the values of certain parameters. Let's introduce the parameter $\gamma(t,x) \equiv \alpha(t,x) - \frac{\tau c^2}{b}$.

Theorem 2.3. *Let $b > 0$ and $\alpha_0 > 0$. Assumption 2.1 is in force. We also assume that the norm of the map defined in part 3 of Assumption 2.1 is sufficiently small. Then,*

- *If $\gamma(x,t) \geq \alpha_0 - \frac{\tau c^2}{b} > 0$, there exist $\omega > 0, C > 0$ such that*
$$\mathcal{E}(t) \leq Ce^{-\omega t}\mathcal{E}(0)$$

- *If, in addition to the previous assumptions, we also impose*

  Assumption 2.4. The map $u \to \alpha(t)u$ is continuous in from $\mathcal{D}(\mathcal{A}^{1/2}) \to L_\infty(R, \mathcal{D}(\mathcal{A}^{1/2}))$
  The map $u \to \dot{\alpha}u$ is bounded from $\mathcal{D}(\mathcal{A}^{1/2}) \to L_\infty(R, \mathcal{H})$

  then we have
$$\mathcal{E}_1(t) \leq C_1 e^{-\omega_1 t}\mathcal{E}_1(0)$$

Corollary 2.5. *Under the assumptions of Theorem 2.3 the evolution operator $U(t,s)$ is exponentially stable on H. If, in addition, Assumption 2.4 holds then $U(t,s)$ is also exponentially stable on H_1.*

The remainder of the manuscript is devoted to the proofs of the main theorems.

3 Proof of Theorem 2.2

Let $b > 0$ and without loss of generality we normalize $\tau = 1$. We introduce the following variable :
$$z \equiv u_t + c^2 b^{-1} u.$$

3.1 Generation of the group on *H*

Consequently $u_t = z - c^2 b^{-1} u$ and
$$\begin{aligned}
u_{ttt} &= -\alpha u_{tt} - b\mathcal{A}z \\
u_{tt} &= z_t - c^2 b^{-1} u_t = z_t - c^2 b^{-1}[z - c^2 b^{-1} u] \\
z_{tt} &= u_{ttt} + c^2 b^{-1} u_{tt} \\
&= -(\alpha - c^2 b^{-1}) u_{tt} - b\mathcal{A}z \\
&= -\gamma(t) z_t - \gamma c^2 b^{-1}[z - c^2 b^{-1} u] - b\mathcal{A}z.
\end{aligned} \tag{3.1}$$
With this notation we introduce the vector
$$Y \equiv \begin{pmatrix} \mathcal{A}^{1/2}u \\ \mathcal{A}^{1/2}z \\ z_t \end{pmatrix}$$

and we consider $Z \equiv \mathcal{H} \times \mathcal{H} \times \mathcal{H}$. The original model can now be rewritten in operator form as:

$$Y_t(t) = B(t)Y(t), \;\; Y(0) = Y_0 \in Z$$

where the matrix operator B with a natural domain takes the form:

$$B(t) \equiv \begin{pmatrix} -c^2b^{-1}I & I & 0 \\ 0 & 0 & \mathcal{A}^{1/2} \\ -\gamma(t)c^4b^{-2}\mathcal{A}^{-1/2} & -b\mathcal{A}^{1/2} + \gamma(t)c^2b^{-1}\mathcal{A}^{-1/2} & -\gamma(t)I \end{pmatrix} \tag{3.2}$$

$$\mathcal{D}(B) = \{Y = (y_1, y_2, y_3) \in Z; y_2 \in \mathcal{D}(\mathcal{A}^{1/2}), y_3 \in \mathcal{D}(\mathcal{A}^{1/2})\}.$$

The operator matrix B can be represented as a bounded perturbation of B_0, ie: $B(t) = B_0 + P(t)$ where

$$B_0 \equiv \begin{pmatrix} -c^2b^{-1}I & I & 0 \\ 0 & 0 & \mathcal{A}^{1/2} \\ 0 & -b\mathcal{A}^{1/2} & 0 \end{pmatrix}$$

and the bounded part takes the form

$$P(t) \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\gamma(t)c^4b^{-2}\mathcal{A}^{-1/2} & \gamma(t)c^2b^{-1}\mathcal{A}^{-1/2} & -\gamma(t)I \end{pmatrix}. \tag{3.3}$$

We shall apply Evolution Hyperbolic Theorem due to Pazy . This will allow us to claim generation of a continuous evolution operator. The task is thus reduced to the verification of conditions (H-1) -(H-3) on p.135, [16]. We recall the statement of the hypotheses. Let $B(t)$ be the infinitesimal generator of a C_0 semigroup $S_t(s), s \geq 0$ on Z. Let $V \subset Z$ with dense injection. We say that V is B admissible if V is invariant subspace of $S(s)$ and the restriction $\tilde{S}(s)$ to V is a C_0 semigroup on V. The following assumptions are made on the family of operators $B(t)$.

(i) $\{B(t)\}_{t\in[0,T]}$ for each t is a stable family with stability constants $\omega > 0, M$.

(ii) V is $B(t)$ - admissible in V for all $t \in [0,T]$ and the family $\{\tilde{B}(t)\}_{t\in[0,T]}$ is a stable family in V with some constants $\tilde{\omega}, \tilde{M}$.

(iii) For $t \in [0,T]$, $D(B(t)) \supset V$, where $B(t)$ is bounded on V and the map $t \to B(t)$ is continuous in the $B(V,Z)$ norm.

Theorem 3.1. *Let $B(t)$, $t \in [0,T]$ be the infinitesimal generator of a C_0 semigroup on Z such that conditions 1-3 above are satisfied. Then $B(t)$ generates a unique evolution system on Z given by the two-parameters family of operators with the properties:*

- $||U(t,s)||_{L(Z)} \leq Me^{\omega(t-s)}$
- $D_t^+U(t,s)x|_{t=s} = B(s)x, \;\; x \in V, 0 \leq s \leq t \leq T,$
- $D_sU(t,s)x = -U(t,s)B(s)x, x \in V, 0 \leq s \leq t \leq T\; U(t,s), 0 \leq s \leq t$ *where the derivatives are in strong Z sense.*

To proceed, we first note that for each $t > 0$ $B(t)$ is a bounded (owing to Assumption 2.1) perturbation of B_0 , where the latter generates continuous semigroup on Z. Thus, by classical semigroup perturbation theorem $B(t)$ generates, for each t, a strongly continuous semigroup $S_t(s)$. These semigroups are stable on the strength of the uniform boundedness of $\gamma(t)$ -Assumption 2.1.

For the second condition, we take $V = \mathcal{D}(B(t))$, where the latter is independent on t. Since Y coincides with the domain of $B(t)$ for each t, the invariance condition for V and $\tilde{S}_t$ is automatically satisfied. The stability parameters $\tilde{\omega}, \tilde{M}$ coincide with the former.

For the third condition, it suffices to argue continuity of $P(t)$. This entails to the continuity (in time) of the following maps:

$$\begin{aligned} \gamma(\cdot,t)\mathcal{A}^{-1/2} &: \mathcal{D}(\mathcal{A}) \to \mathcal{H} \\ \gamma(\cdot,t) &: \mathcal{D}(\mathcal{A}^{1/2}) \to \mathcal{H} \end{aligned} \tag{3.4}$$

It is clear that the second requirement implies the first, which in turn is a consequence of the second statement in Assumption 2.1 .

On the strength of Pazy hyperbolic theorem we conclude that $B(t)$ generates strongly continuous family of evolutions $U(t,s)$ on Z. This is equivalent (via change of variables) to the fact that $A(t)$ generates a strongly continuous evolution on H.

Remark 3.2. The operator matrix $B(t)$ can be represented as a compact perturbation of $B_1(t)$, ie: $B(t) = B_1(t) + K(t)$ where

$$B_1(t) \equiv \begin{pmatrix} -c^2 b^{-1} I & I & 0 \\ 0 & 0 & \mathcal{A}^{1/2} \\ 0 & -b\mathcal{A}^{1/2} & -\gamma(t) I \end{pmatrix}$$

and the bounded part takes the form

$$K(t) \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\gamma(t) c^4 b^{-2} \mathcal{A}^{-1/2} & \gamma(t) c^2 b^{-1} \mathcal{A}^{-1/2} & 0 \end{pmatrix} \tag{3.5}$$

When $\gamma(t)$ is constant in time, then the spectrum of B_1 consists just of eigenvalue $\lambda_1 = -\frac{c^2}{b}$ and the eigenvalues corresponding to a damped wave equation with infintely many eigenvalues located on a vertical line $Re\lambda_n = -1/2\gamma$. These are precisely two limiting points for the essential spectrum of the operator that correspond to infinite dimensional part of PDE. .

3.2 Generation of the group on H_1

Now we turn to generation on H_1. For that a different decomposition -compatible with the topology on H_1 - is more useful. To this end we introduce the following variables

$$z \equiv u_t + c^2 b^{-1} u$$

and

$$v \equiv \mathcal{A}u + b^{-1} z_t.$$

Consequently $u_t = z - c^2 b^{-1} u$ and

$$\begin{aligned} u_{tt} &= z_t - c^2 b^{-1} u_t = z_t - c^2 b^{-1}[z - c^2 b^{-1} u] \\ u_{ttt} &= z_{tt} - c^2 b^{-1}[z_t - c^2 b^{-1} u_t] = z_{tt} - c^2 b^{-1}[z_t - c^2 b^{-1}[z - c^2 b^{-1} u]] \\ v_t &= (\mathcal{A}u + b^{-1} z_t)_t = b^{-1}[-u_{ttt} - \alpha u_{tt} - c^2 \mathcal{A}u] + b^{-1} z_{tt} \\ &= -b^{-1}\alpha u_{tt} - b^{-1} c^2 \mathcal{A}u + c^2 b^{-2}[z_t - c^2 b^{-1} z + c^4 b^{-2} u] \\ &= -c^2 b^{-1} v + [2c^2 b^{-2} - \alpha b^{-1}] z_t - b^{-3} c^4 [z - c^2 b^{-1} u] + \alpha c^2 b^{-2}[z - c^2 b^{-1} u] \\ &= -c^2 b^{-1} v + [2c^2 b^{-2} - \alpha b^{-1}] z_t + \gamma c^2 b^{-2}[z - c^2 b^{-1} u]. \end{aligned} \tag{3.6}$$

With this notation we introduce vector

$$Y \equiv \begin{pmatrix} v \\ \mathcal{A}^{1/2} z \\ z_t \end{pmatrix}$$

and we consider $Z \equiv \mathcal{H} \times \mathcal{H} \times \mathcal{H}$. The original model can now be rewritten in operator form as:

$$Y_t(t) = B(t)Y(t), \;\; Y(0) = Y_0 \in Z$$

where the matrix operator $B(t)$ with a natural domain takes the form $B(t) = B_0(t) + K(t)$ with

$$B_0(t) \equiv \begin{pmatrix} -c^2 b^{-1} I & 0 & 2c^2 b^{-2} - \alpha b^{-1} \\ 0 & 0 & \mathcal{A}^{1/2} \\ 0 & -b\mathcal{A}^{1/2} & -\gamma(t) I \end{pmatrix}$$

$$\mathcal{D}(B) = \{Y = (y_1, y_2, y_3) \in Z; y_2 \in \mathcal{D}(\mathcal{A}^{1/2}), y_3 \in \mathcal{D}(\mathcal{A}^{1/2})\}$$

and K is a suitable compact perturbation composed of lower order terms resulting from the decomposition (3.6) and is given by

$$K(t) \equiv \begin{pmatrix} -\gamma(t) c^4 b^{-3} \mathcal{A}^{-1} & \gamma(t) c^2 b^{-2} \mathcal{A}^{-1/2} & \gamma(t) c^4 b^{-4} \mathcal{A}^{-1} \\ 0 & 0 & 0 \\ -\gamma(t) c^4 b^{-2} \mathcal{A}^{-1} & \gamma(t) c^2 b^{-1} \mathcal{A}^{-1/2} & \gamma(t) c^4 b^{-3} \mathcal{A}^{-1} \end{pmatrix} \tag{3.7}$$

The same argument as in the proof of Theorem 2.2 applies to infer that $B_0(t)$ generates a continuous evolution on Z. Indeed, $B_0(t)$ is a bounded perturbation of the same operator $B_1(t)$ as introduced in the proof of Theorem 2.2. On the other hand, direct inspection reveals that with the given change of variables defined by Y, $Y \in Z$ is equivalent to the topology of vector U in H_1.

On the other hand, it is evident that the topology on Z is equivalent (since $b > 0$) to the topology on H. This completes the proof of Theorem 2.2.

Remark 3.3. The stability of $U(t,s)$ depends on the lower order perturbations K.

4 Proof of Theorem 2.3

The proof of Theorem 2.3 follows through several lemmas. It is convenient to introduce the following notation: Let's recall the parameter $\gamma(t) \equiv [\alpha(t) - \frac{c^2\tau}{b}]$, where the variable coefficient α satisfies Assumption (2.1), and define the following energies:

$$E(t) \equiv \frac{b}{2}|\mathcal{A}^{1/2}(u_t(t) + c^2 b^{-1} u(t))|^2 + \frac{\tau}{2}|u_{tt}(t) + c^2 b^{-1} u_t(t)|^2 + \frac{c^2}{2b}|\gamma(t)^{1/2} u_t(t)|^2$$

$$E_0(t) \equiv \frac{1}{2}|\alpha^{1/2} u_t(t)|^2 + \frac{c^2}{2}|\mathcal{A}^{1/2} u(t)|^2$$

$$\hat{E}(t) = E(t) + E_0(t)$$

Lemma 4.1. *Let $b > 0$ and $\alpha_0 > 0$*

- *If $\gamma > 0$, there exist $\omega > 0, C > 0$ such that $\hat{E}(t) = E(t) + E_0(t)$ satisfies:*

$$\hat{E}(t) \leq C e^{-\omega t} \hat{E}(0).$$

- *If $\gamma = 0$, the energy $E(t)$ remains constant.*

Remark 4.2. Recalling that $\gamma(x,t) \equiv [\alpha(x,t) - \frac{c^2\tau}{b}]$ we have that $E(t) \geq 0$ for $\gamma \geq 0$. Thus $\hat{E}(t)$ is equivalent in norm to the one induced by

$$|u_{tt}(t)|^2 + |\mathcal{A}^{1/2} u_t(t)|^2 + |\mathcal{A}^{1/2} u(t)|^2.$$

In fact, the evolution $(u(t), u_t(t), u_{tt}(t)) \equiv U(t,s)(u_0, u_1, u_2)$ defines a continuous flow on

$$H \equiv D(\mathcal{A}^{1/2}) \times D(\mathcal{A}^{1/2}) \times \mathcal{H}.$$

The proof of Lemma 4.1 follows through a sequence of auxiliary estimates. These are given below.

4.1 The energy dissipation

Lemma 4.3. *The following identity holds*

$$\frac{d}{dt}E(t) + |(\alpha - \frac{\tau c^2}{b})^{1/2} u_{tt}|^2 = \frac{c^2}{b}(\dot{\alpha} u_t, u_t) \tag{4.1}$$

Thus, when $\gamma(x,t) \geq \alpha_0 - \frac{c^2}{b} > 0$ the problem is dissipative with a strict dissipation when $\gamma > 0$. Instead, when $\gamma = 0$ the problem is conservative. This is to say:

- $E(t) + \gamma \int_0^t |u_{tt}|^2 ds = E(0), \gamma > 0$
- $E(t) = E(0), \gamma = 0.$

Proof. Since the energy calculations are justifiable for "smooth" solutions, we consider first regular solutions originating in the domains of respective generators. The existence of regular solutions is guaranteed by Evolution -Semigroup theory [16]. Owing to the fact that the final estimates will not depend on the additional smoothness, we will pass to the limit using density and obtaining final estimates for just finite energy solutions.

Step 1: Multiply (2.2) by u_{tt} and integrate by parts. This gives

$$\frac{d}{dt}[\tau|u_{tt}|^2 + b|\mathcal{A}^{1/2} u_t|^2 + 2c^2(\mathcal{A}u, u_t)]$$
$$+2|\alpha^{1/2} u_{tt}|^2 - 2c^2|\mathcal{A}^{1/2} u_t|^2 = 0. \tag{4.2}$$

Step 2: Multiply (2.2) by u_t and integrate by parts to obtain

$$\frac{d}{dt}[c^2|\mathcal{A}^{1/2}u|^2 + |\alpha^{1/2}u_t|^2 + 2\tau(u_{tt}, u_t)]$$
$$+2b|\mathcal{A}^{1/2}u_t|^2 - 2\tau|u_{tt}|^2 = 2|\dot{\alpha}^{1/2}u_t|^2. \tag{4.3}$$

Step 3. Multiply (4.3) by $\frac{c^2}{b}$ and add to (4.2). After some algebraic manipulations this yields:

$$\frac{d}{dt}[\tau|u_{tt}|^2 + b|\mathcal{A}^{1/2}u_t|^2 + 2c^2(\mathcal{A}u, u_t)$$
$$+\frac{c^2}{b}(c^2|\mathcal{A}^{1/2}u|^2 + |\alpha^{1/2}u_t|^2 + 2\tau(u_{tt}, u_t))]$$
$$+2|(\alpha(t) - \frac{\tau c^2}{b})^{1/2}u_{tt}|^2 = 2\frac{c^2}{b}(\dot{\alpha}u_t, u_t) \tag{4.4}$$

Using the notation introduced before we obtain the inequality claimed by the Lemma . □

4.2 Equipartition of the energy

Multiplying the original equation by u_t and integrating by parts leads to:

Lemma 4.4.

$$b|\mathcal{A}^{1/2}u_t|^2 = \tau|u_{tt}|^2 - \frac{d}{dt}[1/2|\alpha^{1/2}u_t|^2 + 1/2c^2|\mathcal{A}^{1/2}u|^2 + \tau(u_{tt}, u_t)] + (\dot{\alpha}u_t, u_t) \tag{4.5}$$

Our first goal is to establish the *boundedness of the total energy* $E(t) + E_0(t)$ where

$$E_0(t) \equiv 1/2\alpha|u_t(t)|^2 + 1/2c^2|A^{1/2}u(t)|^2.$$

Note that Lemma 4.4 implies

$$b|\mathcal{A}^{1/2}u_t|^2 = \tau|u_{tt}|^2 - \frac{d}{dt}E_0(t) - \frac{d}{dt}\tau(u_{tt}, u_t) + (\dot{\alpha}u_t, u_t) \tag{4.6}$$

We note that while $E(t)$ has already good a priori bounds, however these bounds do not determine topological norm $|\mathcal{A}^{1/2}u|$. It is the purpose of $E_0(t)$ to introduce this missing quantity. From Lemma 4.3 and Lemma 4.4 and (4.6) we conclude

$$\frac{d}{dt}E(t) + 1/2|\gamma^{1/2}u_{tt}|^2 + 1/2|(\gamma - c_0\tau)^{1/2}u_{tt}|^2 + 1/2c_0\tau|u_{tt}|^2 =$$
$$\frac{d}{dt}E(t) + 1/2|\gamma^{1/2}u_{tt}|^2 + 1/2|(\gamma - c_0\tau)^{1/2}u_{tt}|^2 + 1/2c_0[\frac{d}{dt}E_0(t) + b|\mathcal{A}^{1/2}u_t|^2$$
$$+\frac{d}{dt}\tau(u_{tt}, u_t)] = (\frac{c^2}{b} - \frac{c_0}{2})(\dot{\alpha}u_t, u_t) \tag{4.7}$$

Selecting suitably small constant c_0 so that

$$0 < c_0 \leq \frac{\alpha_0}{\tau} - \frac{c^2}{b} = \gamma_0\tau^{-1} \tag{4.8}$$

gives

$$\frac{d}{dt}E(t) + \frac{1}{2}c_0\frac{d}{dt}E_0(t) + 1/2|\gamma^{1/2}u_{tt}|^2 + 1/2c_0b|\mathcal{A}^{1/2}u_t|^2$$
$$\leq \frac{c_0\tau}{2}\frac{d}{dt}(u_{tt}, u_t) + (\frac{c^2}{b} - \frac{c_0}{2})(\dot{\alpha}u_t, u_t) \tag{4.9}$$

In particular, with C denoting a generic constant (independent of t), the following inequality holds for all $s < t$

Lemma 4.5. *Let* $\gamma(x,t) \geq \gamma_0 = \alpha_0 - \frac{c^2\tau}{b} > 0$*. Then the total energy is bounded for all times by the following expression:.*

$$E(t) + \frac{c_0}{2}E_0(t) + 1/2\int_s^t [|\gamma^{1/2}u_{tt}|^2 + bc_0|\mathcal{A}^{1/2}u_t|^2]dz$$
$$\leq E(s) + \frac{c_0}{2}E_0(s) + C_{\gamma,\tau,b,\alpha}[E(t) + E(s)] + |(\frac{c^2}{b} - \frac{c_0}{2})\int_0^t (\dot{\alpha}u_t, u_t)ds|$$
$$\leq C_{\gamma,b,\tau,\alpha}[E(s) + E_0(s)] + |(\frac{c^2}{b} - \frac{c_0}{2})\int_0^t (\dot{\alpha}u_t, u_t)ds| \tag{4.10}$$

This means that the total energy is bounded in time by the initial *total* energy. Our goal is to show that the energy is exponentially decaying provided $\gamma > 0$. In view of Lemma 4.5

4.3 Reconstruction of the potential energy $\int_0^T |\mathcal{A}^{1/2}u|^2 dt$ and decay rates for the energy.

In order to derive exponential decays one needs to reconstruct the integral of $\int_0^T |\mathcal{A}^{1/2}u|^2 dt$. For this we apply the multiplier u which then leads to

$$\frac{b}{2}\frac{d}{dt}|\mathcal{A}^{1/2}u|^2 + c^2|\mathcal{A}^{1/2}u|^2 = \alpha|u_t|^2$$
$$+\frac{d}{dt}[\frac{\tau}{2}|u_t|^2 - \tau(u_{tt},u) - (\alpha u_t,u)] + (\dot\alpha u_t,u) \tag{4.11}$$

Integrating in time (4.11) gives

$$\begin{aligned}
\frac{b}{2}|\mathcal{A}^{1/2}u(t)|^2 + c^2\int_s^t |\mathcal{A}^{1/2}u|^2 &= \frac{b}{2}|\mathcal{A}^{1/2}u(s)|^2 + \alpha\int_s^t |u_t|^2 \\
&+[\frac{\tau}{2}|u_t|^2 - \tau(u_{tt},u) - \alpha(u_t,u)]|_s^t + \int_s^t (\dot\alpha u_t,u)dz \\
\leq \int_s^t |\alpha^{1/2}u_t|^2 + \int_s^t (\dot\alpha u_t,u)dz &+ CE(t) + CE_0(t) + CE(s) + CE_0(s) \\
&\text{by Lemma 4.5} \\
\leq C_{\alpha_1}[E(t)+E_0(t)+E(s)+E_0(s)] + \int_s^t(\dot\alpha u_t,u) &+ \frac{2\alpha_1}{bc_0}(\frac{c^2}{b}-\frac{c_0}{2})\int_s^t(\dot\alpha u_t,u_t) \\
&\leq C[E(0)+E_0(0)] + C_{\alpha_1,\alpha_0,\tau,b}\int_s^t |(\dot\alpha u_t, u_t+u)| \\
\leq C\hat{E}(0) + C_{\alpha_1,\alpha_0,\tau,b}&\int_s^t |(\mathcal{A}^{-1/2}\dot\alpha\mathcal{A}^{-1/2}\mathcal{A}^{1/2}u_t, \mathcal{A}^{1/2}(u_t+u))|
\end{aligned} \tag{4.12}$$

Combining (4.12) with previous lemmas yields:

$$1/2\int_s^t [|\gamma^{1/2}u_{tt}|^2 + b\tau^{-1}|\mathcal{A}^{1/2}u_t|^2]dz + c^2\int_s^t |\mathcal{A}^{1/2}u|^2$$
$$\leq C[E(0)+E_0(0)] + C|\dot\alpha|_{L_\infty(R,L(D(\mathcal{A}^{1/2}\to[D(\mathcal{A}^{1/2})]'}\int_s^t |\mathcal{A}^{1/2}|u_t|^2| + |\mathcal{A}^{1/2}u|^2 dz$$

which after accounting for the Assumption 2.1 implies

$$\int_0^T [E(t)+E_0(t)] \leq C_{\gamma,b,\alpha,\tau,c}[E(0)+E_0] < \infty$$

From here one also obtains (with sufficiently small $\dot\alpha$) and large enough T that

$$T\tilde{E}(T) \leq \tilde\rho\tilde{E}(0), \ \ \rho < 1$$

which then proves exponential decay of the energy on the strength of Pazy-Datko's Theorem [16]. This completes the proof of the first part of Theorem 2.2

4.4 Decay rates for the higher energy -proof of the second part of Theorem 2.2

We built upon decays of the low energy. The correct multiplier exhibiting higher energy is $\mathcal{A}u$. We thus multiply equation by $\mathcal{A}u$ and integrate by parts. This gives

$$\begin{aligned}
(u_{ttt},\mathcal{A}u) &= -\frac{1}{2}\frac{d}{dt}|\mathcal{A}^{1/2}u_t|^2 + \frac{d}{dt}(u_{tt},\mathcal{A}u) \\
b(\mathcal{A}u_t,\mathcal{A}u) &= b\frac{1}{2}\frac{d}{dt}|\mathcal{A}^{1/2}u_t|^2 \\
(\alpha u_{tt},\mathcal{A}u) &= \frac{d}{dt}(\alpha u_t,\mathcal{A}u) - (\dot\alpha u_t,\mathcal{A}u) - (\alpha u_t,\mathcal{A}u_t) = \\
= \frac{d}{dt}(\alpha u_t,\mathcal{A}u) &- (\dot\alpha\mathcal{A}^{-1/2}\mathcal{A}^{1/2}u_t,\mathcal{A}u) - (\mathcal{A}^{1/2}(\alpha u_t),\mathcal{A}^{1/2}u_t)
\end{aligned} \tag{4.13}$$

Combining the above leads to the inequality

$$\frac{1}{2}\frac{d}{dt}[b|\mathcal{A}u|^2 - |\mathcal{A}^{1/2}u_t|^2] + c^2|\mathcal{A}u|^2 = -\frac{d}{dt}(u_{tt}, \mathcal{A}u) - (\mathcal{A}^{1/2}\alpha u_t, \mathcal{A}^{1/2}u_t)$$
$$+(\dot{\alpha}\mathcal{A}^{-1/2}\mathcal{A}^{1/2}u_t, \mathcal{A}u)$$

Thus we obtain:

$$\mathcal{E}_1(T) + \int_0^T \mathcal{E}_1(s)ds \leq C\mathcal{E}_1(0) + M_\alpha \int_0^T \mathcal{E}_0(s)ds$$

where M_α denotes the norm in multiplier space

$$\begin{aligned} u \to \alpha u &: \mathcal{D}(\mathcal{A}^{1/2} \to L_\infty(R, \mathcal{D}(\mathcal{A}^{1/2}) \\ u \to \dot{\alpha} u &: \mathcal{D}(\mathcal{A}^{1/2} \to L_\infty(R, \mathcal{H}) \end{aligned} \tag{4.14}$$

By applying Lower Energy Theorem we obtain that there exist $T > 0$ such that

$$\mathcal{E}_1(T) \leq \rho_1\mathcal{E}_1(0), \rho_1 < 1$$

Since the dynamics is invariant with the respect to $\mathcal{E}_1$ norm (generation of evolution), we conclude exponential decays under the additional assumption that the norms in (4.14) are bounded.

Acknowledgment

The first author wishes to thank Pedro Jordan, US Naval Research Laboratory, Stennis Space Center for stimulating discussion on nonlinear acoustic as well as for providing her with topical literature on modelling.

References

[1] F. Coulouvrat, *On the equations of nonlinear acoustics*, Journal de Acoustique 5 (1992), 321âĂŞ359.

[2] D.G. Crighton, *Model equations of nonlinear acoustics*, Ann.Rev.Fluid Mech. 11 (1979), 11-33.

[3] H. Fattorini. The Cauchy Problem, Encyclopedia of Mathematics, Addison Wesley , 1983.

[4] M. F. Hamilton, D. T. Blackstock, Nonlinear Acoustics, Academic Press, New York, 1997.

[5] P.M. Jordan, *An analytical study of Kuznetsov's equation: diffusive solitons, shock formation, and solution bifurcation*, Physics Letters A, vol. 326, pp. 77-84, 2004.

[6] P.M. Jordan, *Nonlinear acoustic phenomena in viscous thermally relaxing fluids: Shock bifurcation and the emergence of diffusive solitons. (A)* (Lecture). The 9th International Conf. on Theoretical and Computational Acoustics (ICTCA 2009). Dresden, Germany, 9 September 2009. see also J. Acoust. Soc. Am., vol 124, issue 4, pp. 2491-2491, 2008.

[7] B. Kaltenbacher and I. Lasiecka, *Global existence and exponential decay rates for the Westervelt equation*, Discrete and Continuous Dynamical Systems Series S, vol 2, pp. 503-525, 2009.

[8] B. Kaltenbacher and I. Lasiecka, *An analysis of nonhomogeneous Kuznetsov's equation: Local and global well-posedness; exponential decay*, to appear (Editors Choice) *Mathematische Nachrichten*, 2011.

[9] B. Kaltenbacher and I. Lasiecka, *Well-posedness of the Westervelt and the Kuznetsov equation with nonhomogeneous Neumann boundary conditions*, to appear Proceedings of AIMS Conference in DResden, *DCDS* 2011.

[10] B. Kaltenbacher, I. Lasiecka, S. Veljović, *Well-posedness and exponential decay for the Westervelt equation with inhomogeneous Dirichlet boundary data*, to appear in: J. Escher et al (Eds): *Parabolic Problems: Herbert Amann Festschrift*, Birkhaeuser, Basel 2011.

[11] B. Kaltenbacher and I. Lasiecka and R. Marchand. Wellposedness and exponential decay rates for the Moore-Gibson-Thompson equationn arising in high intensity ultrasound. To appear in the invited volume dedicated to F. Troeltsch , *Control and Cybernetics*.

[12] S. Makarov, M. Ochmann, *Nonlinear and Thermoviscous Phenomena in Acoustics* Part. II. Acta Acustica united with Acustica 83 (1997), 197-222.

[13] M. Kaltenbacher, *Numerical Simulations of Mechatronic Sensors and Actuators*, Springer, Berlin, 2004.

[14] V. P. Kuznetsov, *Equations of nonlinear acoustics*, Sov. Phys. Acoust., vol. 16, pp. 467âĂŞ470, 1971.

[15] F. K. Moore and W. E. Gibson, *Propagation of weak disturbances in a gas subject to relaxation effects*, Journal of Aerospace Science and Technologies 27 (1960), 117-127.

[16] A. Pazy, Semigroups of Operators and Applications to Partial Differential Equations, Springer, New York, 1983.

[17] P. A. Thompson, Compressible-fluid Dynamics, McGraw-Hill, New York, 1972.

[18] S. Tjøtta *Higher order model equations in nonlinear acoustics*, Acta Acustica united with Acustica 87 (2001), 316-321.

[19] A.Rozanova, *The Khokhlov-Zabolotskaya-Kuznetsov equation*, Comptes Rendus Mathematique 344 (2007), 337-342.

[20] P.J. Westervelt, *Parametric acoustic array*, The Journal of the Acoustic Society of America, vol. 35, pp 535–537, 1963.

Author information

Barbara Kaltenbacher, University of Graz, Graz, Austria.
E-mail: barbara.kaltenbacher@uni-graz.au

Irena Lasiecka, Department of Mathematics , University of Virginia and Department of Mathematics and Statistics, KFUPM, Dhahran, USA and KSA.
E-mail: il2v@virginia.edu

Received: July 31, 2011
Accepted: October 18, 2011