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On Nearly uniformly convex metric spaces

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1 Abstract

In this work, we define the nearly uniform convexity and the Δ-uniform convexity in metric spaces, and prove their equivalence. We also prove the nonlinear version of some classical results related to nearly uniformly convex metric spaces.

I dedicate this paper to Ghiyath al-Din Jamshid Mas'ud al-Kashi (or al-Kashani) (c. 1380 Kashan - 22 June 1429 Samarkand) who was an astronomer and one of the best mathematicians in the Islamic world.

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2 Introduction and Preliminaries

One of the basic concepts of geometry of Banach spaces is that of uniform convexity. It was introduced by J. A. Clarkson [1] in 1936. Five years later V. L. Smulian [18] characterized the property dual to uniform convexity. It is called uniform smoothness. The infinite dimensional counterparts of uniform convexity and uniform smoothness were studied in [14]. Since then these two notions have been extensively studied. They turned out to be useful for instance in the metric fixed point theory (see [3, 10]). It is pointed out that uniform convexity and uniform smoothness describe geometric properties of finite dimensional subspaces of a normed space. However, most of the problems in the fixed point theory have global character. This was a motivation for considering infinite dimensional counterparts of the classical geometric notions. One of them is nearly uniform convexity introduced by R. Huff in [7]. Independently, a similar idea appeared in [5], where the modulus corresponding to nearly uniform convexity was defined.

Extension to metric spaces of this kind of linear properties was sought very early on. The early attempts were based on the use of Menger convexity (see for example [4, 6, 12]). Recently, Khamsi and Khan [9] have used the classical definition of uniform convexity in metric spaces to obtain an analogue of the Hilbert parallelogram identity and another well-known inequality and have cited a number of applications of the new inequalities obtained.

In this work, we use Penot's formulation of metric convexity based on the notion of convexity structures [16]. It is worth to mention that Penot's use of convexity structures was successful in obtaining some positive fixed point results in metric spaces [8], similar to the classical Kirk's fixed point result [13]. In particular we extend the notions of nearly uniform convexity (U.N.C) and Δ-uniform convexity (Δ-U.C.) to metric spaces. We introduce these properties and prove their equivalence.

Throughout this note X denotes a metric space and $B(x, r)$ will be the closed ball centered at x with radius r.

Definition 2.1. A nonempty family $\mathcal{F}$ of subsets of X is said to define a convexity structure if and only if $\mathcal{F}$ is stable by intersection. The elements of $\mathcal{F}$ are called convex.

We will always assume that the closed balls and X are convex. In other words, the closed balls and X are in $\mathcal{F}$. Let $A \subset X$, set

$$co(A) = \bigcap \{C; C \in \mathcal{F}; A \subset C\}.$$

Next we define N.U.C. in metric spaces.

Definition 2.2. Let $x \in X, r > 0$ and $\varepsilon > 0$, define

$$\delta_N(\varepsilon) = \sup\{\eta > 0; B(x, (1-\eta)r) \cap co(x_n) \neq \emptyset\}$$

for any $\{x_n\} \subset X$ such that $x_n \in B(x, r)$ and the separation constant

$$sep(x_n) = \inf_{n \neq m} d(x_n, x_m) \geq r\,\varepsilon.$$

Set $\varepsilon_N(X) = \sup\{\varepsilon > 0; \delta_N(\varepsilon) = 0\}$. We say that X is $\mathcal{F}$-N.U.C if and only if $\varepsilon_N(X) = 0$.

In this definition we scale the N.U.C. property in a way which allows a classification of metric spaces and specially Banach spaces from their geometrical structure point of view.

In the next definition we introduce the Δ-U.C. in metric spaces. Recall that the Δ-U.C. was initiated by Goebel and Sekowski [5] in Banach spaces. They proved that Δ-U.C. property hold in finite dimensional spaces as well as in uniformly convex Banach spaces. The main interesting result concerns the link between reflexivity and normal structure with Δ-U.C.

Let us recall the definitions of measures of noncompactness α and χ in a metric space X:

$$\alpha(A) = \inf\{r > 0; A \subset \bigcup_{1\leq i\leq n} A_i \ \ with \ \ diam(A_i) \leq r\}$$

and $\chi(A) = \inf\{r > 0; A \subset \bigcup_{1\leq i\leq n} B_i\}$, where the B_i are closed balls.

Note that α and χ are defined for every bounded subset A of X.

Definition 2.3. Let $x \in X, r > 0$ and $\varepsilon > 0$, set

$$\Delta_\alpha(\varepsilon) = \sup\{\eta > 0; B(x, (1-\eta)r) \cap A \neq \emptyset\}$$

for every $A \in \mathcal{F}$ such that $A \subset B(x, r)$ and $\alpha(A) \geq r\varepsilon$. Put

$$\varepsilon_\alpha(X) = \sup\{\varepsilon > 0; \Delta_\alpha(\varepsilon) = 0\}.$$

Define $\Delta_\chi(\varepsilon)$ and $\varepsilon_\chi(X)$ with the same formulas except that one has to replace α by χ. We will say that X is $\mathcal{F} - \Delta$-U.C. if and only if $\varepsilon_\alpha(X) = 0$.

Another measures of noncompactness which will be useful in our work is the separation measure of noncompactness [2].

Definition 2.4. Let X be a metric space. For any bounded nonempty subset A of X, we define the separation constant $\beta(A)$ by

$$\beta(A) = \sup\{sep(x_n);\ \{x_n\} \subset A\}.$$

The function $\beta(.)$ is also known as the separation measure of noncompactness.

Clearly, the following properties are satisfied by $\beta(.)$ (see [2]):

(i) For any bounded nonempty subset A of X, we have $\beta(A) \leq \delta(A)$, where $\delta(A)$ is the diameter of A.

(ii) If $A \subset B$, then $\beta(A) \subset \beta(B)$.

(iii) For any bounded nonempty subset A of X, we have $\beta(A) = \beta(\overline{A})$, where $\overline{A}$ is the closure of A.

(iv) For any bounded nonempty subset A of X, we have $\beta(A) = \beta(A \setminus F)$, where F is any finite subset of A.

(v) For any bounded nonempty subset A of X, we have $\beta(A) = 0$ if and only if A is totally bounded. Recall that A is totally bounded if and only if for any $\varepsilon > 0$, there exists $a_1, \cdots, a_k$ in A such that $A \subset \bigcup_{1\leq i\leq k} B(a_i, \varepsilon)$, where $B(a_i, \varepsilon)$ is either closed or open ball.

(vi) Assume X is complete. For any bounded nonempty subset A of X, we have $\beta(A) = 0$ if and only if $\overline{A}$ is compact. This result explains why β is a measure of noncompactness.

(vii) Assume X is compete. Let $\{x_n\}$ be a bounded sequence in X with no convergent subsequence. Then $\beta(\{x_n;\ n \geq 1\}) > 0$. In particular, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $sep(x_{n_k}) > 0$.

The following technical result will be needed in the next section.

Lemma 2.5. *Let X be a complete metric space. If $\{C_n\}$ is a decreasing bounded sequence of nonempty closed subsets with $\inf_{n\geq 1} \beta(C_n) = 0$, then $\bigcap_{n\geq 1} C_n$ is a nonempty compact subset of X.*

Proof. Pick $x_n \in C_n$, for any $n \geq 1$. Set $C = \{x_n;\ n \geq 1\}$. We have

$$\beta(C) = \beta\Big(\{x_m;\ m \geq n\}\Big) \leq \beta(C_n),$$

for any $n \geq 1$. This will imply $\beta(C) = 0$. Hence $\overline{C}$ is compact. Therefore $\{x_n\}$ has a convergent subsequence. Clearly its limit belongs to C_n, for all $n \geq 1$. Therefore $\bigcap_{n\geq 1} C_n$ is not empty. The compactness follows from the properties of $\beta(.)$. □

3 Nearly Uniformly Convex Metric Spaces are Reflexive

In this section we show an analogous result to the famous linear theorem which states that N.U.C. Banach spaces are reflexive. Penot [16] gave a definition of reflexivity for convexity structures as follows.

Definition 3.1. A convexity structure $\mathcal{F}$ is said to be compact if and only if any family of closed convex subsets $(C_\alpha)_{\alpha\in\Gamma} \subset \mathcal{F}$ has a nonempty intersection provided any finite subfamily of $(C_\alpha)_{\alpha\in\Gamma}$ has a nonempty intersection.

Next we show that if X is $\mathcal{F}$-N.U.C., then $\mathcal{F}$ is compact. First let us prove some needed technical lemmas.

Lemma 3.2. *Let X be a metric space and $\mathcal{F}$ be a convexity structure in X. Assume that X is $\mathcal{F}$-N.U.C. Then any closed convex nonempty $C \in \mathcal{F}$ is proximinal; i.e. for any $x \in X$, we have*

$$P_C(x) = \{c \in C;\ d(x,c) = d(x,C) = \inf_{a\in C} d(x,a)\} \neq \emptyset.$$

Moreover the set $P_C(x)$ is totally bounded and convex. In particular, if X is complete, then $P_C(x)$ is compact.

Proof. Without loss of any generality we may assume $d(x,C) > 0$. Then there exists $\{c_n\} \subset C$ such that $\lim_{n\to\infty} d(x,c_n) = d(x,C)$. If $\{c_n\}$ has a convergent subsequence, then its limit will belong to $P_C(x)$. Otherwise let us assume that $\{c_n\}$ has no convergent subsequence. In this case there exists a subsequence $\{c_{n_k}\}$ such that $sep(c_{n_k}) > 0$. Without loss of any generality, we may assume $sep(c_n) > 0$. Since X is $\mathcal{F}$-N.U.C., so there exists $\eta > 0$ such that

$$B\Big(x, (1-\eta)\sup_{m\geq n} d(x,c_m)\Big) \cap co(\{c_m;\ m \geq n\}) \neq \emptyset,$$

for any $n \geq 1$. Since $co(\{c_m;\ m \geq n\}) \subset C$, we get

$$d(x,C) \leq (1-\eta)\sup_{m\geq n} d(x,c_m)$$

for any $n \geq 1$. If we let $n \to \infty$, we get

$$d(x,C) \leq (1-\eta)d(x,C).$$

This will contradict the fact $d(x,C) > 0$. Therefore $P_C(x)$ is not empty. In fact, the above proof shows that $\beta(P_C(x)) = 0$ which implies that $P_C(x)$ is totally bounded. Moreover, it is easy to see that $P_C(x)$ is a closed subset of X. Since $\mathcal{F}$ contains closed balls, and $P_C(x) = C \cap B(x, d(x,C))$, therefore $P_C(x) \in \mathcal{F}$. If X is complete, then $P_C(x)$ is also complete. Hence $P_C(x)$ being complete and totally bounded is compact. □

In the linear case, a Banach space is reflexive if and only if any decreasing sequence of nonempty bounded closed convex subsets has a nonempty intersection. For convexity structures, it is not clear that this conclusion is equivalent to compactness. In fact, for more on this question, we refer to [11].

Lemma 3.3. *Let X be a complete metric space and $\mathcal{F}$ be a convexity structure in X. Assume that X is $\mathcal{F}$-N.U.C. Then any decreasing bounded sequence of nonempty closed convex subsets $\{C_n\} \subset \mathcal{F}$ has a nonempty intersection.*

Proof. Let $x \in C_1$. Then for any $n \geq 2$, choose $c_n \in P_{C_n}(x)$. Assume that $\{c_n\}$ has no convergent subsequence. In this case there exists a subsequence $\{c_{n_k}\}$ such that $sep(c_{n_k}) > 0$. Without loss of any generality, we may assume $sep(c_n) > 0$. As X is $\mathcal{F}$-N.U.C., so there exists $\eta > 0$ such that

$$B\Big(x, (1-\eta)\sup_{m\geq n} d(x,c_m)\Big) \cap co(\{c_m;\ m \geq n\}) \neq \emptyset,$$

for any $n \geq 2$. Since $\{d(x,C_n)\} = \{d(x,c_n)\}$ is an increasing sequence and bounded (since C_1 is bounded), therefore it converges. Note that $\lim_{n\to\infty} d(x,c_n) = R > 0$ because $\{c_n\}$ has no convergent subsequence. Since $co(\{c_m;\ m \geq n\}) \subset C_n$, we get

$$d(x,c_n) = d(x,C_n) \leq (1-\eta)\sup_{m\geq n} d(x,c_m)$$

for any $n \geq 1$. If we let $n \to \infty$, we get

$$R \leq (1-\eta)R.$$

This will contradict the fact $R > 0$. Therefore $\{c_n\}$ has a convergent subsequence and its limit will belong to C_n, for any $n \geq 1$. This implies that $\{C_n\}$ has a nonempty intersection. □

Now we prove the main result of this section.

Theorem 3.4. *Let X be a complete metric space and $\mathcal{F}$ be a convexity structure in X. Assume that X is $\mathcal{F}$-N.U.C. Then $\mathcal{F}$ is compact.*

Proof. Let $\{C_\alpha\}_{\alpha\in\Gamma} \subset \mathcal{F}$ be a nonincreasing family of nonempty, convex, closed bounded subsets, where Γ is a directed index set. Let us prove that $\bigcap_{\alpha\in\Gamma} C_\alpha \neq \emptyset$. Recall that Γ is directed if there exists an order $\preceq$ defined on Γ such that for any $\alpha, \beta \in \Gamma$, there exists $\gamma \in \Gamma$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$. $\{C_\alpha\}_{\alpha\in\Gamma}$ is nonincreasing if and only if for any $\alpha, \beta \in \Gamma$ with $\alpha \preceq \beta$, then $C_\beta \subset C_\alpha$. Note that for any $\alpha_0 \in \Gamma$, we have

$$\bigcap_{\alpha\in\Gamma} C_\alpha = \bigcap_{\alpha_0\preceq\alpha} C_\alpha.$$

Therefore without loss of any generality we may assume that there exists $C \in \mathcal{F}$ which is nonempty closed bounded convex subset and $C_\alpha \subset C$ for any $\alpha \in \Gamma$. Choose $x \in C$. Set $R = \sup_{\alpha\in\Gamma} d(x, C_\alpha)$. Without loss of any generality we may assume $R > 0$. Otherwise $x \in C_\alpha$ for any $\alpha \in \Gamma$. For any $\alpha \in \Gamma$ pick $c_\alpha \in P_{C_\alpha}(x)$. Set $A_\alpha = \{c_\gamma;\ \alpha \preceq \gamma\}$. Let us prove that $\inf_{\alpha\in\Gamma} \beta(A_\alpha) = 0$. Assume not. Then there exists $\varepsilon_0 < \beta(A_\alpha)$, for any $\alpha \in \Gamma$. Then for any $\alpha \in \Gamma$, there exists a sequence $\{c_{\beta_n}\} \in A_\alpha$ such that $sep(c_{\beta_n}) \geq \varepsilon_0$. Since X is $\mathcal{F}$-N.U.C., there exists $\eta > 0$ such that

$$B\Big(x, (1-\eta)R\Big) \cap co(\{c_{\beta_n};\ n \geq 1\}) \neq \emptyset.$$

Since $co(\{c_{\beta_n};\ n \geq 1\}) \subset C_\alpha$, we get

$$d(x, C_\alpha) \leq R(1-\eta),$$

for any $\alpha \in \Gamma$. This clearly implies $R \leq R(1-\eta)$ which is a contradiction to the assumption $R > 0$. Therefore we have $\inf_{\alpha\in\Gamma} \beta(A_\alpha) = 0$. Hence there exists a sequence $\{\alpha_n\} \subset \Gamma$ such that $\lim_{n\to\infty} \beta(A_{\alpha_n}) = 0$. Since Γ is a directed set, we may assume that $\alpha_n \preceq \alpha_{n+1}$. Hence $\{A_{\alpha_n}\}$ is a decreasing sequence of nonempty sets of X such that $\inf_{n\geq 1} \beta(A_{\alpha_n}) = 0$. By Lemma 2.5, $A = \bigcap_{n\geq 1} \overline{A_{\beta_n}}$ is a nonempty compact subset of X. Let $\alpha \in \Gamma$. Then there exists $\gamma_n \in \Gamma$ such that $\alpha \preceq \gamma_n$ and $\alpha_n \preceq \gamma_n$. Hence $A_{\gamma_n} \subset A_\alpha \cap A_{\alpha_n}$, which implies $\overline{A_\alpha} \cap \overline{A_{\alpha_n}}$ is not empty. Again using Lemma 2.5, we conclude that $\bigcap_{n\geq 1} \Big(\overline{A_\alpha} \cap \overline{A_{\alpha_n}}\Big)$ is not empty. In other words, we have $\overline{A_\alpha} \cap A$ is not empty for any $\alpha \in \Gamma$. Since A is compact and the family $\{A_\alpha\}$ is downwards directed, we conclude that

$$\bigcap_{\alpha\in\Gamma} \Big(A \cap \overline{A_\alpha}\Big) \neq \emptyset.$$

Since $\overline{A_\alpha} \subset C_\alpha$ for any $\alpha \in \Gamma$, we conclude that the family $\{C_\alpha\}$ has a nonempty intersection. □

4 Main result

In the following result we compare the modulus of Δ-U.C. and the modulus of N.U.C.

Theorem 4.1. *Let X be a metric space and $\mathcal{F}$ be convexity structure in X. For any $x \in X, r > 0$ and $\varepsilon > 0$, we have*

(a) $\Delta_\chi\left(\frac{\varepsilon}{2}\right) \leq \delta_N(\varepsilon) \leq \Delta_\chi(\varepsilon)$,

(b) $\delta_N\left(\frac{\varepsilon}{2}\right) \leq \Delta_\alpha(\varepsilon) \leq \delta_N(\varepsilon)$.

In particular, we have $\varepsilon_\alpha(X) = 0$ if and only if $\varepsilon_\chi(X) = 0$.

Proof. First notice that if $A \in \mathcal{F}, A \neq \emptyset$ and $\alpha(A) > r\varepsilon$, so A contains a sequence $\{a_n\}$ such that $sep(a_n) \geq r\frac{\varepsilon}{2}$. If $\chi(A) > r\varepsilon$, then A contains a sequence $\{a_n\}$ such that $sep(a_n) \geq r\varepsilon$. Conversely, assume that $\{a_n\} \subset X$ and $sep(a_n) > r\varepsilon$. Then $\alpha(co(a_n)) \geq r\varepsilon$ and $\chi(co(a_n)) \geq r\frac{\varepsilon}{2}$.
Let us show how (a) holds. (The proof of (b) will follow using the same argument).
Let $x \in X$ and $A \subset X, A \in \mathcal{F}$ such that $\chi(A) \geq r\varepsilon$ and $A \subset B(x, r)$. Then one can find $\{a_n\} \subset A$ such that $sep(a_n) \geq r\varepsilon$. Since $\{a_n\} \subset B(x, r)$, so by using the definition of δ_N we get

$$B(x, (1-\delta_N(\varepsilon)r) \cap co(a_n) \neq \emptyset.$$

Therefore $B(x, (1-\delta_N(\varepsilon)r) \cap A \neq \emptyset$ because $co(a_n) \subset A$. Using the definition of Δ_χ we obtain

$$\delta_N(\varepsilon) \leq \Delta_\chi(\varepsilon).$$

Conversely, let $\{a_n\} \subset X$ be such that $sep(a_n) \geq r\varepsilon$, for $r > 0$ and $\varepsilon > 0$. Assume that $\{a_n\} \subset B(x,r)$ for $x \in X$. Then $co(a_n) \subset B(x,r)$ and $\chi(co(a_n)) \geq r\frac{\varepsilon}{2}$ since $sep(a_n) \geq r\varepsilon$. By definition of Δ_χ we deduce that

$$co(a_n) \cap B\left(x, \left(1 - \Delta_\chi\left(\frac{\varepsilon}{2}\right) r\right)\right) \neq \emptyset.$$

This yields

$$\Delta_\chi\left(\frac{\varepsilon}{2}\right) \leq \delta_N(\varepsilon).$$

Hence the proof of Theorem 4.1 is complete.

□

As a direct corollary of Theorem 4.1, we get the following result.

Corollary 4.2. *Let X be a metric space and $\mathcal{F}$ be a convexity structure. The following statements are equivalent.*

(i) X is N.U.C. with respect to $\mathcal{F}$,

(ii) X is Δ-U.C. with respect to $\mathcal{F}$.

Translating this result in normed linear spaces, we get the following theorem.

Theorem 4.3. *Let X be a Banach space. Let $\mathcal{F}$ be the family of convex subsets of X. Then, the following statements are equivalent.*

(i) X is nearly uniformly convex,

(ii) X is Δ-uniformly convex (in the sense of Goebel and Sekowski [5]).

Note that Theorem 4.3 was known to Montesinos [15] and Prus [17].

The main result proved by Goebel and Sekowski [5] concerns Banach spaces with $\varepsilon_\alpha(X) < 1$.
In the next result we compare the three coefficients introduced in our definitions.

Corollary 4.4. *In a metric space X, we have:*

(i) if $\varepsilon_\chi(X) < 2$, then $\varepsilon_N(X) < 1$,

(ii) if $\varepsilon_\alpha(X) < 1$, then $\varepsilon_N(X) < 1$.

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