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# ON SPECTRAL THEORY IN EFFECT ALGEBRAS

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## 1 Abstract

If one supposes a quantum logic $L$ to be a $\sigma$-effect algebra, then the observables on $L$ are identified with the $L$-valued measures defined on the Borel subsets of the real line. In this structure (and without the aid of Hilbert space formalism) we show that (1) the spectrum of an observable $A$ can be completely characterized by studying the observable $(A-\lambda)^{-1}$, and (2) corresponding to every observable $A$ there is a spectral resolution uniquely determined by $A$.

## 2 Introduction

In the last two decades the notion of effect algebra has received much attention within the studies on the mathematical foundation of quantum mechanics [2, 7, 8, 10]. Effect algebras appear to be the natural outcome in the search of a mathematical structure that captures the fundamental aspects of the elementary two-valued physical quantities, or effects, pertaining to a physical system. The notion of an effect algebra is sufficiently general to encompass the traditional order structure accompanying classical systems (Boolean algebras), and it is sufficiently structured to carry a meaningful interplay with the physically relevant notions of states and of observables [2, 7, 8, 10].

Until quite recently the observables in non-relativistic quantum mechanics have been identified with the set of self-adjoint operators on a separable, infinite dimensional, complex Hilbert space. Likewise, on the same Hilbert space, the states have been identified with the trace operators of trace class 1 [6]. However, with the advent of Mackey's book on the mathematical foundation of quantum mechanics [16], both observables and states have assumed a more abstract character having no overt connection with Hilbert space. This has led some investigators to consider the problem of deciding which quantum mechanical results are essentially consequences of Hilbert space formalism and which can be obtained without involving Hilbert space [11, 21, 22]. In this paper we will show that most of the desirable theorems involving spectra can be obtained without the use of Hilbert space. Indeed, we shall study the relationship between the notions of observables and the so-called spectral resolutions on an effect algebra. In 1968, D. Catlin [6] studied this relationship on an orthomodular poset (OMP). Our work mimics Catlin's work, and our results extend the results of Catlin to the more general setting of an effect algebra, a generalization of an OMP. Some related results concerning spectral resolutions for $\sigma$-complete lattice effect algebras and a spectral theorem for $\sigma$-MV-algebras can be found in [18, 19].

## 3 Basic Definitions

In this section we summarize the basic definitions concerning $\sigma$-effect algebras [10], $\sigma$-orthoalgebras [13], and $\sigma$-OMPs [3, 5, 6, 16, 17], and we present some of the basic and important facts that link these structures together. We further give the definitions of states and observables on such structures, and we define the spectrum of an observable. An *effect algebra* is a system $(E, 0, 1, \oplus)$ consisting of a set $E$ containing two special elements 0, 1 and equipped with a partially defined binary operation $\oplus$ satisfying the following conditions $\forall\, a, b, c \in E$:

(EA1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(EA2) If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ is defined, $(a \oplus b) \oplus c$ is defined, and $a \oplus (b \oplus c)$= $(a \oplus b) \oplus c$.

(EA3) For every $a \in E$ there exists a unique element $a' \in E$ such that $a \oplus a'$ is defined and $a \oplus a' = 1$. The unique element $a'$ is called the *orthosupplement* of $a$.

(EA4) If $1 \oplus a$ is defined, then $a = 0$.

We shall write $E$ for the effect algebra $(E, 0, 1, \oplus)$ if there is no danger of misunderstanding. Let $E$ be an effect algebra and $a, b \in E$. Following [6], we say that $a$ is *orthogonal* to $b$ in $E$ and write $a \perp b$ if and only if $a \oplus b$ is defined in $L$. We define $a \leq b$ to mean that there exists $c \in E$ such that $a \perp c$ and $b = a \oplus c$. For any effect algebra $E$, it can be easily proved (see [6]) that $0 \leq a \leq 1$ holds for all $a \in E$, that $a \perp b$ iff $a \leq b'$, that, with $\leq$ as defined above, $(E, \leq, 0, 1)$ is a partially ordered set.

Recall that an *orthoalgebra* may be regarded as an effect algebra $E$ in which condition (EA4) is replaced by the stronger *consistency law*:

$$a \in E,\, a \perp a \quad \Rightarrow \quad a = 0.$$

An *orthomodular poset* (OMP) may be regarded as an effect algebra $E$ that satisfies the following additional condition (see [6]):

$$a, b \in E,\, a \perp b \quad \Rightarrow \quad a \vee b \text{ exists in } E \text{ and } a \vee b = a \oplus b.$$

An *orthomodular lattice* (OML) may be defined as an OMP $E$ in which $a \vee b$ (and hence $a \wedge b$) exists $\forall\, a, b \in E$. Let $L$ be an effect algebra. A subset $A \subseteq L$ is called a *subeffect algebra* if $0, 1 \in A$ and, whenever $a, b \in A$ and $a \perp b$, it follows that $a' \in A$ and $a \oplus b \in A$. A subeffect algebra $A$ of $L$ is called a *Boolean subalgebra* if it is a distributive suborthomodular lattice. An effect algebra $L$ is *sharp* if every element $a \in L$ is *sharp*; that is, $a \wedge a' = 0$. It can be easily checked that every sharp effect algebra $L$ is an orthoalgebra. Indeed, if $a \oplus a$ is defined in $L$, then $a \leq a'$ and $a \leq a$; hence $a \leq a \wedge a' = 0$, since $a$ is sharp. Therefore $a = 0$.

Throughout this paper, the symbol $\mathbb{R}$ denotes the set of all real numbers, and the notation $:=$ means "equals by definition".

**Example 3.1.** Consider the unit interval $[0, 1] \subseteq \mathbb{R}$, and for $a, b \in [0, 1]$, define $a \perp b$ if $a + b \leq 1$ in which case $a \oplus b := a + b$. It is easy to check that $([0, 1], 0, 1, \oplus)$ is an effect algebra.

The following example plays an important role for unsharp measurements of quantum mechanics [4, 10].

**Example 3.2.** Consider the set $\mathcal{E}(H)$ of all self-adjoint operators $A$ on a Hilbert space $H$ with $O \leq A \leq I$, where $O$ and $I$ are the zero and the identity operators, respectively, on $H$. For $A, B \in \mathcal{E}(H)$, define

$$A \oplus B := A + B \quad \text{iff} \quad A + B \leq I.$$

It is not difficult to show that, under this $\oplus$, the system $(\mathcal{E}(H), O, I, \oplus)$ forms an effect algebra [10].

Let $L$ be an effect algebra and let $F = \{a_1, \ldots, a_n\}$ be a finite sequence in $L$. Recursively, we define for $n \geq 3$

$$a_1 \oplus a_2 \oplus \ldots \oplus a_n := (a_1 \oplus \ldots \oplus a_{n-1}) \oplus a_n \tag{3.1}$$

supposing that $a_1 \oplus \ldots \oplus a_{n-1}$ and $(a_1 \oplus \ldots \oplus a_{n-1}) \oplus a_n$ exist in $L$. From the associativity of $\oplus$ in $L$, we conclude that (3.1) is correctly defined. By definition, we put $a_1 \oplus \ldots \oplus a_n = a_1$ if $n = 1$ and $a_1 \oplus \ldots \oplus a_n = 0$ if $n = 0$. Then for any permutation $(i_1, \ldots, i_n)$ of $\{1, 2, \ldots, n\}$ and any $k$ with $1 \leq k \leq n$, we have

$$a_1 \oplus a_2 \oplus \ldots \oplus a_n = a_{i_1} \oplus \ldots \oplus a_{i_n}, \tag{3.2}$$

$$a_1 \oplus a_2 \oplus \ldots \oplus a_n = (a_1 \oplus \ldots \oplus a_k) \oplus (a_{k+1} \oplus a_{k+2} \oplus \ldots \oplus a_n). \tag{3.3}$$

We say that a finite sequence $F = \{a_1, \ldots, a_n\}$ in $L$ is *$\oplus$-orthogonal* if $a_1 \oplus \ldots \oplus a_n$ exists in $L$. In this case we say that $F$ has a $\oplus$-sum, $\bigoplus_{i=1}^n a_i$, defined via

$$\bigoplus_{i=1}^{n} a_i = a_1 \oplus \ldots \oplus a_n. \tag{3.4}$$

It is clear that two elements $a, b$ of $L$ are orthogonal; i.e., $a \perp b$, iff $\{a, b\}$ is $\oplus$-orthogonal.

An arbitrary system $G = \{a_i\}_{i \in I}$ of not necessarily different elements of $L$ is *$\oplus$-orthogonal* iff, for every finite subset $F$ of $I$, the system $\{a_i\}_{i \in F}$ is $\oplus$-orthogonal. If $G = \{a_i\}_{i \in I}$ is $\oplus$-orthogonal, so is any $\{a_i\}_{i \in J}$ for any $J \subseteq I$. An $\oplus$-orthogonal $G = \{a_i\}_{i \in I}$ of $L$ has a $\oplus$-sum in $L$, written as $\bigoplus_{i \in I} a_i$, iff in $L$ there exists the join

$$\bigoplus_{i \in I} a_i := \bigvee_F \bigoplus_{i \in F} a_i \tag{3.5}$$

where $F$ runs over all finite subsets in $I$. In this case, we also write $\bigoplus G = \bigoplus_{i \in I} a_i$. It is evident that if $G = \{a_1, \ldots, a_n\}$ is $\oplus$-orthogonal, then the $\oplus$-sums defined by (3.4) and (3.5) coincide.

The proof of the following lemma is similar to that for orthoalgebras, and the latter can be found in [13].

**Lemma 3.3** *Let $L$ be an effect algebra and let $a, b, x, y \in L$ be such that $a \leq x$, $b \leq y$, $x \perp y$ and $a \perp b$. If $a \oplus b = x \oplus y$, then $a = x$ and $b = y$.*

We say that an effect algebra $L$ is *$\sigma$-orthocomplete* iff $\bigoplus_{i \in I} a_i$ belongs to $L$ for any countable system $\{a_i : i \in I\}$ of $\oplus$-orthogonal elements from $L$. A $\sigma$-orthocomplete effect algebra is called a *$\sigma$-effect algebra*. An OMP $L$ is called *$\sigma$-orthocomplete* iff every countable orthogonal subset of $L$ has a supremum in $L$, we also call $L$ a *$\sigma$-OMP*. The proof of the following theorem is similar to that for orthoalgebras, and the latter can be found in [13].

**Theorem 3.4** *An OMP $L$ is $\sigma$-orthocomplete iff it is $\sigma$-orthocomplete as an effect algebra.*

A subeffect algebra $M$ of an effect algebra $L$ is called a *$\sigma$-subeffect algebra* iff $\bigoplus_{i\in I} a_i$ belongs to $M$ for any countable system $\{a_i : i \in I\}$ of $\oplus$-orthogonal elements from $M$. The next lemma was proved in [19] for orthoalgebras. The same proof can be used for effect algebras.

**Lemma 3.5** *Let $L$ be an effect algebra, and let $M \subseteq L$ be such that*

*(i) $0 \in M$, and $a \in M \;\Rightarrow\; a' \in M$;*

*(ii) if $p, q, r$ are pairwise orthogonal elements of $M$, then $p \oplus q \oplus r$ exists in $L$ and belongs to $M$; and*

*(iii) if $(a_i)_{i\in\mathbb{N}} \subseteq M$ is an increasing sequence, then it has a supremum in $M$.*

*Then $M$ is a $\sigma$-subeffect algebra of $L$ which is a $\sigma$-OMP.*

For the remainder of this paper, unless otherwise stated, we will assume that $L$ is a $\sigma$-effect algebra. Let $\mathcal{B}$ denote the Borel subsets of the real line $\mathbb{R}$. By an *observable (or $L$-valued measure)* on $L$ we mean a mapping $A : \mathcal{B} \to L$ such that

(i) $A(\mathbb{R}) = 1$, and

(ii) $A(\bigcup_{i=1}^{\infty} E_i) = \bigoplus_{i=1}^{\infty} A(E_i)$ whenever $E_i \in \mathcal{B}$ and $E_i \cap E_j = \emptyset$ for $i \neq j$.

We denote the set of all observables on $L$ by $\mathcal{O}$.

By a *state* on an effect algebra $L$ we mean a function $\alpha : L \to [0, 1]$ such that

(i) $\alpha(1) = 1$, and

(ii) $\alpha(a \oplus b) = \alpha(a) + \alpha(b)$ whenever $a \perp b$ in $L$.

If for a state $\alpha : L \to [0, 1]$ we have

$$\alpha\Big(\bigoplus_{i=1}^{\infty} a_i\Big) = \sum_{i=1}^{\infty} \alpha(a_i)$$

whenever $\bigoplus_{i=1}^{\infty} a_i$ exists in $L$, then $\alpha$ is called a *$\sigma$-additive state*. It is clear that for any state $\alpha$, $\alpha(0) = 0$. Moreover, for any observable $A$ on $L$, if $B_1, B_2 \in \mathcal{B}$ and $B_1 \subseteq B_2$, then $A(B_1) \leq A(B_2)$. Indeed, we have $B_2 = B_1 \cup (B_2 - B_1)$ implies $A(B_2) = A(B_1) \oplus A(B_2 - B_1)$; hence $A(B_1) \leq A(B_2)$.

The following lemma was proved in [1].

**Lemma 3.6** *Let $L$ be a $\sigma$-effect algebra. If $A$ is an observable on $L$ and if $(B_i)_{i\in\mathbb{N}} \subseteq \mathcal{B}$ is such that $B_i \subseteq B_{i+1}$ $\forall\, i \in \mathbb{N}$, then*

$$A\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \bigvee_{i=1}^{\infty} A(B_i).$$

The following result was proved in [5, page 75].

**Theorem 3.7** *Let $L$ be a $\sigma$-OMP. If $A$ is an observable on $L$, then the range of $A$ is a Boolean $\sigma$-subalgebra of $L$.*

Let $L$ be a $\sigma$-effect algebra and let $A : \mathcal{B} \to L$ be an observable on $L$. If $A(E) \wedge (A(E))' = 0$ for every $E \in \mathcal{B}$, then $A$ is called a *sharp observable*.

The following result was proved in [15, Theorem 4.1].

**Theorem 3.8** *Let $L$ be a $\sigma$-effect algebra and let $A$ be a sharp observable on $L$. Then the range of $A$, $R(A)$, is a Boolean $\sigma$-subeffect algebra of $L$.*

The *spectrum* $s(A)$ of an observable $A : \mathcal{B} \to L$ is the smallest closed subset $F$ of $\mathbb{R}$ such that $A(F) = 1$ (see [9, 17]).

The following theorem is a generalization of the result 4.1.12 in [17] to $\sigma$-effect algebras.

**Theorem 3.9** *Every observable $A$ on $L$ has a spectrum.*

**Proof:** Let $T = \{F_\alpha : \alpha \in I\}$ be the collection of all closed subsets of $\mathbb{R}$ such that $A(F_\alpha) = 1$. Then $T \neq \emptyset$ since $\mathbb{R} \in T$. Set $F = \bigcap_{\alpha\in I} F_\alpha$. Then the set $\{\mathbb{R} \setminus F_\alpha : \alpha \in I\}$ is an open covering of the second countable space $\mathbb{R} \setminus F$ and, therefore, there is a countable subcollection $\{F_n : n \in \mathbb{N}\} \subseteq \{F_\alpha : \alpha \in I\}$ such that $\bigcup_{n\in\mathbb{N}}(\mathbb{R} \setminus F_n) = \mathbb{R} \setminus F$. But we can find $(B_i)_{i\in\mathbb{N}}$ such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and $B_i \subseteq \mathbb{R} \setminus F$ such that $\bigcup_{i\in\mathbb{N}} B_i = \bigcup_{i\in\mathbb{N}} \mathbb{R} \setminus F_i$. Hence $A(\mathbb{R} \setminus F) = A(\bigcup_{i\in\mathbb{N}} \mathbb{R} \setminus F_i = A(\bigcup_{i\in\mathbb{N}} B_i) = \bigoplus_{i\in\mathbb{N}} A(B_i) = 0$. Therefore $A(F) = 1$. ■

**Theorem 3.10** *For an observable $A$, $\lambda \in s(A)$ if and only $A(\lambda - \epsilon, \lambda + \epsilon) \neq 0$ for all $\epsilon > 0$.*

**Proof:** Assume, first, that $\lambda \in s(A)$. If $\exists\, \epsilon > 0$ such that $A(\lambda - \epsilon, \lambda + \epsilon) = 0$, then $\lambda$ does not belong to the closed set $F := \mathbb{R} \setminus (\lambda - \epsilon, \lambda + \epsilon)$, where $A(F) = 1$, which is a contradiction.

Conversely, assume that $A(\lambda-\epsilon,\lambda+\epsilon)\neq 0\;\forall\epsilon>0$, and let $F$ be a closed set such that $A(F)=1$. Then $A(\mathbb{R}\setminus F)=0$. To show that $\lambda\in s(A)$, it suffices to show that $\lambda\in F$. To see this, assume on the contrary that $\lambda\in\mathbb{R}\setminus F=\bigcup_{k=1}^{\infty}I_k$, where the $I_k$ are disjoint open intervals. Then $\lambda\in I_k$ for some $k\in\mathbb{N}$, so we can find an $\epsilon>0$ such that $(\lambda-\epsilon,\lambda+\epsilon)\subseteq I_k$, and therefore

$$0\leq A(\lambda-\epsilon,\lambda+\epsilon)\leq A(I_k)\leq A(\mathbb{R}\setminus F)=0,$$

which is a contradiction. ■

The *point spectrum* of an observable $A:\mathcal{B}\to L$ is is the set

$$p(A):=\{\lambda\in\mathbb{R}:A(\{\lambda\})\neq 0\}.$$

It can be easily seen that $p(A)\subseteq s(A)$. The *continuous spectrum* of an observable $A$ is the set

$$c(A):=s(A)\setminus p(A).$$

## 4 Spectral Mapping Theorems and the Observable $A-\lambda I$

If the logic $L$ is taken to be the projection lattice of a complex Hilbert space $\mathcal{H}$, then via the spectral theorem, the set of observables can be identified with the self adjoint operators on $\mathcal{H}$. In 1968, D. Catlin [6] classified the spectra of an observable $A$ by considering the character of $(A-\lambda I)^{-1}$ on a $\sigma$-OMP without the aid of Hilbert space formalism. Then he used this classification to prove some spectral mapping theorems. In this section we will extend the above-mentioned results to a $\sigma$-effect algebra, a generalization of a $\sigma$-OMP. Throughout this section, $(L,0,1,\oplus)$ is assumed to be a $\sigma$-effect algebra.

Let $f$ be a real valued function whose domain, dom$f$, is a subset of the reals $\mathbb{R}$. We say that $f$ is a *Borel function* providing dom$f\in\mathcal{B}$ and for each $G\in\mathcal{B}$, $f^{-1}(G)\in\mathcal{B}$. Clearly any Borel function $f$ can be extended to a Borel function $\hat{f}$ where dom$\hat{f}=\mathbb{R}$. (Just define $\hat{f}(x)=0$ for $x\in\mathbb{R}\setminus\text{dom}f$.) If $f$ is a Borel function with dom$f=\mathbb{R}$ and if $A\in\mathcal{O}$, we define $f(A)$ to be the observable $A\circ f^{-1}$. If dom$f\neq\mathbb{R}$ we say that $f(A)$ *exists* or is *defined* providing it is the case that for every pair of Borel extensions $f_1,f_2$ of $f$ with dom$(f_1)=$ dom$(f_2)=\mathbb{R}$ we have $f_1(A)=f_2(A)$. If $f(A)$ exists, we define $f(A)$ to be $\hat{f}(A)$ for any extension $\hat{f}$ of $f$ with dom$\hat{f}=\mathbb{R}$.

**Theorem 4.1** *Let $A\in\mathcal{O}$ and let $f$ be a Borel function. Then $f(A)$ exists if and only if $A(domf)=1$.*

**Proof:** If $f(A)$ exists, let $\lambda_2\in f(\text{dom}f)$ and let $\lambda_1\neq\lambda_2$. Define

$$f_i(\lambda):=\begin{cases} f(\lambda) & \text{if } \lambda\in\text{dom}f\\ \lambda_i & \text{if } \lambda\in\mathbb{R}\setminus\text{dom}f,\end{cases}$$

$i=1,2$. Each $f_i$ is a Borel extension of $f$. Thus

$$A\circ f_1^{-1}(\{\lambda_2\})=A\circ f_2^{-1}(\{\lambda_2\}),$$

and so

$$A(f_1^{-1}(\{\lambda_2\})=A(f_2^{-1}(\{\lambda_2\}).$$

But

$$f_1^{-1}(\{\lambda_2\})=f^{-1}(\{\lambda_2\})\text{ and }f_2^{-1}(\{\lambda_2\})=(\mathbb{R}\setminus\text{dom}f)\cup f^{-1}(\{\lambda_2\}).$$

Thus

$$A(f^{-1}(\{\lambda_2\})=A(\mathbb{R}\setminus\text{dom}f)\oplus A(f^{-1}(\{\lambda_2\})).$$

Whence $A(\mathbb{R}\setminus\text{dom}f)=0$, and therefore $A(\text{dom}f)=1$.

Conversely, suppose that $A(\text{dom}f)=1$ so that $A(\mathbb{R}\setminus\text{dom}f)=0$. Let $f_1,\ f_2$ be Borel extensions of $f$. Then

$$f_1^{-1}(E)\cap\text{dom}f=f_2^{-1}(E)\cap\text{dom}f$$

for all $E\in\mathcal{B}$. Now $\forall\,E\in\mathcal{B}$, we have

$$\begin{aligned} f_i(A)(E) &= A(f_i^{-1}(E))\\ &= A(f_i^{-1}(E)\cap(\mathbb{R}\setminus\text{dom}f))\oplus A(f_i^{-1}(E)\cap\text{dom}f)\\ &= A(f_i^{-1}(E)\cap\text{dom}f).\end{aligned}$$

Therefore $f_1(A)=f_2(A)$. ■

**Theorem 4.2** *Let $A\in\mathcal{O}$ and suppose that for a Borel function $f$, $f(A)$ is defined.*

*(i)* $A(domf \cap s(A)) = 1$.

*(ii)* $s(A) \subseteq \overline{domf}$.

*(iii)* $f(A) = A \circ f^{-1}$.

*(iv) If* $g(f(A))$ *is defined, then so is* $(g \circ f)(A)$ *and* $g(f(A)) = (g \circ f)(A)$.

**Proof:** (i) We have

$$\begin{aligned} 1 &= A(\text{dom}f \cup s(A)) = A(\text{dom}f \setminus s(A)) \oplus A(\text{dom}f \cap s(A)) \oplus A(s(A) \setminus \text{dom}f) \\ &= A(\text{dom}f \cap s(A)). \end{aligned}$$

(ii) Since $A(\text{dom}f) = 1$, $A(\overline{\text{dom}f}) = 1$. Now by definition of $s(A)$, we have $s(A) \subseteq \overline{\text{dom}f}$.
(iii) Let $\hat{f}$ be any extension of $f$. Then for all $E \in \mathcal{B}$, we have

$$\begin{aligned} f(A)(E) &= \hat{f}(A)(E) = A(\hat{f}^{-1}(E)) \\ &= A(\hat{f}^{-1}(E) \cap \text{dom}f) \oplus A(\hat{f}^{-1}(E) \cap (\mathbb{R} \setminus \text{dom}f)) \\ &= A(\hat{f}^{-1}(E) \cap \text{dom}f) = A(f^{-1}(E)). \end{aligned}$$

(iv) This follows at once from (iii). ■

**Lemma 4.3** *Let* $A \in \mathcal{O}$ *and let* $f$ *be a Borel function such that* $f(A)$ *is defined. Then*

$$s(f(A)) \subseteq \overline{f(s(A))}.$$

**Proof:** We have $f(A)(\overline{f(s(A))}) = A(f^{-1}\overline{f(s(A))}) \geq A(\text{dom}f \cap s(A)) = 1$. Hence $f(A)(\overline{f(s(A))}) = 1$ and, by definition of $s(A)$, $s(f(A)) \subseteq \overline{f(s(A))}$. ■

**Theorem 4.4** *If* $f$ *is continuous on* $s(A)$*, or if* $f$ *has a continuous extension to* $s(A)$ *and if* $f(A)$ *exists, then*

$$s(f(A)) = \overline{f(s(A))}.$$

**Proof:** By Lemma 4.3 it suffices to prove that $\overline{f(s(A))} \subseteq s(f(A))$. If $\hat{f}$ is a continuous extension of $f$ to $s(A)$, then $\hat{f}(A) = f(A)$ and $\overline{f(s(A))} \subseteq \overline{\hat{f}(s(A))}$. Thus it would suffice to show in this case that $\overline{\hat{f}(s(A))} \subseteq s(\hat{f}(A))$. In other words, we can suppose that $f$ is defined and continuous on all of $s(A)$.

Let $\xi \in \overline{f(s(A))}$. Then there exists a sequence $\{\lambda_i\} \subseteq s(A)$ such that $f(\lambda_i) \longrightarrow \xi$. By Theorem 3.10, we have that $\forall\, \delta > 0$, $A(\lambda_i - \delta, \lambda_i + \delta) \neq 0$. Therefore, by continuity, $\forall\, \epsilon > 0$, we have

$$\begin{aligned} f(A)(f(\lambda_i) - \epsilon, f(\lambda_i) + \epsilon) &= A(f^{-1}((f(\lambda_i) - \epsilon, f(\lambda_i) + \epsilon)) \\ &\geq A((\lambda_i - \delta_f(\epsilon, \lambda_i), \lambda_i + \delta_f(\epsilon, \lambda_i)) \cap s(A)) \neq 0. \end{aligned}$$

Thus, by Theorem 3.10, $f(\lambda_i) \in s(f(A))\ \forall\, i$. Since $s(f(A))$ is closed, $\xi \in s(f(A))$. ■

An observable $A$ is said to be *bounded* providing $s(A)$ is compact. For bounded observables, we obtain the following result, which generalizes Corollary 3.5 of [6].

**Theorem 4.5** (**Spectral Mapping Theorem**). *Let* $A$ *be a bounded observable, and let* $f$ *be a Borel function defined and continuous on* $s(A)$*. If* $f(A)$ *exists, then* $s(f(A)) = f(s(A))$.

**Proof:** By continuity of $f$, $f(s(A))$ is compact; hence it is closed and bounded. Now apply Theorem 4.4. ■

An observable $A$ is said to be *invertible* providing $f(A)$ exists for the function $f(\lambda) = \frac{1}{\lambda}$. In this case we write $A^{-1} = f(A)$. According to Theorem 4.1, $A$ is invertible iff $A(\{0\}) = 0$. In particular, if $0 \notin s(A)$, then $A^{-1}$ exists. Indeed, if $0 \notin s(A)$, then there exists a closed set $F$ such that $A(F) = 1$ and $0 \notin F$. Hence $\{0\} \subseteq \mathbb{R} \setminus F$ and $A(\mathbb{R} \setminus F) = 0$. Therefore, $A(\{0\}) = 0$; that is, $A$ is invertible.

**Theorem 4.6** *Let* $A \in \mathcal{O}$ *be invertible. Then*

*(i)* $(A^{-1})^{-1} = A$.

*(ii) If* $0 \notin s(A)$*, then* $A^{-1}$ *is bounded.*

*(iii) If* $A$ *is bounded, then* $0 \notin s(A^{-1})$.

**Proof:** (i) Let

$$f(\lambda) := \begin{cases} \frac{1}{\lambda} & \text{if } \lambda \neq 0 \\ 0 & \text{if } \lambda = 0. \end{cases}$$

Then $A^{-1} = f(A)$. Now $A^{-1}(\{0\}) = f(A)(\{0\}) = A(f^{-1}(0)) = A(\{0\}) = 0$. Hence $A^{-1}$ is invertible and

$$(A^{-1})^{-1} = f(f(A)) = (f \circ f)(A) = A.$$

(ii) Suppose that $0 \notin s(A)$. Then, by Theorem 3.10, there exists an open interval $I = (-\gamma, \gamma)$ such that $A(I) = 0$ and $0 \in \mathbb{R} \setminus s(A)$. Indeed, we have $C := \mathbb{R} \setminus (-\gamma, \gamma)$ is closed and $A(C) = 1$, so by defintion of $s(A)$ we have that $s(A) \subseteq \mathbb{R} \setminus I$. Since $0 \notin s(A)$, $f(\lambda) = \frac{1}{\lambda}$ is continuous on $s(A)$ and so, by Theorem 3.4,

$$s(A^{-1}) = \overline{f(s(A))} \subseteq \overline{f(\mathbb{R} \setminus I)} = [-\frac{1}{\gamma}, \frac{1}{\gamma}].$$

Thus $s(A^{-1})$ is bounded. Since $s(A^{-1})$ is also closed, it is compact. Therefore $A^{-1}$ is bounded.

(iii) Suppose that $A$ is bounded. Then $s(A)$ is bounded, and so we can find $k > 0$ such that $s(A) \subseteq [-k, k]$. Since $A(s(A)) = 1$, $A([-k, k]) = 1$ and so $A((-\infty, -k) \cup (k, \infty)) = 0$. Hence $A(f^{-1}(-\frac{1}{k}, \frac{1}{k}) = 0$; that is, $f(A)(-\frac{1}{k}, \frac{1}{k}) = 0$, and therefore $A^{-1}(-\frac{1}{k}, \frac{1}{k}) = 0$. Thus, by Theorem 3.10, we have $0 \notin s(A^{-1})$. ■

Define $f_\lambda : \mathbb{R} \to \mathbb{R}$ by $f_\lambda(\xi) := \xi - \lambda$. Then it is natural to write $A - \lambda = f_\lambda(A)$. Now we show that spectra can be classified by using $A - \lambda$ in exactly the same manner as usually done in operator theory.

**Theorem 4.7** *Let $A \in \mathcal{O}$.*

*(i)* $\lambda \in \mathbb{R} \setminus s(A) \;\Leftrightarrow\; (A - \lambda)^{-1}$ *exists and is bounded.*

*(ii)* $\lambda \in p(A) \;\Leftrightarrow\; (A - \lambda)^{-1}$ *does not exist.*

*(iii)* $\lambda \in c(A) \;\Leftrightarrow\; (A - \lambda)^{-1}$ *exists and is not bounded.*

**Proof:** We, first, observe that $\forall \lambda$, $f_\lambda$ is continuous on $s(A)$. It follows that

$$s(A - \lambda) = s(A) - \{\lambda\}.$$

(i) By Theorem 4.6, $(A - \lambda)^{-1}$ exists and is bounded

$$\Leftrightarrow\; 0 \notin s(A - \lambda) \;\Leftrightarrow\; 0 \notin s(A) \setminus \{\lambda\} \;\Leftrightarrow\; \lambda \notin s(A).$$

(ii) $(A - \lambda)^{-1}$ fails to exist $\Leftrightarrow\; A(f_\lambda^{-1}(\{0\})) \neq 0 \;\Leftrightarrow\; A(\{\lambda\}) \neq 0 \;\Leftrightarrow\; \lambda \in p(A)$.

(iii) By definition of $c(A)$, $\lambda \in c(A) \;\Leftrightarrow\; \lambda \in s(A) \setminus p(A) \;\Leftrightarrow\; (A - \lambda)^{-1}$ exists and is not bounded. ■

## 5 The Spectral Theorem

As mentioned in Section 4, if the logic $L$ is taken to be the projection lattice of a Hilbert space $\mathcal{H}$, then the observables on $L$ can be identified with the self adjoint operators on $\mathcal{H}$. Then, via the spectral theorem for self-adjoint operators, there is a one-to-one correspondence between observables and spectral resolutions of the identity. In [6], D. Catlin showed that the same result is possible when $L$ is taken to be any $\sigma$-OMP. In this section we will extend this result to any $\sigma$-effect algebra, a generalization of a $\sigma$-OMP.

Let $L$ be an effect algebra. By a *real* (resp., *rational*) *resolution* in $L$ we mean a function $e : \mathbb{R} \to L$ (resp., $e : \mathbb{Q} \to L$) such that the following conditions ar satisfied:

(i) $\lambda \leq \mu$ in $\mathbb{R}$ (resp., in $\mathbb{Q}$) $\;\Rightarrow\; e_\lambda \leq e_\mu$,

(ii) $\wedge_\lambda e_\lambda = 0$,

(iii) $\vee_\lambda e_\lambda = 1$,

(iv) $\wedge_{\mu<\lambda} e_\lambda = e_\mu \;\forall \mu \in \mathbb{R}$ (resp., $\forall \mu \in \mathbb{Q}$).

**Theorem 5.1** *If $A$ is an observable on a $\sigma$-effect algebra $L$, then the function $e^A : \mathbb{R} \to L$ defined by $e^A_\lambda := A((-\infty, \lambda])$ is a spectral resolution in $L$.*

**Proof:** We shall verify the four conditions of the above-mentioned definition.

(i) If $\lambda \leq \mu$ in $\mathbb{R}$, then $(-\infty, \lambda] \subseteq (-\infty, \mu]$, and hence $A((-\infty, \lambda]) \leq A((-\infty, \mu])$; that is, $e^A_\lambda \leq e^A_\mu$.

(ii) Since $(-\infty, n] \subseteq (-\infty, n+1]\ \forall n \in \mathbb{N}$, then, by Lemma 3.6,

$$\begin{aligned}\bigvee_{n=1}^{\infty} A((-\infty, n]) &= A(\bigcup_{n=1}^{\infty}(-\infty, n]) \\ &= A(\mathbb{R}) = 1.\end{aligned}$$

Now let $t$ be any upper bound of $(e_\lambda^A)_{\lambda\in\mathbb{R}}$. Then $e_n^A \le t\ \forall n \in \mathbb{N}$ implies that $1 = \bigvee_n^{\infty} e_n^A \le t$; hence $t = 1$, and therefore $\bigvee_{\lambda\in\mathbb{R}} e_\lambda^A = 1$.

(iii) Since $(-n, \infty) \subseteq (-(n+1), \infty)\ \forall n \in \mathbb{N}$, we have $A(-n, \infty) \le A(-(n+1), \infty)\ \forall n \in \mathbb{N}$. Hence, by Lemma 3.6, $\bigvee_{n=1}^{\infty} A(-n, \infty) = A(\bigcup_{n=1}^{\infty}(-n, \infty)) = A(\mathbb{R}) = 1$; that is, $\bigvee_{n=1}^{\infty} (e_{-n}^A)' = 1$. Using De Morgan laws, we get $\bigwedge_{n=1}^{\infty} e_{-n}^A = 0$. Now let $t$ be a lower bound of $(e_\lambda^A)_{\lambda\in\mathbb{R}}$. Then $t \le e_{-n}^A\ \forall n \in \mathbb{N}$ implies that $t \le \bigwedge_{n=1}^{\infty} e_{-n}^A = 0$; that is, $t = 0$. Therefore $\bigwedge_{\lambda\in\mathbb{R}} e_\lambda^A = 0$.

(iv) If $\mu \in \mathbb{R}$, then $(\mu + \frac{1}{n}, \infty) \subseteq (\mu + \frac{1}{n+1}, \infty)\ \forall n \in \mathbb{N}$; so, by Lemma 3.6, $\bigvee_{n=1}^{\infty} A(\mu + \frac{1}{n}, \infty) = A(\bigcup_{n=1}^{\infty}(\mu + \frac{1}{n}, \infty)) = A(\mu, \infty)$. Hence, by De Morgan law, $\bigwedge_{n=1}^{\infty} A((-\infty, \mu + \frac{1}{n}]) = A((-\infty, \mu])$; that is, $\bigwedge_{n=1}^{\infty} e_{\mu+\frac{1}{n}}^A = e_\mu^A$. Finally, let $t$ be a lower bound of $(e_\lambda^A)_{\mu<\lambda}$ where $\lambda \in \mathbb{R}$. Then $t \le \bigwedge_{n=1}^{\infty} e_{\mu+\frac{1}{n}}^A = e_\mu^A$, and therefore $\bigwedge_{\mu<\lambda} e_\lambda^A = e_\mu$. ■

Theorem 5.1 showed that to each observable $A$ on a $\sigma$-effect algebra $L$, there corresponds a spectral resolution $(e_\lambda^A)_{\lambda\in\mathbb{R}}$ in $L$. On the other hand, if $(e_\lambda)_{\lambda\in\mathbb{R}}$ is a spectral resolution in $L$, where $L$ is a $\sigma$-effect algebra, does there exist an observable $A$ on $L$ such that $(e_\lambda)_{\lambda\in\mathbb{R}} = (e_\lambda^A)_{\lambda\in\mathbb{R}}$? We will show that the answer is yes providing $(e_\lambda)_{\lambda\in\mathbb{R}} \subseteq B$ where $B$ is a Boolean $\sigma$-subeffect algebra.

For the remainder of this section, we shall assume that $B$ is a Boolean $\sigma$-algebra. By Loomis Theorem [14, p. 171, 15c], there exists a measurable space $(X, \mathcal{M})$ and a $\sigma$-ideal $\mathcal{K} \subseteq \mathcal{M}$ such that $B \cong \mathcal{M}/\mathcal{K}$. Let $\phi : \mathcal{M} \to \mathcal{M}/\mathcal{K}$ be the natural $\sigma$-epimorphism, where $\phi(M) = [M]\ \forall M \in \mathcal{M}$. Since $B \cong \mathcal{M}/\mathcal{K}$, there exists a Boolean $\sigma$-isomorphism $\theta : \mathcal{M}/\mathcal{K} \to B$. Hence $\eta := \theta \circ \phi$ is a $\sigma$-epimorphism from $\mathcal{M}$ onto $B$. Let $e : \mathbb{R} \to B$ be a real spectral resolution and let $f : \mathbb{Q} \to B$ be the restriction of $e$ to $\mathbb{Q}$. For each rational number $\lambda \in \mathbb{Q}$, choose a set $\widetilde{F}_\lambda \in \mathcal{M}$ such that $\eta(\widetilde{F}_\lambda) = f_\lambda$. This can be done because $\eta$ is surjective. Define

$$\overline{F}_\lambda := \bigcap_{\lambda<\rho} \widetilde{F}_\lambda,\ \rho, \lambda \in \mathbb{Q}.$$

**Remarks 5.2** (i) The definition of $\overline{F}_\lambda$ yields that $\lambda, \mu \in \mathbb{Q},\ \lambda \le \mu\ \Rightarrow\ \overline{F}_\lambda \subseteq \overline{F}_\mu$. Also for each $\lambda \in \mathbb{Q}$, $\eta(\overline{F}_\lambda) = f_\lambda$. Define

$$\hat{F}_\lambda := \overline{F}_\lambda \setminus \bigcap_{\sigma\in\mathbb{Q}} \overline{F}_\sigma,\ \ \lambda \in \mathbb{Q}.$$

(ii) If $\lambda, \mu \in \mathbb{Q}$ with $\lambda \le \mu$, it then follows that $\hat{F}_\lambda \subseteq \hat{F}_\mu$. Also, we have $\bigcap_{\lambda\in\mathbb{Q}} \hat{F}_\lambda = \emptyset$ and for each $\lambda \in \mathbb{Q}$, $\eta(\hat{F}_\lambda) = f_\lambda$. Indeed, if $x \in \bigcap_{\lambda\in\mathbb{Q}} \hat{F}_\lambda \neq \emptyset$, then $x \in \hat{F}_\lambda$ for each $\lambda \in \mathbb{Q}$. From the definition of $\hat{F}_\lambda$, $x \in \overline{F}_\lambda$ for each $\lambda \in \mathbb{Q}$ and $x \notin \overline{F}_\sigma$ for some $\sigma \in \mathbb{Q}$, which is a contradiction.

(iii) We have

$$\bigcap_{\lambda<\mu} \hat{F}_\mu = \hat{F}_\lambda.$$

Indeed, the definition of $\overline{F}_\sigma$ gives $\bigcap_{\lambda<\mu} \overline{F}_\mu = \bigcap_{\lambda<\mu}(\bigcap_{\mu<\rho} \widetilde{F}_\rho) = \bigcap_{\lambda<\rho} \widetilde{F}_\rho = \overline{F}_\lambda$, and hence

$$\begin{aligned}\bigcap_{\lambda<\mu} \hat{F}_\mu &= \bigcap_{\lambda<\mu}(\overline{F}_\mu \setminus \bigcap_{\sigma\in\mathbb{Q}} \overline{F}_\sigma) = (\bigcap_{\lambda<\mu} \overline{F}_\mu) \setminus \bigcap_{\rho\in\mathbb{Q}} \overline{F}_\sigma \\ &= \overline{F}_\lambda \setminus \bigcap_{\sigma\in\mathbb{Q}} \overline{F}_\sigma = \hat{F}_\lambda.\end{aligned}$$

Finally, for $\lambda \in \mathbb{Q}$, define

$$F_\lambda := \begin{cases} \hat{F}_\lambda & \text{if } \lambda < 0,\ \lambda \in \mathbb{Q} \\ \hat{F}_\lambda \cup (X \setminus \bigcup_{\sigma\in\mathbb{Q}} \hat{F}_\sigma) & \text{if } \lambda \ge 0,\ \lambda \in \mathbb{Q}. \end{cases}$$

**Theorem 5.3** [6] $\{F_\lambda : \lambda \in \mathbb{Q}\}$ *is a rational spectral resolution in* $\mathcal{M}$ *and* $\eta(F_\lambda) = f_\lambda\ \forall \lambda \in \mathbb{Q}$. *That is, the rational spectral resolution* $\{f_\lambda : \lambda \in \mathbb{Q}\}$ *in* $B$ *is lifted through* $\eta$ *to the rational spectral resolution* $\{F_\lambda : \lambda \in \mathbb{Q}\}$ *in* $\mathcal{M}$.

Next, using Lemma 4.2 of [6], it can be easily checked that the rational spectral resolution $\{F_\lambda : \lambda \in \mathbb{Q}\}$ in $\mathcal{M}$ can be extended to a real spectral resolution $\{E_\lambda : \lambda \in \mathbb{R}\}$ by defining

$$E_\lambda := \bigcap_{\lambda\le\rho} F_\rho,\ \ \rho \in \mathbb{Q},\ \lambda \in \mathbb{R}.$$

Moreover, $\eta(E_\lambda) = e_\lambda\ \forall\,\lambda \in \mathbb{R}$. Indeed, we have $\eta(E_\lambda) = \eta(\bigcap_{\lambda\leq\rho} F_\rho) = \bigwedge_{\lambda\leq\rho} \eta(F_\rho) = \bigwedge_{\lambda\leq\rho} f_\rho = e_\lambda$.

**Lemma 5.4** [6] *Let $(X, \mathcal{M})$ be a measurable space and let $\{E_\lambda : \lambda \in \mathbb{R}\}$ be a spectral resolution in $\mathcal{M}$. Then there exists a unique measurable function $f : X \to \mathbb{R}$ such that*

$$E_\lambda = f^{-1}((-\infty, \lambda])\ \forall\,\lambda \in \mathbb{R}.$$

**Corollary 5.5** *Let $(X, \mathcal{M})$ be a measurable space and let $\{E_\lambda : \lambda \in \mathbb{R}\}$ be a spectral resolution in $\mathcal{M}$. Then there exists a unique $\mathcal{M}$-valued measure $\widetilde{A} : \mathcal{B} \to \mathcal{M}$ such that*

$$E_\lambda = \widetilde{A}((-\infty, \lambda])\ \forall\,\lambda \in \mathbb{R}.$$

**Proof:** Define $\widetilde{A} : \mathcal{B} \to \mathcal{M}$ by

$$\widetilde{A}(B) := f^{-1}(B)\ \forall\,B \in \mathcal{B},$$

where $f$ is the unique function given in Lemma 5.4. Then

$$\widetilde{A}(\mathbb{R}) = f^{-1}(\mathbb{R}) = X,\ \ \widetilde{A}(\emptyset) = f^{-1}(\emptyset) = \emptyset,$$

and if $B_i \in \mathcal{B}$, $i \in \mathbb{N}$, then

$$\widetilde{A}(\bigcup_{i=1}^{\infty} B_i) = f^{-1}(\bigcup_{i=1}^{\infty} B_i) = \bigcup_{i=1}^{\infty} f^{-1}(B_i) = \bigcup_{i=1}^{\infty} \widetilde{A}(B_i). \quad \blacksquare$$

**Theorem 5.6** [6] *Let $B$ be a Boolean $\sigma$-algebra, and let $e : \mathbb{R} \to B$ be a real spectral resolution in $B$. Then there exists a unique $B$-valued measure $A : \mathcal{B} \to B$ such that*

$$A((-\infty, \lambda]) = e_\lambda\ \forall\,\lambda \in \mathbb{R}.$$

**Theorem 5.7** *Let $L$ be a $\sigma$-effect algebra. If $(e_\lambda)_{\lambda\in\mathbb{R}}$ is a spectral resolution in $L$ and $B$ is a Boolean $\sigma$-subeffect algebra of $L$ containing $(e_\lambda)_{\lambda\in\mathbb{R}}$, then there exists a unique observable $A$ on $L$ such that*

*(i) range $A \subseteq B$,*

*(ii) $A((-\infty, \lambda]) = e_\lambda\ \forall\,\lambda \in \mathbb{R}$.*

**Proof:** By Theorem 5.6, there exists a unique observable $A$ on $B$ such that

$$A((-\infty, \lambda]) = e_\lambda\ \forall\,\lambda \in \mathbb{R}.$$

It remains to show that $A$ is an observable on $L$. Let $\{E_i : i \in \mathbb{N}\} \subseteq \mathcal{B}$ be pairwise disjoint. Since $A$ is an observable on $B$, $\{A(E_i) : i \in \mathbb{N}\}$ is a countable $\oplus$-orthogonal subset of $B$. So, by Theorem 3.4, we have

$$A(\bigcup_{i\in\mathbb{N}} E_i) = \bigvee_{i\in\mathbb{N}}^{B} A(E_i) = \bigoplus_{i\in\mathbb{N}} A(E_i),$$

where $\bigvee^B$ denotes the supremum in $B$. $\blacksquare$

The converse of Theorem 5.7 need not be true; that is, if $A$ is an observable on $L$, then the spectral resolution defined by Theorem 5.7 may fail to exist in any Boolean $\sigma$-subeffect algebra of $L$, as the following example shows.

**Example 5.8** Let $L = [0, 1]$ be the unit interval effect algebra of Example 2.1 and define $A : \mathcal{B} \to [0, 1]$ by

$$A(E) := m^*(E \cap [0, 1]) \qquad \forall\,E \in \mathcal{B},$$

where $m^*$ is Lebesgue's outer measure on $\mathbb{R}$. Then $A$ is an observable on $[0, 1]$. Indeed, if $(E_i)_{i\in\mathbb{N}} \subseteq \mathcal{B}$ is pairwise disjoint, then

$$\begin{aligned} A(\textstyle\bigcup_{i=1}^{\infty} E_i) &= m^*((\textstyle\bigcup_{i=1}^{\infty} E_i) \cap [0, 1]) \\ &= \textstyle\sum_{i=1}^{\infty} m^*(E_i \cap [0, 1]) \\ &= \textstyle\sum_{i=1}^{\infty} A(E_i) = \bigoplus_{i=1}^{\infty} A(E_i). \end{aligned}$$

Define $e : \mathbb{R} \to [0,1]$ by

$$e_\lambda := A(-\infty, \lambda] \qquad \forall \lambda \in \mathbb{R}.$$

By Theorem 5.1, $(e_\lambda)_{\lambda\in\mathbb{R}}$ is a spectral resolution in $[0,1]$. We now claim that $(e_\lambda)_{\lambda\in\mathbb{R}} = [0,1]$. Indeed, let $\lambda \in \mathbb{R}$. If $\lambda < 0$, then $e_\lambda = A(-\infty,\lambda] = m^*([0,1] \cap (-\infty,\lambda]) = m^*(\emptyset) = 0$, and if $\lambda = 0$, then $e_\lambda = A(-\infty,\lambda] = m^*([0,1] \cap (-\infty,0]) = m^*(\{0\}) = 0$. If $0 < \lambda < 1$, then $e_\lambda = A(-\infty,\lambda] = m^*([0,1] \cap (-\infty,\lambda]) = m^*([0,\lambda]) = \lambda$. Finally, if $\lambda > 1$, then $e_\lambda = A(-\infty,\lambda] = m^*([0,1]) = 1$. It follows that $(e_\lambda)_{\lambda\in\mathbb{R}} = [0,1]$.

We finally show that the effect algebra $([0,1], \oplus, 0, 1)$ is not Boolean. In fact, we have $\frac{1}{2} + \frac{1}{4} \le 1 \Rightarrow \frac{1}{2} \perp \frac{1}{4}$, but $\frac{1}{2} \oplus \frac{1}{4} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \neq \frac{1}{2} = \frac{1}{2} \vee \frac{1}{4}$; so that $([0,1], \oplus, 0, 1)$ is not an OMP and, hence, not a Boolean effect algebra.

The following theorem shows, however, that the converse of Theorem 5.7 would be true if the observable $A$ is sharp.

**Theorem 5.9** *If $A$ is an observable on a $\sigma$-effect algebra $L$, then the function $e : \mathbb{R} \to L$ defined by $e_\lambda := A((-\infty,\lambda])$ is a spectral resolution in $L$. Moreover, if $A$ is sharp, then range $A$ is a Boolean $\sigma$-subeffect algebra of $L$.*

**Proof:** It follows immediately from Theorems 5.1 and 3.8. ■

Theorems 5.7 and 5.9 are combined in the following main theorem.

**Theorem 5.10 (The Spectral Theorem)** *Let $L$ be a $\sigma$-effect algebra. If $(e_\lambda)_{\lambda\in\mathbb{R}}$ is a spectral resolution in $L$ and $B$ is a Boolean $\sigma$-subeffect algebra of $L$ containing $(e_\lambda)_{\lambda\in\mathbb{R}}$, then there exists a unique observable $A$ on $L$ such that*

*(i) range $A \subseteq B$,*

*(ii) $A((-\infty,\lambda]) = e_\lambda \ \forall \lambda \in \mathbb{R}$.*

*Conversely, if $A$ is an observable on $L$, then $e_\lambda := A((-\infty,\lambda])$ defines a real spectral resolution in $L$. Moreover, if $A$ is sharp, then range $A$ is a Boolean $\sigma$-subeffect algebra of $L$.*

Finally, since each orthoalgebra is a sharp effect algebra, it follows that every observable on a $\sigma$-orthoalgebra is sharp. Hence, as a consequence of Theorem 5.10, we obtain the following theorem which generalizes Theorem 5.5 of [6].

**Theorem 5.11 (The Spectral Theorem)** *Let $L$ be a $\sigma$-orthoalgebra. If $(e_\lambda)_{\lambda\in\mathbb{R}}$ is a spectral resolution in $L$ and $B$ is a Boolean $\sigma$-suborthoalgebra of $L$ containing $(e_\lambda)_{\lambda\in\mathbb{R}}$, then there exists a unique observable $A$ on $L$ such that*

*(i) range $A \subseteq B$,*

*(ii) $A((-\infty,\lambda]) = e_\lambda \ \forall \lambda \in \mathbb{R}$.*

*Conversely, if $A$ is an observable on $L$, then $e_\lambda := A((-\infty,\lambda])$ defines a real spectral resolution in $L$ and range $A$ is a Boolean $\sigma$-suborthoalgebra of $L$.*

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