Palestine Journal of Mathematics

Vol. 1 (2012) , 27–31

SPACELIKE ELASTIC BIHARMONIC CURVES WITH TIMELIKE M_1 ACCORDING TO BISHOP FRAME IN MINKOWSKI 3-SPACE

Essin TURHAN and Talat KÖRPINAR

Communicated by Jose Luis Lopez-Bonilla

2000 Mathematics Subject Classification: 31B30, 58E20.
Key words: Biharmonic curve, Elastic curve, Minkowski 3-space.

1 Abstract

In this paper, we study spacelike elastic biharmonic curves with timelike M_1 according to Bishop frame in Minkowski 3-space $\mathbb{E}_1^3$. We give some characterizations for curvature and torsion of a biharmonic curve in Minkowski 3-space $\mathbb{E}_1^3$. Finally, we use Noether's theorem to derive some equations.

2 Introduction

Let (N,h) and (M,g) be Riemannian manifolds. Denote by R^N and R the Riemannian curvature tensors of N and M, respectively. We use the sign convention:

$$R^N(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad X,Y \in \Gamma(TN).$$

For a smooth map $\phi : N \longrightarrow M$, the Levi-Civita connection ∇ of (N,h) induces a conncetion ∇^ϕ on the pull-back bundle

$$\phi^* TM =_{p \in N} T_{\phi(p)} M.$$

The section $\mathcal{T}(\phi) := \mathrm{tr}\nabla^\phi d\phi$ is called the tension field of ϕ. A map ϕ is said to be harmonic if its tension field vanishes identically.

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} \left|\mathcal{T}(\phi)\right|^2 dv_h.$$

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \mathrm{tr}R(\mathcal{T}(\phi), d\phi)\, d\phi, \tag{2.1}$$

where R the Riemannian curvature tensors of N,and called the bitension field of ϕ. The operator Δ_ϕ is the rough Laplacian acting on $\Gamma(\phi^*TM)$ defined by

$$\Delta_\phi := -\sum_{i=1}^{n} \left(\nabla_{e_i}^\phi \nabla_{e_i}^\phi - \nabla_{\nabla_{e_i}^N e_i}^\phi \right),$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field of N. Obviously, every harmonic map is biharmonic. Non-harmonic biharmonic maps are called proper biharmonic maps.

In particular, if the target manifold M is the Euclidean space $\mathbb{E}^m$, the biharmonic equation of a map $\phi : N \to \mathbb{E}^m$ is

$$\Delta_h \Delta_h \phi = 0,$$

where Δ_h is the Laplace–Beltrami operator of (N,h).

Recently, there have been a growing interest in the theory of biharmonic maps which can be divided into two main research directions. On the one side, the differential geometric aspect has driven attention to the construction of examples and classification results. The other side is the analytic aspect from the point of view of PDE: biharmonic maps are solutions of a fourth order strongly elliptic semilinear PDE.

On the other hand, one of the oldest topics in the calculus of variations is the study of the elastic rod which, according to Daniel Bernoulli's idealization, minimizes total squared curvature among curves of the same length and first order boundary data. The classical term elastica refers to a curve in the plane or $\mathbb{R}^3$ which represents such a rod in equilibrium. While the elastica and its generalizations have long been (and continue to be) of interest in the context of elasticity theory, the elastica as a purely geometrical entity seems to have been largely ignored.

In this paper, we study spacelike elastic biharmonic curves according to Bishop frame in Minkowski 3-space $\mathbb{E}_1^3$. We give some characterizations for curvature and torsion of a biharmonic curve in Minkowski 3-space $\mathbb{E}_1^3$. Finally, we use Noether's theorem to derive some equations.

3 Preliminaries

The Minkowski 3-space $\mathbb{E}_1^3$ is the Euclidean 3-space $\mathbb{E}^3$ provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}_1^3$.

Minkowski space is originally from the relativity in Physics. In fact, a timelike curve corresponds to the path of an observer moving at less than the speed of light, a null curves correspond to moving at the speed of light and a spacelike curves to moving faster than light.

Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= \tau\mathbf{N}, \end{aligned}$$

where $\kappa = |\mathcal{T}(\gamma)| = |\nabla_{\mathbf{T}}\mathbf{T}|$ is the curvature of γ and τ its torsion and

$$\begin{aligned} g(\mathbf{T}, \mathbf{T}) &= 1, \ g(\mathbf{N}, \mathbf{N}) = -1, \ g(\mathbf{B}, \mathbf{B}) = 1, \\ g(\mathbf{T}, \mathbf{N}) &= g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned}$$

In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. One can express parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 - k_2\mathbf{M}_2, \\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= k_2\mathbf{T}, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} g(\mathbf{T}, \mathbf{T}) &= 1, \ g(\mathbf{M}_1, \mathbf{M}_1) = -1, \ g(\mathbf{M}_2, \mathbf{M}_2) = 1, \\ g(\mathbf{T}, \mathbf{M}_1) &= g(\mathbf{T}, \mathbf{M}_2) = g(\mathbf{M}_1, \mathbf{M}_2) = 0. \end{aligned} \tag{3.2}$$

Here, we shall call the set $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2\}$ as Bishop trihedra and k_1 and k_2 as Bishop curvatures.

One can show that

$$\begin{aligned} \kappa(s) &= \sqrt{\left|-k_1^2 + k_2^2\right|}, \\ \theta(s) &= \arg\tanh \frac{k_2}{k_1}, \\ \tau(s) &= \theta'(s), \end{aligned}$$

so that k_1 and k_2 effectively correspond to a cartesian coordinate system for the polar coordinates κ, θ with $\theta = \int \tau(s)ds$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0, which disappears from τ (and hence from the Frenet frame) due to the differentiation.

4 Spacelike Biharmonic Curves with Timelike M_1 in $\mathbb{E}_1^3$

Biharmonic equation for the curve γ reduces to

$$\nabla_{\mathbf{T}}^3\mathbf{T} - R(\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T})\mathbf{T} = 0, \tag{4.1}$$

that is, γ is called a biharmonic curve if it is a solution of the equation

Theorem 4.1. $\gamma : I \longrightarrow \mathbb{E}_1^3$ *is a unit speed spacelike biharmonic curve with timelike* $\mathbf{M}_1$ *if and only if*

$$\begin{aligned} k_1^2 - k_2^2 &= C, \\ k_1'' + k_1^3 - k_1 k_2^2 &= 0, \\ k_2'' - k_2^3 + k_2 k_1^2 &= 0, \end{aligned} \tag{4.2}$$

where C *is non-zero constant of integration.*

Corollary 4.2. $\gamma : I \longrightarrow \mathbb{E}_1^3$ *is a unit speed spacelike biharmonic curve with timelike* $\mathbf{M}_1$ *if and only if*

$$\begin{aligned} k_1^2(s) - k_2^2(s) &= C \neq 0, \\ k_1''(s) + Ck_1(s) &= 0, \\ k_2''(s) + Ck_2(s) &= 0, \end{aligned} \tag{4.3}$$

where C *is constant of integration.*

5 Spacelike Elastic Biharmonic Curves According to Bishop Frame in $\mathbb{E}_1^3$

Consider regular curves (curves with nonvanishing velocity vector) in Minkowski 3-space $\mathbb{E}_1^3$ defined on a fixed interval $I = [a_1, a_2]$:

$$\gamma : I \longrightarrow \mathbb{E}_1^3.$$

We will assume (for technical reasons) that the curvature κ of γ is nonvanishing.

The elastica minimizes the bending energy

$$\wp(X) = \int_\gamma \kappa(s)^2 \, ds$$

with fixed length and boundary conditions. Accordingly, let α_1 and α_2 be points in $\mathbb{E}_1^3$ and α_1', α_2' nonzero vectors. We will consider the space of smooth curves

$$\Omega = \{\gamma : \gamma(a_i) = \alpha_i, \ \gamma'(a_i) = \alpha_i'\},$$

and the subspace of unit-speed curves

$$\Omega_u = \{\gamma \in \Omega : \|\gamma'\| = 1\}.$$

Later on we need to pay more attention to the precise level of differentiability of curves, but we will ignore that for now.

$\wp^\lambda : \Omega \longrightarrow \mathbb{R}$ is defined by

$$\wp^\lambda(\gamma) = \frac{1}{2} \int_\gamma [\|\gamma''\| + \Lambda(t)(\|\gamma'\| - 1)] \, dt.$$

One version of the Lagrange multiplier principle says a minimum of $\wp$ on Ω_u is a stationary point for $\wp^\lambda$ for some $\Lambda(t)$. ($\Lambda(t)$ is a pointwise multiplier, constraining speed.) The name $\wp^\lambda$ for the function will be justified later, when we will see that $\Lambda(t)$ depends on a constant λ.

Theorem 5.1. *(Noether's Theorem) If* γ *is a solution curve and* W *is an infinitesimal symmetry, then*

$$\gamma''.W' + (\Lambda\gamma' - \gamma''').W$$

is constant. In particular, for a translational symmetry, W *is constant; so*

$$(\Lambda\gamma' - \gamma''').W = \text{constant}.$$

Letting W range over all translations, we get

$$\Lambda\gamma' - \gamma''' = J, \tag{4.1}$$

for J some constant field.

Theorem 5.2. *Let* $\gamma : I \longrightarrow \mathbb{E}_1^3$ *be a spacelike elastic biharmonic curve with timelike* $\mathbf{M}_1$ *according to Bishop frame. Then, the elastica minimizes the bending energy*

$$\wp(X) = C(a_2 - a_1).$$

Theorem 5.3. *Let $\gamma : I \longrightarrow \mathbb{E}_1^3$ be a spacelike elastic biharmonic curve with timelike $\mathbf{M}_1$ according to Bishop frame. Then,*

$$\begin{aligned} \Lambda_s(s) &= 0, \\ k_1'' + (C - \Lambda(s)) k_1 &= 0, \\ k_2'' + (C - \Lambda(s)) k_2 &= 0, \end{aligned} \tag{5.1}$$

where $k_1^2 - k_2^2 = C$.

Proof. Now it is helpful to assume spacelike biharmonic curve γ is parametrized by arclength s. Then,

$$\begin{aligned} \gamma' &= \gamma_s = \mathbf{T}, \\ \gamma'' &= k_1\mathbf{M}_1 - k_2\mathbf{M}_2, \\ \gamma''' &= C\mathbf{T} + k_1'\mathbf{M}_1 - k_2'\mathbf{M}_2, \end{aligned} \tag{5.2}$$

where $k_1^2 - k_2^2 = C$.

Using (4.1) and above equation (5.2), we get

$$J = \gamma''' - \Lambda(s)\gamma' = (C - \Lambda(s))\mathbf{T} + k_1'\mathbf{M}_1 - k_2'\mathbf{M}_2, \tag{5.3}$$

for J some constant field.

Differentiate J to get

$$\begin{aligned} J_s = {} & (-\Lambda_s(s) + k_1'k_1 - k_2'k_2)\mathbf{T} + (k_1'' + (C - \Lambda(s)) k_1)\mathbf{M}_1 \\ & + (-k_2'' - (C - \Lambda(s)) k_2)\mathbf{M}_2. \end{aligned} \tag{5.4}$$

These, together with (4.1), complete the proof of the theorem.

Theorem 5.4. *Let $\gamma : I \longrightarrow \mathbb{E}_1^3$ be a spacelike elastic biharmonic curve with timelike $\mathbf{M}_1$ according to Bishop frame. Then,*

$$\Lambda(s) = 0.$$

Proof. Using (5.3) and constant field J,

$$\begin{aligned} -\Lambda_s(s) + k_1'k_1 - k_2'k_2 &= 0, \\ -k_2'' - (C - \Lambda(s)) k_2 &= 0, \\ -k_2'' - (C - \Lambda(s)) k_2 &= 0. \end{aligned}$$

From (4.1) and above equation, we have $\Lambda(s) = 0$. The proof is completed.

Corollary 5.5. *Let $\gamma : I \longrightarrow \mathbb{E}_1^3$ be a spacelike elastic biharmonic curve according to Bishop frame. Then,*

$$J = C\mathbf{T} + k_1'\mathbf{M}_1 - k_2'\mathbf{M}_2. \tag{5.4}$$

Corollary 5.6. *Let $\gamma : I \longrightarrow \mathbb{E}_1^3$ be a spacelike elastic biharmonic curve with timelike $\mathbf{M}_1$. Then,*

$$\|J\|^2 = C^2 - [k_1']^2 + [k_2']^2. \tag{5.5}$$

Proof. Using (4.1) and (4.6), we have

$$\|J\|^2 = C^2 \langle \mathbf{T}, \mathbf{T} \rangle + [k_1']^2 \langle \mathbf{M}_1, \mathbf{M}_1 \rangle + [k_2']^2 \langle \mathbf{M}_2, \mathbf{M}_2 \rangle.$$

Substituting (3.2) into above equation, we get (5.5). The proof is completed.

References

[1] A.T. Ali, M. Turgut, *Position vector of a time-like slant helix in Minkowski 3-space*, J. Math. Anal. Appl. 365 (2010), 559–569.

[2] J. Inoguchi, *Submanifolds with harmonic mean curvature in contact 3-manifolds*, Colloq. Math. 100 (2004), 163–179.

[3] G.Y. Jiang, *2-harmonic isometric immersions between Riemannian manifolds,* Chinese Ann. Math. Ser. A 7 (1986), 130–144.

[4] V. Jurdjevic, *Non-Euclidean elastica,* Am. J. Mathematics 117 (1995), 93–124.

[5] M.K. Karacan, B. Bükcü, N. Yuksel, *On the dual Bishop Darboux rotation axis of the dual space curve*, Appl. Sci. 10 (2008), 115–120.

[6] M.K. Karacan, B. Bükcü, *An alternative moving frame for tubular surfaces around timelike curves in the Minkowski 3-space,* Balkan J. Geom. Appl. 12 (2) (2007), 73–80.

[7] T. Körpınar, E. Turhan, *On Horizontal Biharmonic Curves In The Heisenberg Group* $Heis^3$, Arab. J. Sci. Eng. Sect. A Sci., 35 (1) (2010), 79-85.

[8] J. Langer, D. Singer, *Knotted elastic curves in* R^3, J. London Math. Soc. 30 (1984), 512–520.

[9] B. O'Neill, , *Semi-Riemannian Geometry*, Academic Press, New York (1983).

[10] C. Oniciuc, *On the second variation formula for biharmonic maps to a sphere,* Publ. Math. Debrecen 61 (2002), 613–622.

[11] E. Turhan and T. Körpınar: *Position vectors of spacelike biharmonic curves with spacelike bihormal in Lorentzian Heisenberg group* $Heis^3$, Int. J. Open Problems Compt. Math. 3 (3) (2010), 413-422.

[12] E. Turhan, T. Körpınar, *Characterize on the Heisenberg Group with left invariant Lorentzian metric*, Demonstratio Mathematica 42 (2) (2009), 423-428.

[13] E. Turhan, T. Körpınar, *On Characterization Of Timelike Horizontal Biharmonic Curves In The Lorentzian Heisenberg Group* $Heis^3$, Zeitschrift für Naturforschung A- A Journal of Physical Sciences 65a (2010), 641-648.

[14] S. Yılmaz, *Position vectors of some special space-like curves according to Bishop frame in Minkowski space* $\mathbb{E}_1^3$, Sci. Magna 5 (1) (2009), 48–50.

Author information

Essin TURHAN and Talat KÖRPINAR, Fırat University, Department of Mathematics 23119, Elazığ, Turkey.
E-mail: essin.turhan@gmail.com and talatkorpinar@gmail.com

Received: November 10, 2011
Accepted: January 12, 2012