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On generalized *-derivations in *-rings

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Abstract. The objective of this paper is to define the notions of generalized $*$-derivation & generalized reverse $*$-derivation, and to prove some theorems involving these mappings. As an application, generalized $*$-derivations of C^*-algebra are characterized.

1. Introduction

Throughout the discussion, unless otherwise mentioned, R will denote an associative ring with center $Z(R)$ and A will represents a C^*-algebra. For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$. We shall make extensive use of the following basic commutator identities without any specific mention:

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z \quad \text{for all } x, y, z \in R.$$

Recall that a ring R is prime if for $x, y \in R,\ xRy = \{0\}$ implies either $x = 0$ or $y = 0$, and R is semiprime if $xRx = \{0\}$ implies $x = 0$. An additive mapping $x \longmapsto x^*$ of R into itself is called an involution if the following conditions are satisfied: $(i)\ \ (xy)^* = y^*x^*$, and $(ii)\ \ (x^*)^* = x$ for all $x, y \in R$. A ring equipped with an involution is called a $*$-ring or ring with involution. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $\delta : R \longrightarrow R$ is said to be a left derivation (resp. Jordan left derivation) if $\delta(xy) = x\delta(y) + y\delta(x)$ (resp. $\delta(x^2) = 2x\delta(x)$) holds for all $x, y \in R$. An additive mapping $T : R \longrightarrow R$ is called a left (resp. right) centralizer if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) holds for all $x, y \in R$. Following [7, 15], an additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Thus, the concept of generalized derivations covers both the concepts of derivations and left multipliers (viz., [21] and [22]) that is, additive maps F satisfying $F(xy) = F(x)y$ for all $x, y \in R$.

Let R be a $*$-ring. Following [1], an additive mapping $d : R \longrightarrow R$ is said to be a $*$-derivation if $d(xy) = d(x)y^* + xd(y)$ holds for all $x, y \in R$. An additive mapping $d : R \longrightarrow R$ is called a reverse $*$-derivation if $d(xy) = d(y)x^* + yd(x)$ holds for all $x, y \in R$. An additive mapping $T : R \longrightarrow R$ is said to be a left (resp. right) $*$-centralizer if $T(xy) = T(x)y^*$ (resp. $T(xy) = x^*T(y)$) holds for all $x, y \in R$. An additive mapping $T : R \longrightarrow R$ is called a reverse left $*$-centralizer if $T(xy) = T(y)x^*$ holds for all $x, y \in R$. In [11], Brešar and Vukman prove that if a prime $*$-ring R admits a $*$-derivation(resp. reverse $*$-derivation) d, then either $d = 0$ or R is commutative. Further, the author together with Ashraf [4] extended the above mentioned result for semiprime $*$-rings. During the last some decades many authors have studied derivations in the context of prime and semiprime rings with involution (viz., [1], [4], [11], [13], [14], and [16]).

Motivated by definition of $*$-derivation, the notion of generalized $*$-derivation and generalized reverse $*$-derivation defined as follows: an additive mapping $F : R \longrightarrow R$ is called a generalized $*$-derivation if there exists a $*$-derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y^* + xd(y)$ holds for all $x, y \in R$. An additive mapping $F : R \longrightarrow R$ is said to be a generalized reverse $*$-derivation if there exists a reverse $*$-derivation $d : R \longrightarrow R$ such that $F(xy) = F(y)x^* + yd(x)$ holds for all $x, y \in R$. Hence, the concept of generalized $*$-derivations(resp. generalized reverse $*$-derivations) covers the concepts of $*$-derivations(resp. reverse $*$-derivations). Moreover, generalized $*$-derivation with $d = 0$ covers the concept of left $*$-multipliers $i.e.$, additive maps F satisfying $F(xy) = F(x)y^*$ for all $x, y \in R$(see [1], where further references can be found).

The aim of the present paper is to establish some results involving generalized $*$-derivations and generalized reverse $*$-derivations. The obtained results generalizes the result given by Brešar and Vukman [11] to a large class of $*$-rings. Further, some related results on $*$-rings has also been given. Moreover, the last section of this paper is devoted to applications of our results to a ring with involution, in particular to a C^*-algebra.

2. Main Results

Over the last few decades, several authors have investigated the relationship between the commutativity of a ring R and the existence of certain specified derivations of R (cf., [4], [5], [12] and [16], where further references can be looked). The first result in this direction is due to Posner [19] who proved that if a prime ring R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. An analogous result for centralizing automorphisms on prime rings was obtained by Mayne [17]. A number of authors have extended these theorems of Posner and Mayne; they have showed that derivations, automorphisms, and some related maps cannot be centralizing on certain subsets of noncommutative prime (and some other) rings. For these results we refer the reader to ([5], [8], [16] where further references can be found). In [6], the description of all centralizing additive maps of a prime ring R of characteristic not 2 was given and subsequently in [3] the characterization for semiprime rings of characteristic not 2 was given. It was shown that every such map f is of the form $f(x) = \lambda x + \mu(x)$, where $\lambda \in C$, the extended centroid of R, and μ is an additive map of R into C (see also [8] where similar results for some other rings are presented). In the year 1990, Brešar and Vukman [12] established that a prime ring must be commutative if it admits a nonzero left derivation. Further, Vukman [20] extended the above mentioned result for semiprime rings and prove that if a 2-torsion free semiprime ring admits a Jordan left derivation δ, then δ is a derivation which maps R into $Z(R)$. In this section, our objective is to explore similar types of problems in the setting of $*$-rings. More precisely, we prove the following theorem:

Theorem 2.1. Let R be a semiprime $*$-ring. If R admits a generalized $*$-derivation F with an associated nonzero $*$-derivation d, then F maps R into $Z(R)$.

Proof. Notice that F is a generalized $*$-derivation with an associated nonzero $*$-derivation d, we have

$$F(xy) = F(x)y^* + xd(y) \text{ for all } x, y \in R. \tag{2.1}$$

Replacing y by yz in (2.1) we obtain

$$F(xyz) = F(x)z^*y^* + xd(yz) \text{ for all } x, y, z \in R. \tag{2.2}$$

Since d is a $*$ -derivation, the last equation yields that

$$F(xyz) = F(x)z^*y^* + xd(y)z^* + xyd(z) \text{ for all } x, y, z \in R. \tag{2.3}$$

On the other hand, we have

$$F(xyz) = F((xy)z) = F(x)y^*z^* + xd(y)z^* + xyd(z) \text{ for all } x, y, z \in R. \tag{2.4}$$

Comparing (2.3) and (2.4), we obtain

$$F(x)[z^*, y^*] = 0 \text{ for all } x, y, z \in R. \tag{2.5}$$

Replacing y by y^* and z by z^* in relation (2.5), we get

$$F(x)[z, y] = 0 \text{ for all } x, y, z \in R. \tag{2.6}$$

Replace z by $zF(x)$ in (2.6) to get

$$F(x)z[F(x), y] + F(x)[z, y]F(x) = 0 \text{ for all } x, y, z \in R.$$

In view of (2.6) the last expression yields that

$$F(x)z[F(x), y] = 0 \text{ for all } x, y, z \in R. \tag{2.7}$$

Left multiplication by y to relation (2.7) forces that

$$yF(x)z[F(x), y] = 0 \text{ for all } x, y, z \in R. \tag{2.8}$$

Again, replacing z by yz in (2.7), we obtain

$$F(x)yz[F(x), y] = 0 \text{ for all } x, y, z \in R. \tag{2.9}$$

Subtracting (2.8) from (2.9), we find that

$$[F(x), y]z[F(x), y] = 0 \text{ for all } x, y, z \in R. \tag{2.10}$$

The last relation can rewritten as $[F(x), y]R[F(x), y] = \{0\}$ for all $x, y \in R$. Thus, the semiprimeness of R forces that $[F(x), y] = 0$ for all $x, y \in R$. Hence, F maps R into $Z(R)$. This completes the proof of our theorem.

Our next result is related to left $*$-centralizers on semiprime $*$-rings.

Theorem 2.2. Let R be a semiprime $*$-ring. If $T : R \longrightarrow R$ is an additive mapping such that $T(xy) = T(x)y^*$ for all $x, y \in R$, then T maps R into $Z(R)$.

Proof. By the assumption, we have

$$T(xy) = T(x)y^* \text{ for all } x, y \in R. \tag{2.11}$$

Now, compute $T(xzy)$ in two different ways. On the one hand we have

$$T(xzy) = T(x)z^*y^* \text{ for all } x, y, z \in R. \tag{2.12}$$

On the other hand, we obtain

$$T(xzy) = T(x(zy)) = T(x)y^*z^* \text{ for all } x, y, z \in R. \tag{2.13}$$

Comparing (2.12) and (2.13), we find that

$$T(x)[z^*, y^*] = 0 \text{ for all } x, y, z \in R. \tag{2.14}$$

This expression is same as the equation (2.5) with the exception that we have left $*$-centralizer T instead of generalized $*$-derivation F. Hence, using same approach as we have used after (2.5) in the proof of the last paragraph of Theorem 2.1, we get the required result. That is, $[T(x), y] = 0$ for all $x, y \in R$. Hence, T maps R into $Z(R)$.

Following are immediate consequences of Theorem 2.1:

Corollary 2.1. Let R be a prime $*$-ring. If R admits a generalized $*$-derivation F with an associated nonzero $*$-derivation d, then either $F = 0$ or R is commutative.

Proof. In view of Theorem 2.1, we have $F(x)[y, z] = 0$ for all $x, y, z \in R$. Replacing y by yt, we find that $F(x)y[t, z] + F(x)[y, z]t = 0$ for all $x, y, z, t \in R$, and hence $F(x)y[t, z] = 0$ for all $x, y, z, t \in R$. That is, $F(x)R[t, z] = \{0\}$ for all $x, z, t \in R$. The primeness of R forces that either $F(x) = 0$ or $[t, z] = 0$ for all $x, z, t \in R$. Hence, either $F = 0$ or R is commutative. This proves the corollary.

Corollary 2.2. Let R be a semisimple $*$-ring. If R admits a generalized $*$-derivation F with an associated nonzero $*$-derivation d, then F maps R into $Z(R)$.

Proof. As consequence of Theorem 2.1 and of the fact that every semisimple $*$-ring is semiprime $*$-ring.

Corollary 2.3. Let R be a semiprime $*$-ring. If R admits a nonzero $*$-derivation d, then d maps R into $Z(R)$.

Proof. Taking $F = d$ in the proof of Theorems 2.1.

We now prove another theorem in the spirit of Theorem 2.1, that is,

Theorem 2.3. Let R be a semiprime $*$-ring. If R admits a generalized reverse $*$-derivation F with an associated nonzero reverse $*$-derivation d, then $[d(x), z] = 0$ for all $x, z \in R$.

Proof. We are given that F is a generalized reverse $*$-derivation with an associated nonzero reverse $*$-derivation d, we have

$$F(xy) = F(y)x^* + yd(x) \text{ for all } x, y \in R. \tag{2.15}$$

Substituting xz for x in (2.15) and using the fact that d is a reverse $*$-derivation, we obtain

$$F(xzy) = F(y)z^*x^* + yd(z)x^* + yzd(x) \text{ for all } x, y, z \in R. \tag{2.16}$$

Also, we find that

$$F(xzy) = F(x(zy)) = F(y)z^*x^* + yd(z)x^* + zyd(x) \text{ for all } x, y, z \in R. \tag{2.17}$$

On comparing last two equations, we get

$$[y, z]d(x) = 0 \text{ for all } x, y, z \in R. \tag{2.18}$$

Replace y by $d(x)y$ in (2.18) to get

$$d(x)[y, z]d(x) + [d(x), z]yd(x) = 0 \text{ for all } x, y, z \in R. \tag{2.19}$$

Application of relation (2.18) yields that

$$[d(x), z]yd(x) = 0 \text{ for all } x, y, z \in R. \tag{2.20}$$

Replacing y by yz in (2.20), we get

$$[d(x), z]yzd(x) = 0 \text{ for all } x, y, z \in R. \tag{2.21}$$

Multiplying by z from right to (2.20) gives that

$$[d(x), z]yd(x)z = 0 \text{ for all } x, y, z \in R. \tag{2.22}$$

On subtracting (2.21) from (2.22), we obtain

$$[d(x), z]y[d(x), z] = 0 \text{ for all } x, y, z \in R. \tag{2.23}$$

This implies that $[d(x), z]R[d(x), z] = \{0\}$ for all $x, z \in R$, whence it follows finally that $[d(x), z] = 0$ for all $x, z \in R$. This proves the theorem.

The above theorem has two very interesting corollaries, namely:

Corollary 2.4. Let R be a noncommutative prime $*$-ring. If R admits a generalized reverse $*$-derivation F with an associated reverse $*$-derivation d, then F is a reverse left $*$-centralizer(multiplier) on R.

Proof. From Theorem 2.3, we have $[y, z]d(x) = 0$ for all $x, y, z \in R$. Replace y by xy in the last expression to get $x[y, z]d(x) + [x, z]yd(x) = 0$ for all $x, y, z \in R$. We conclude that $[x, z]yd(x) = 0$ for all $x, y, z \in R$. That is, $[x, z]Rd(x) = \{0\}$ for all $x, z \in R$. The primeness of R yields that either $[x, z] = 0$ or $d(x) = 0$. Now, we put $U = \{x \in R \mid [x, z] = 0 \text{ for all } z \in R\}$ and $V = \{x \in R \mid d(x) = 0\}$. Then, U and V are additive subgroups of R such that $U \cup V = R$. Thus by using Braure's trick, we find that either $U = R$ or $V = R$. If $U = R$, then $[x, z] = 0$ for all $x, z \in R$ and hence R is commutative, a contradiction. On the other hand if $V = R$, then $d(x) = 0$ for all $x \in R$. Hence, $F(xy) = F(y)x^*$ for all $x, y \in R$. With this the proof is complete.

Corollary 2.5. Let R be a semiprime $*$ -ring. If R admits a nonzero reverse $*$-derivation d, then d maps R into $Z(R)$.

Proof. Choose $F = d$ in the proof of Theorems 2.3.

The following example demonstrates that the condition of primeness in the hypothesis of Corollary 2.1 is crucial.

Example 2.1. Let S be a commutative ring. Next, let R=$\left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} |a, b, c \in S \right\}$. Define maps F, d, $*$: $R \to R$ by $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & c & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$. Then, it is straightforward to check that F and d satisfies the requirments of Corollary 2.1. However, neither $F = 0$ nor R is commutative.

3. Applications

In the present section, we give application of Theorem 2.1 & 2.2 to certain special classes of algebras, some of which are related to C^*- algebra. A Banach algebra is a linear associative algebra which, as a vector space, is a Banach space with norm $\| \cdot \|$ satisfying the multiplicative inequality; $\|xy\| \leq \|x\|\|y\|$ for all x and y in A. An additive mapping $x \longmapsto x^*$ of A into itself is called an involution if the following conditions are satisfied: (i) $(xy)^* = y^*x^*$, (ii) $(x^*)^* = x$, and (iii) $(\lambda x)^* = \bar{\lambda}x^*$ for all $x, y \in A$ and $\lambda \in \mathbf{C}$, where $\bar{\lambda}$ is the conjugate of λ. An algebra equipped with an involution is called a $*$-algebra or algebra with involution. A C^*-algebra A is a Banach $*$-algebra with the additional norm condition $\|x^*x\| = \|x\|^2$ for all $x \in A$. A C^*-algebra A is primitive if its zero ideal is primitive, that is, if A has a faithful nonzero irreducible representation. Throughout the present section, C^*-algebras are assumed to be nonunital unless indicated otherwise.

Theorem 3.1. *Let A be a C^*-algebra. If $F : A \to A$ is a linear generalized $*$-derivation with an associated linear $*$-derivation $d : A \to A$, then $[F(x), x] = 0$ for all $x \in A$.*

Proof. It is well known that every C^*-algebra is a semiprime ring (viz., [2] for details). First we suppose that an associated linear $*$-derivation d is nonzero i.e., $d \neq 0$. Then in view of Theorem 2.1, F maps A into $Z(A)$ and hence we conclude that $[F(x), x] = 0$ for all $x \in A$. On the other hand, if $d = 0$, then F is a left $*$-multiplier(centralizer) on A i.e., $F(xy) = F(x)y^*$ for all $x, y \in A$. Thus by Theorem 2.2 we get the required result. Thereby the proof is completed.

Theorem 3.2. *Let A be a primitive C^*-algebra. If $F : A \to A$ is a linear generalized reverse $*$-derivation with an associated nonzero linear reverse $*$-derivation $d : A \to A$, then A is commutative.*

Proof. Using the fact that every primitive C^*-algebra is prime (viz., [18, Theorem 5.4.5]). Hence, A is a prime C^*-algebra. Suppose F is a linear generalized reverse $*$-derivation on A. Then by equation (2.18), we have $[y, z]d(x) = 0$ for all $x, y, z \in A$. Replacing x by xz in the last expression and using it, we obtain $[y, z]xd(z) = 0$ for all $x, y, z \in A$ i.e., $[y, z]Ad(z) = \{0\}$ for all $y, z \in A$. The primeness of A yields that either $[y, z] = 0$ or $d(z) = 0$ for all $y \in A$. Now, we set $A_1 = \{z \in A \mid [y, z] = 0 \text{ for all } y \in A\}$ and $A_2 = \{z \in A \mid d(z) = 0\}$. Then, clearly A_1 and A_2 are additive subgroups of A. Moreover, by the discussion given, A is the set-theoretic union of A_1 and A_2. But a group can not be the set-theoretic union of two of its proper subgroups, hence $A_1 = A$ or $A_2 = A$. If $A_1 = A$, then $[y, z] = 0$ for all $y, z \in A$ and hence A is commutative. On the other hand if $A_2 = A$, then $d(z) = 0$ for all $z \in A$, a contradiction. This completes the proof of our theorem.

We conclude this paper with some immediate consequences of the above theorems:

Corollary 3.1. *Let A be a C^*-algebra. If $d : A \to A$ is a nonzero linear $*$-derivation on A, then $[d(x), x] = 0$ for all $x \in A$.*

Corollary 3.2. *Let A be a primitive C^*-algebra. If A admits a nonzero linear reverse $*$-derivation $d : A \to A$, then A is commutative.*

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