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A study on two coupled modified KdV systems with time-dependent and constant coefficients

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1 Abstract

This work concerns itself on two new coupled mKdV systems, the first with time-dependent coefficients, and the second with constant coefficients. The soliton ansatz will be used for the first system to obtain 1-bright soliton solution. The simplified form of the bilinear method will be used to derive multiple-soliton solutions and multiple singular soliton solutions for the second coupled mKdV system.

2 Introduction

In solitary waves theory, the integrable coupled mKdV systems attract the interest of many mathematicians and physicists. The coupled variable coefficient mKdV is usually derived from a two-layer fluid model by using the multiple scale scheme with the reductive perturbation method.

Various methods [1–19] have been used to conduct analysis on the coupled nonlinear KdV and mKdV equations. Examples of the methods that have been used so far are the Hirota bilinear method, the Bäcklund transformation method, Darboux transformation, Pfaffian technique, the inverse scattering method, the Painlevé analysis, the generalized symmetry method, the subsidiary ordinary differential equation method (sub-ODE for short), the coupled amplitude-phase formulation, sine-cosine method, sech-tanh method, the mapping and the deformation approach, and many other methods. The Hirota's bilinear method [7–14], and the Hereman's simplified form [15] are rather heuristic and significant. These approaches possess powerful features that make it practical for the determination of multiple soliton solutions [16–31] for a wide class of nonlinear evolution equations. Moreover, the soliton ansatz [3] has been used for the determination of the bright soliton solutions.

In this work, two coupled modified KdV (mKdV) equations, the first with time-dependent coefficients, and the second with constant coefficients, will be investigated for the determination of multiple soliton solutions and bright soliton solutions. The coupled mKdV equations that we will examine are given by

$$\begin{aligned} u_t + f(t)u_{xxx} + g(t)u^2u_x + h(t)uvw_x = 0, \\ v_t + f(t)v_{xxx} + g(t)v^2u_x + h(t)vwu_x = 0, \\ w_t + f(t)w_{xxx} + g(t)w^2w_x + h(t)wuv_x = 0, \end{aligned} \tag{1}$$

where $f(t), g(t)$ and $h(t)$ are time-dependent coefficients. If we set $f(t) = 1, g(t) = 6$, and $h(t) = 3$, Eq. (1) will be reduced to

$$\begin{aligned} u_t + u_{xxx} + 6u^2u_x + 3uvw_x = 0, \\ v_t + v_{xxx} + 6v^2u_x + 3vwu_x = 0, \\ w_t + w_{xxx} + 6w^2w_x + 3wuv_x = 0. \end{aligned} \tag{2}$$

The first coupled KdV equation (1) will be studied for bright solitons, whereas the latter will be investigated for complete integrability and for the determination of multiple soliton solutions and multiple singular soliton solutions. The computer algebra systems such as Maple and Mathematica allow us to perform complicated and tedious calculations.

This work is organized as follows. In Section 2, we make use of the soliton ansatz to determine bright solitons for the first model of coupled mKdV equations with time-dependent coupling coefficients. In Section 3, we derive multiple soliton solutions and multiple singular soliton solutions for the second model of coupled mKdV equations with constant linear coupling coefficients. Conclusions are contained in Section 4.

3 The first coupled mKdV system

We first begin our analysis on the first coupled mKdV system with time-dependent coefficients

$$\begin{aligned} u_t + f(t)u_{xxx} + g(t)u^2u_x + h(t)uvw_x = 0, \\ v_t + f(t)v_{xxx} + g(t)v^2u_x + h(t)vwu_x = 0, \\ w_t + f(t)w_{xxx} + g(t)w^2w_x + h(t)wuv_x = 0, \end{aligned} \tag{1}$$

where $u(x,t), v(x,t)$ and $w(x,t)$ are real functions of the spatial variable x and the temporal variable t, $f(t), g(t)$ and $h(t)$ are real coupling coefficients which are time-dependent functions, and subscripts denote partial derivatives.

As stated before, this model is derived from a two-layer fluid model which is used to study the interaction between the atmosphere and oceanic phenomena. The derivation of (1) was achieved by using the multiple scale approach with the reductive perturbation method [1].

In this section, our goal is to find the exact bright soliton solutions for the coupled mKdV system (1), governing the mutual interaction in addition to nonlinear self-interaction of two solitary waves in a mKdV system. For definiteness, we look to study the problem when linear and nonlinear coupling terms are present. It should be mentioned that consideration of linear coupling terms to study the propagation of soliton pulses is just a particular case because most of real physical systems exhibit nonlinear interaction between waves.

To obtain the bright soliton solutions of (1), we assume the solitary wave ansätze of the form [3–5]

$$\begin{aligned} u(x,t) &= A(t)\text{sech}^p\left\{\eta(t)\left(x-vt\right)\right\}, \\ v(x,t) &= B(t)\text{sech}^q\left\{\eta(t)\left(x-vt\right)\right\}, \\ w(x,t) &= C(t)\text{sech}^r\left\{\eta(t)\left(x-vt\right)\right\}, \end{aligned} \tag{2}$$

where $v = v(t)$ is a time- dependent coefficient that will be determined. Here A, B, or C is the amplitude of solitons, η and v are the inverse width and the velocity of the solitons respectively. The exponents p, q and r are unknown at this point and will be determined later.

From the first part of ansätz (2), one obtains

$$u_t = \frac{dA}{dt}\text{sech}^p\theta - Ap\left\{\frac{d\eta}{dt}\left(x-vt\right) - \eta\left(v + t\frac{dv}{dt}\right)\right\}\text{sech}^p\theta\tanh\theta, \tag{3}$$

$$u_x = -Ap\eta\,\text{sech}^p\theta\tanh\theta, \tag{4}$$

$$u_{xxx} = -A\left(p\eta\right)^3\text{sech}^p\theta\tanh\theta + p(p+1)(p+2)\eta^3 A\text{sech}^{p+2}\theta\tanh\theta. \tag{5}$$

From the second part of ansätz (2) we find

$$
\begin{aligned}
v_t &= \frac{dB}{dt}\operatorname{sech}^q\theta - Bq\left\{\frac{d\eta}{dt}(x-vt) - \eta\left(v + t\frac{dv}{dt}\right)\right\}\operatorname{sech}^q\theta\tanh\theta, && (6)\\
v_x &= -Bq\eta\,\operatorname{sech}^q\theta\tanh\theta, && (7)\\
v_{xxx} &= -B\,(q\eta)^3\,\operatorname{sech}^q\theta\tanh\theta + q(q+1)(q+2)\eta^3 B\operatorname{sech}^{q+2}\theta\tanh\theta. && (8)
\end{aligned}
$$

From the third part of ansätz (2) we find

$$
\begin{aligned}
w_t &= \frac{dC}{dt}\operatorname{sech}^r\theta - Cr\left\{\frac{d\eta}{dt}(x-vt) - \eta\left(v + t\frac{dv}{dt}\right)\right\}\operatorname{sech}^r\theta\tanh\theta, && (9)\\
w_x &= -Cr\eta\,\operatorname{sech}^r\theta\tanh\theta, && (10)\\
w_{xxx} &= -C\,(r\eta)^3\,\operatorname{sech}^r\theta\tanh\theta + r(r+1)(r+2)\eta^3 C\operatorname{sech}^{r+2}\theta\tanh\theta, && (11)
\end{aligned}
$$

where

$$
\theta = \eta(t)\,(x - v(t)t)\,. \qquad (12)
$$

Substituting Eqs. (2)–(12) into Eqs. (1) gives

$$
\begin{aligned}
&\operatorname{sech}^p\theta\frac{dA}{dt} - pA\left\{\frac{d\eta}{dt}(x-vt) - \eta\left(v + t\frac{dv}{dt}\right)\right\}\operatorname{sech}^p\theta\tanh\theta\\
&+f(t)\left\{-A\,(p\eta)^3\,\operatorname{sech}^p\theta\tanh\theta + p(p+1)(p+2)\eta^3 A\operatorname{sech}^{p+2}\theta\tanh\theta\right\}\\
&+g(t)\left\{-A^3 p\eta\operatorname{sech}^{3p}\theta\tanh\theta\right\}\\
&+h(t)\left\{-ABCr\eta\operatorname{sech}^{p+q+r}\theta\tanh\theta\right\} = 0, \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
&\operatorname{sech}^q\theta\frac{dB}{dt} - qB\left\{\frac{d\eta}{dt}(x-vt) - \eta\left(v + t\frac{dv}{dt}\right)\right\}\operatorname{sech}^q\theta\tanh\theta\\
&+f(t)\left\{-B(q\eta)^3\operatorname{sech}^q\theta\tanh\theta + q(q+1)(q+2)\eta^3 B\operatorname{sech}^{q+2}\theta\tanh\theta\right\}\\
&+g(t)\left\{-AB^2 p\eta\operatorname{sech}^{2p+q}\theta\tanh\theta\right\}\\
&+h(t)\left\{-ABCp\eta\operatorname{sech}^{p+q+r}\theta\tanh\theta\right\} = 0, \qquad (14)
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{sech}^r\theta\frac{dC}{dt} - rC\left\{\frac{d\eta}{dt}(x-vt) - \eta\left(v + t\frac{dv}{dt}\right)\right\}\operatorname{sech}^r\theta\tanh\theta\\
&+f(t)\left\{-C(r\eta)^3\operatorname{sech}^r\theta\tanh\theta + r(r+1)(r+2)\eta^3 C\operatorname{sech}^{r+2}\theta\tanh\theta\right\}\\
&+g(t)\left\{-C^3 r\eta\operatorname{sech}^{3r}\theta\tanh\theta\right\}\\
&+h(t)\left\{-ABCq\eta\operatorname{sech}^{p+q+r}\theta\tanh\theta\right\} = 0. \qquad (15)
\end{aligned}
$$

Equating the exponents of $\operatorname{sech}^{p+2}\theta\tanh\theta$ and $\operatorname{sech}^{3p}\theta\tanh\theta$ terms in Eq. (13), one gets

$$
p + 2 = 3p, \qquad (16)
$$

so that

$$
p = 1. \qquad (17)
$$

We next match the exponents of $\operatorname{sech}^{q+2}\theta \tanh\theta$ and $\operatorname{sech}^{p+2q}\theta \tanh\theta$ in (14) to obtain

$$q+2=p+2q, \tag{18}$$

which gives

$$q=1. \tag{19}$$

We finally match $\operatorname{sech}^{r+2}\theta \tanh\theta$ and $\operatorname{sech}^{3r}\theta \tanh\theta$ in (15) to obtain

$$r+2=3r, \tag{20}$$

which gives

$$r=1. \tag{21}$$

If we put $p=1$ in Eq. (13), we can determine the inverse width η of the soliton pulses by setting the corresponding coefficients of $\operatorname{sech}^3\theta \tanh\theta$ to zero, hence we find

$$6Af(t)\eta^3 - A^3 g(t)\eta - ABCh(t)\eta = 0, \tag{22}$$

which gives

$$\eta = \sqrt{\frac{A^2 g(t) + BCh(t)}{6f(t)}}, f(t) \neq 0. \tag{23}$$

Similarly we find from setting the coefficients of $\sec h^3\theta \tanh\theta$ terms to zero in Eq. (14) that

$$\eta = \sqrt{\frac{ABg(t) + ACh(t)}{6f(t)}}, f(t) \neq 0. \tag{24}$$

Proceeding as before, we find from (15) that

$$\eta = \sqrt{\frac{C^2 g(t) + ABh(t)}{6f(t)}}, f(t) \neq 0. \tag{25}$$

Equating the three values of η from (23)–(25) gives

$$A(t) = B(t) = C(t). \tag{26}$$

This in turn gives the inverse width by

$$\eta = A(t)\sqrt{\frac{g(t) + h(t)}{6f(t)}}, f(t) \neq 0. \tag{27}$$

Setting the coefficients of $\operatorname{sech}^p\theta \tanh\theta$, $\operatorname{sech}^q\theta \tanh\theta$, and $\operatorname{sech}^r\theta \tanh\theta$ terms to zero in Eqs. (13) – (15), respectively, we get

$$-pA\left\{\frac{d\eta}{dt}(x-vt) - \eta\left(v + t\frac{dv}{dt}\right)\right\} - f(t)A(p\eta)^3 = 0, \tag{28}$$

$$-qB\left\{\frac{d\eta}{dt}(x-vt) - \eta\left(v + t\frac{dv}{dt}\right)\right\} - f(t)B(q\eta)^3 = 0, \tag{29}$$

and

$$-rC\left\{\frac{d\eta}{dt}(x-vt) - \eta\left(v + t\frac{dv}{dt}\right)\right\} - f(t)C(r\eta)^3 = 0. \tag{30}$$

Taking into account the fact that the soliton velocity $v(t)$ we want to determine from Eqs. (28) – (30) is a function of time, one can split these three equations into four equations as follows:

$$\frac{d\eta}{dt} = 0, \tag{31}$$

$$\frac{d(vt\eta)}{dt} - f\eta^3 = 0, \tag{32}$$

$$\frac{d(vt\eta)}{dt} - f\eta^3 = 0, \tag{33}$$

$$\frac{d(vt\eta)}{dt} - f\eta^3 = 0. \tag{34}$$

It is clear that (32)–34) are equivalent. By integrating the above equations with respect to the time variable t, one obtains

$$\eta(t) = k, \tag{35}$$

and

$$v(t) = \frac{k^2}{t}\int_0^t f(t^{'})dt^{'}, \tag{36}$$

where k is an integral constant related to the initial pulse inverse width as defined in (35). From Eq. (35), it is apparent that the inverse width of the solitons remains constant when the pulse propagates in the varying mKdV system. We remark also from Eq. (36) that the pulse velocity is affected by the time-dependent coupling coefficients $f(t)$.

Lastly, we find from setting the coefficients of $\text{sech}^p\theta$ and $\text{sech}^q\theta$ terms to zero in Eqs. (13)–(15), respectively, that

$$\frac{dA}{dt} = 0, \tag{37}$$

$$\frac{dB}{dt} = 0, \tag{38}$$

$$\frac{dC}{dt} = 0, \tag{39}$$

which gives after integration

$$A(t) = A_0, \tag{40}$$

$$B(t) = B_0, \tag{41}$$

$$C(t) = C_0, \tag{42}$$

where A_0, B_0 and C_0 are integral constants related to the initial pulse amplitudes of the three solitons. As $A(t) = B(t) = C(t)$ from (26), we get $A_0 = B_0 = C_0 = \alpha$ where α is a constant.

It is interesting to note also from (26) that, since $A(t) = \alpha$, and the inverse width $\eta = k$, then it is necessary that

$$\frac{g(t) + h(t)}{6f(t)} = \beta. \tag{43}$$

Finally we find that the three bright soliton solutions are symmetric and given by

$$u(x,t) = v(x,t) = w(x,t) = A_0\text{sech}\{k(x - v(t)t)\}, \tag{44}$$

where the velocity $v\,(t)$ is given by (36) and k is given by (35).

One concludes that the first coupled equation (1) possesses the soliton solution and anti-soliton solutions since $u(x,t) = \pm v(x,t)$.

4 The second coupled mKdV system

In this section, we will study the second coupled mKdV system, with constant coefficients, given by

$$\begin{aligned} u_t + u_{xxx} + 6u^2 u_x + 3uvw_x &= 0, \\ v_t + v_{xxx} + 6v^2 u_x + 3vwu_x &= 0, \\ w_t + w_{xxx} + 6w^2 w_x + 3wuv_x &= 0. \end{aligned} \tag{1}$$

4.1 Multiple soliton solutions

Substituting

$$\begin{aligned} u(x,t) &= e^{\theta_i}, \theta_i = k_i x - c_i t, \\ v(x,t) &= Ae^{\theta_i}, \\ w(x,t) &= Be^{\theta_i}, \end{aligned} \tag{2}$$

where A and B are constants, into the linear terms of (1) gives the dispersion relation by

$$c_i = k_i^3, \tag{3}$$

and as a result we obtain

$$\theta_i = k_i x - k_i^3 t. \tag{4}$$

The multi-soliton solutions of the coupled mKdV system are given by

$$\begin{aligned} u(x,t) &= R\left(\arctan(\tfrac{f(x,t)}{g(x,t)})\right)_x = R\tfrac{f_x g - g_x f}{f^2+g^2}, \\ v(x,t) &= R_1\left(\arctan(\tfrac{f(x,t)}{g(x,t)})\right)_x = R_1\tfrac{f_x g - g_x f}{f^2+g^2}, \\ w(x,t) &= R_2\left(\arctan(\tfrac{f(x,t)}{g(x,t)})\right)_x = R_2\tfrac{f_x g - g_x f}{f^2+g^2}, \end{aligned} \tag{5}$$

where the auxiliary functions $f(x,t)$ and $g(x,t)$ for the single soliton solution are given by

$$\begin{aligned} f(x,t) &= 1 + e^{\theta_1} = 1 + e^{k_1 x - k_1^3 t}, \\ g(x,t) &= 1 - e^{\theta_1} = 1 - e^{k_1 x - k_1^3 t}. \end{aligned} \tag{6}$$

Substituting (5) and (6) into (1) and solving for R, R_1 and R_2 we find three sets of solutions given by

$$R = \frac{2}{\sqrt{7}},\ R_1 = \frac{12}{\sqrt{7}},\ R_2 = \frac{4}{\sqrt{7}}, \tag{7}$$

$$R = -\frac{4}{\sqrt{3}},\ R_1 = -\frac{4}{\sqrt{3}},\ R_2 = \frac{2}{\sqrt{3}}, \tag{8}$$

and

$$R = 2\sqrt{\frac{2}{3}},\ R_1 = 2\sqrt{\frac{2}{3}},\ R_2 = 2\sqrt{\frac{2}{3}}. \tag{9}$$

Combining (5)–(9) gives the following three sets of single soliton solutions

$$\begin{aligned} u(x,t) &= \frac{2k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{7}(1+e^{2k_1(x-(k_1^2+\alpha)t)})}, \\ v(x,t) &= \frac{12k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{7}(1+e^{2k_1(x-(k_1^2+\alpha)t)})}, \\ w(x,t) &= \frac{4k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{7}(1+e^{2k_1(x-(k_1^2+\alpha)t)})}, \end{aligned} \tag{10}$$

$$\begin{aligned} u(x,t) &= -\frac{4k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{3}(1+e^{2k_1(x-(k_1^2+\alpha)t)})}, \\ v(x,t) &= -\frac{4k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{3}(1+e^{2k_1(x-(k_1^2+\alpha)t)})}, \\ w(x,t) &= \frac{2k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{3}(1+e^{2k_1(x-(k_1^2+\alpha)t)})}, \end{aligned} \tag{11}$$

and

$$\begin{aligned} u(x,t) &= \frac{2\sqrt{2}k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{3}(1+e^{2k_1(x-(k_1^2+\alpha)t)})}, \\ v(x,t) &= \frac{2\sqrt{2}k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{3}(1+e^{2k_1(x-(k_1^2+\alpha)t)})}, \\ w(x,t) &= \frac{2\sqrt{2}k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{3}(1+e^{2k_1(x-(k_1^2+\alpha)t)})}. \end{aligned} \tag{12}$$

The last set gives symmetric solutions similar to what we obtained before for coupled mKdV system with time-dependent coefficients.

To determine the two-soliton solutions we set

$$\begin{aligned} f(x,t) &= 1 + e^{\theta_1} + e^{\theta_2} - a_{12}e^{\theta_1+\theta_2}, \\ g(x,t) &= 1 - e^{\theta_1} - e^{\theta_2} - a_{12}e^{\theta_1+\theta_2}. \end{aligned} \tag{13}$$

Substituting (13) into (5) and using the obtained result in the coupled mKdV system (1), one obtains the phase shift a_{12} by

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{14}$$

and this can be generalized to

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, 1 \le i < j \le 3. \tag{15}$$

Substituting (14) and (13) into (5), and using the three sets (7)–(9) we obtain three sets of two solitons solutions. It is interesting to point out that the system (1) does not show any resonant phenomenon [8] because the phase shift term a_{12} in (14) cannot be 0 or ∞ for $|k_1| \neq |k_2|$.

To determine the three soliton solutions, we set

$$\begin{aligned} f(x,t) &= 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} \\ &- a_{12}e^{\theta_1+\theta_2} - a_{13}e^{\theta_1+\theta_3} - a_{23}e^{\theta_2+\theta_3} - b_{123}e^{\theta_1+\theta_2+\theta_3}, \\ g(x,t) &= 1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_3} \\ &- a_{12}e^{\theta_1+\theta_2} - a_{13}e^{\theta_1+\theta_3} - a_{23}e^{\theta_2+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3}, \end{aligned} \tag{16}$$

where the phase shifts a_{ij} are derived above in (15). Proceeding as before we find

$$b_{123} = a_{12}a_{13}a_{23}. \tag{17}$$

This shows that the coupled mKdV system (1) is completely integrable and N-soliton solutions can be obtained for finite N, where $N \geq 1$. Three sets of three soliton solutions for the coupled mKdV equation (1) are therefore obtained by substituting (16) into (5).

4.2 Multiple singular soliton solutions

In this section, we will determine multiple singular soliton solutions for the coupled mKdV system (1). Following [13–16], the singular soliton solution of the coupled mKdV system (1) is assumed to be of the form

$$\begin{aligned} u(x,t) &= R\left(\ln(\tfrac{f(x,t)}{g(x,t)})\right)_x = R\tfrac{gf_x-fg_x}{gf}, \\ v(x,t) &= R_1\left(\ln(\tfrac{f(x,t)}{g(x,t)})\right)_x = R_1\tfrac{gf_x-fg_x}{gf}, \\ w(x,t) &= R_2\left(\ln(\tfrac{f(x,t)}{g(x,t)})\right)_x = R_2\tfrac{gf_x-fg_x}{gf}, \end{aligned} \tag{18}$$

where R, R_1 and R_2 are constants that will be determined. The auxiliary functions $f(x,t)$ and $g(x,t)$ have expansions of the form

$$\begin{aligned} f(x,t) &= 1+\textstyle\sum_{n=1}^{\infty} f_n(x,t), \\ g(x,t) &= 1-\textstyle\sum_{n=1}^{\infty} g_n(x,t). \end{aligned} \tag{19}$$

Following the discussion presented in the previous section, the dispersion relation is given by

$$c_i = k_i^3, \tag{20}$$

and as a result we obtain

$$\theta_i = k_i x - k_i^3 t. \tag{21}$$

The obtained results give a new definition to (19) in the form

$$\begin{aligned} f(x,t) &= 1+e^{k_1(x-k_1^2 t)}, \\ g(x,t) &= 1-e^{k_1(x-k_1^2 t)}. \end{aligned} \tag{22}$$

Substituting (22) into (18), and using the outcome in (1), one obtains three sets of solutions given by

$$R = \frac{1}{\sqrt{7}},\ R_1 = \frac{6}{\sqrt{7}},\ R_2 = \frac{2}{\sqrt{7}}, \tag{23}$$

$$R = -\frac{2}{\sqrt{3}},\ R_1 = -\frac{2}{\sqrt{3}},\ R_2 = \frac{1}{\sqrt{3}}, \tag{24}$$

and

$$R = \sqrt{\frac{2}{3}},\ R_1 = \sqrt{\frac{2}{3}},\ R_2 = \sqrt{\frac{2}{3}}. \tag{25}$$

Combining the previous results gives the following three sets of single singular soliton solutions

$$\begin{aligned} u(x,t) &= \tfrac{2k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{7}(1-e^{2k_1(x-(k_1^2+\alpha)t)})}, \\ v(x,t) &= \tfrac{12k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{7}(1-e^{2k_1(x-(k_1^2+\alpha)t)})}, \\ w(x,t) &= \tfrac{4k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{7}(1-e^{2k_1(x-(k_1^2+\alpha)t)})}, \end{aligned} \tag{26}$$

$$\begin{aligned} u(x,t) &= -\tfrac{4k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{3}(1-e^{2k_1(x-(k_1^2+\alpha)t)})}, \\ v(x,t) &= -\tfrac{4k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{3}(1-e^{2k_1(x-(k_1^2+\alpha)t)})}, \\ w(x,t) &= \tfrac{2k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{3}(1-e^{2k_1(x-(k_1^2+\alpha)t)})}, \end{aligned} \tag{27}$$

and

$$
\begin{aligned}
u(x,t) &= \frac{2\sqrt{2}k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{3}(1-e^{2k_1(x-(k_1^2+\alpha)t)})},\\
v(x,t) &= \frac{2\sqrt{2}k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{3}(1-e^{2k_1(x-(k_1^2+\alpha)t)})},\\
w(x,t) &= \frac{2\sqrt{2}k_1\, e^{k_1(x-(k_1^2+\alpha)t)}}{\sqrt{3}(1-e^{2k_1(x-(k_1^2+\alpha)t)})}.
\end{aligned}
\tag{28}
$$

The last set gives symmetric solutions. The singularity behavior of each solution is clearly observed from the denominators.

To determine the singular two soliton solutions, we set

$$
\begin{aligned}
f(x,t) &= 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2},\\
g(x,t) &= 1 - e^{\theta_1} - e^{\theta_2} + b_{12}e^{\theta_1+\theta_2}.
\end{aligned}
\tag{29}
$$

Substituting (29) into (19) and using the outcome into (1), we find that (29) is a solution of this equation if the phase shifts a_{12} and b_{12}, and therefore a_{ij} and b_{ij}, are equal and given by

$$
a_{ij} = b_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}.
\tag{30}
$$

For the singular two soliton solutions we use $1 \le i < j \le 2$ to obtain

$$
\begin{aligned}
f(x,t) &= 1 + e^{k_1(x-k_1^2t)} + e^{k_2(x-k_2^2t)} + \tfrac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-(k_1^3+k_2^3)t},\\
g(x,t) &= 1 - e^{k_1(x-k_1^2t)} - e^{k_2(x-k_2^2t)} + \tfrac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-(k_1^3+k_2^3)t}.
\end{aligned}
\tag{31}
$$

This in turn gives the singular two soliton solutions if we substitute (31) into (18).

To determine the singular three soliton solutions, we can proceed in a similar manner and set

$$
\begin{aligned}
f(x,t) &= 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3}\\
&+ a_{12}e^{\theta_1+\theta_2} + a_{23}e^{\theta_2+\theta_3} + a_{13}e^{\theta_1+\theta_3} + f_3(x,t),\\
g(x,t) &= 1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_3}\\
&+ a_{12}e^{\theta_1+\theta_2} + a_{23}e^{\theta_2+\theta_3} + a_{13}e^{\theta_1+\theta_3} + g_3(x,t).
\end{aligned}
\tag{32}
$$

Substituting (32) into (18) and using the result into (1) to find that

$$
\begin{aligned}
f_3(x,t) &= b_{123}e^{\theta_1+\theta_2+\theta_3},\\
g_3(x,t) &= -b_{123}e^{\theta_1+\theta_2+\theta_3},\\
b_{123} &= a_{12}a_{13}a_{23}.
\end{aligned}
\tag{33}
$$

For the singular three-soliton solutions we use $1 \le i < j \le 3$, we therefore obtain

$$
\begin{aligned}
f(x,t) &= 1 + e^{k_1(x-k_1^2t)} + e^{k_2(x-k_2^2t)} + e^{k_3(x-k_3^2t)}\\
&+\tfrac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-(k_1^3+k_2^3)t} + \tfrac{(k_1-k_3)^2}{(k_1+k_3)^2}e^{(k_1+k_3)x-(k_1^3+k_3^3)t}\\
&+\tfrac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{(k_2+k_3)x-(k_2^3+k_3^3)t}\\
&+\tfrac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}e^{(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t},\\
g(x,t) &= 1 - e^{k_1(x-k_1^2t)} - e^{k_2(x-k_2^2t)} - e^{k_3(x-k_3^2t)}\\
&+\tfrac{(k_1-k_2)^2}{(k_1+k_2)^2}e^{(k_1+k_2)x-(k_1^3+k_2^3)t} + \tfrac{(k_1-k_3)^2}{(k_1+k_3)^2}e^{(k_1+k_3)x-(k_1^3+k_3^3)t}\\
&+\tfrac{(k_2-k_3)^2}{(k_2+k_3)^2}e^{(k_2+k_3)x-(k_2^3+k_3^3)t}\\
&-\tfrac{(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2}{(k_1+k_2)^2(k_1+k_3)^2(k_2+k_3)^2}e^{(k_1+k_2+k_3)x-(k_1^3+k_2^3+k_3^3)t}.
\end{aligned}
\tag{34}
$$

The singular three-soliton solutions follow immediately upon substituting (34) into (18).

5 Discussion

In this work we studied two coupled mKdV systems. The first system includes time-dependent coefficients, where we obtained only symmetric solutions with the wave speed $v(t)$ depends on the function $f(t)$. The second model with constant coefficients was handled by the simplified bilinear method. Three sets of multiple soliton solutions were obtained, one set consists of symmetric solutions whereas the other two sets are asymmetric solutions. Moreover, three sets of multiple singular soliton solutions, one of them is symmetric, were obtained. The approach used for the second system is different than the approach used by Hietarinta [14]. We believe that this approach will be applicable to other applications.

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