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Deformation of immersed Legendrian curves along a pseudo-gradient for the action functional: the H_0^1-flow at infinity

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1 Introduction

Let (M^3, α) be a three-dimensional compact orientable manifold without boundary and let α be a contact form on M^3. We consider in this paper the Legendrian framework developed in [1], [2], [3]: we assume that there is a non-singular vector-field v in $\ker \alpha$ such that $\beta = d\alpha(v, \cdot)$ is a contact form with the same orientation than α. We introduce the action functional $J(x) = \int_0^1 \alpha_x(\dot{x})dt$ on the space of Legendrian curves $C_\beta = \{x \in H^1(S^1, M)\, s.t.\, \beta_x(\dot{x}) \equiv 0, \alpha(\dot{x}) = \text{positive constant}\}$. Its critical points are the periodic orbits of the Reeb vector-field of α, which we denote ξ.

The variational problem is non-compact and there are critical points at infinity in this problem. They have been thoroughly described in [1], [2], [3].

Despite several positive results, starting from the work of Paul Rabinowitz [12] and continuing with results of C.Viterbo [14], H.Hofer [8] and others, the general framework, the framework of the exotic contact forms of S^3 and the variations in the large (the existence of **multiple** periodic orbits) for the Reeb vector-fields of these forms are still not fully understood.

On the other hand, the understanding of the central issue of the periodic orbits of the Reeb vector-fields for these three dimensional contact forms and structures is not the only interesting issue of the field.

Equally interesting are the variational flows, spaces etc associated to these problems. For example, in a direction that is sharply different from the present one, the understanding of the moduli spaces of pseudo-holomorphic curves [7] is in itself an interesting problem.

In the present Legendrian approach, there are two main directions of research:

The first one is related to the understanding of the topology of the variational space C_β defined above. The topology of the space of Legendrian curves $L_\beta = \{x = x(t) \in H^1(S^1, M); d\alpha(\dot{x}, v) = 0\}$ is fully understood, see [6] and [13]; it is the topology of the loop space of M. The topology of L_β when the assumption that β is a contact form is removed is a wide open problem; and there should be a large variety of results depending on how much β is far from a contact form and close to a foliation.

The topology of C_β is understood in only very few cases. There is a conjecture about this topology, which states that if $ker\alpha$ "turns well" along v, see [1], p I.11, then the injection of C_β in the loop space of M should be a homotopy equivalence. This conjecture is currently studied by Ali Maalaoui [9].

The second interesting problem relates to the variational flow of $J(x)$ on C_β. It is a flow that deforms immersed curves of a three-dimensional manifold, whereas it decreases the area defined by $d\alpha$ on the surfaces that these curves bound (assuming eg that the manifold M is eg simply connected). In order to introduce this flow, we need to describe in more detail our framework of study. This will be completed in the next section.

Throughout the remainder of this paper, numerous references are made to the monographs [1], [2], [3]. We provide here a brief account of how the results of the present paper fit in the these monographs:

The construction of the variational flow for (J, C_β) is completed [3], pp 25-86 and pp 184-186 ("A direct way to reach ν or $\tilde{\nu}$-stretched curves"). The main steps of the construction of the flow in [3] are Lemma 1 p 26, Lemma 3 p34, Lemma D p46 and Lemma 3' p55. Existence for the (semi)-flow Z_ν is established pp 59-84 of [3]. The definition of a ν-stretched curve is provided p 119 of [3]. After this construction, we use the argument developed pp 184-186 of [3] and the deformation along this Z_ν semi-flow leads to the union of the unstable manifolds of the periodic orbits and the so-called ν-stretched curves. In order to transform these curves into curves of $\cup\Gamma_{2k} = \{\text{curves made of } \xi\text{-pieces of orbits alternating with } \pm v\text{-pieces of orbits}\}$, we need to change the value of ν from ν to $\frac{\nu}{2}$, that is to use a combination of the Z_ν and the $Z_{\frac{\nu}{2}}$-flows. This is a rather delicate argument that requires the control of the curves as their "nearly ξ-pieces (pieces of curves tangent to ξ up to $O(\nu)$) change. The idea for this convergence process is explained pp 147-152 of [3]. The deformation makes also use of the so-called "small-normals flow" of Appendix 4, pp 297-324 of [3]. There are several typos in [3] and the proof of Lemma 1 should be rewritten.

Pages 55-69 of [2] give a summary of this deformation process, with a slightly different use of the "small-normals flow" of Appendix 4 of [3].

The present paper completes the technical details for this deformation. It also replaces the use of the combination of the $Z_\nu/Z_{\frac{\nu}{2}}$-flows in [2], [3] with the use of the so-called H_0^1-flow, pp39-55 of [2].

The author is aware that the results of [1], [2], [3] and of the present paper should be rewritten in a single paper/monograph. In this monograph, the whole deformation should be explained in a unified and a continuous way.

2 Tangent spaces, Variational Flow

In our framework, α is a contact form, v is a non-zero vector-field in $ker\alpha$, ξ is the Reeb vector-field of α. We are assuming that $\beta = d\alpha(v, .)$ is a contact form with the same orientation than α. We re-scale v so that

$$\beta \wedge d\beta = \alpha \wedge d\alpha.$$

Under these assumptions, the following technical results hold, see [2], [3]:

Proposition 2.1. *i)* $d\alpha(v, [\xi, v]) = -1$,

ii) $[\xi, [\xi, v]] = -\tau v$,

iii) The Reeb vector-field of β *is* $w = -[\xi, v] + \bar{\mu}\xi$ *where* $\bar{\mu} = \alpha(w) = d\alpha(v, [v, [\xi, v]])$.

Let

$$C_\beta = \{x \in H^1(S^1, M) \text{ s.t. } \beta_x(\dot{x}) \equiv 0, \alpha_x(\dot{x}) = \text{a positive constant.}\}$$

A tangent vector z to M reads

$$z = \lambda\xi + \mu v + \eta w.$$

If x belongs to C_β, then $\dot{x} = a\xi + bv$, with a being a positive constant. A tangent vector z at x to C_β reads $z = \lambda\xi + \mu v + \eta w$ with:

$$\frac{d}{dt}(\lambda + \overline{\mu}\eta) = \dot{\overline{\lambda + \overline{\mu}\eta}} = b\eta - \int_0^1 b\eta$$

$$\lambda, \mu, \eta \quad 1-\text{periodic}$$

$$\dot{\eta} = \mu a - \lambda b.$$

There are two ways of producing tangent vectors to C_β. One is rigorous and involves the rewriting of the equations on λ and η as a system of ordinary differential equations of first order, with "forcing" involving the function μ. The other one is "a priori" not rigorous: given a function η, one can compute μ using the first equation of the system above, up to a constant of integration. Going then to the second equation, one can compute μ as $\frac{\dot{\eta}+\lambda b}{a}$. The non-rigorous issue is then related to the regularity of the functions λ, η, μ thereby produced. λ and η can be taken to be H^1, but μ is then only L^2.

This issue can be resolved using the smoothing effect that is hidden in the associated evolution equations. Let us describe this now:

3 The variational problem and the variational flows

Let $J(x) = \int_0^1 \alpha_x(\dot{x})dt$. The first variation of $J(x)$ along a tangent vector to is $\int_0^1 b\eta dt$.

This strongly suggests to take the function $\eta = b$ and build with it a vector field on C_β, a decreasing pseudo-gradient for J, $Z_0(x)$. $Z_0(x)$ then reads:

$$Z_0(x) = \lambda\xi + \frac{\dot{b} + \lambda b}{a}v + bw$$

$$\lambda = -\overline{\mu}b + \int_0^t b^2 - t\int_0^1 b^2$$

The smoothing effect does indeed exist. The evolution equation

$$\frac{\partial x}{\partial s} = Z_0(x), x(0) \in C_\beta$$

has a solution for a short positive time $s \leq \epsilon(x(0)), \epsilon(x(0)) \gneq 0$, see [3]. The smoothing effect is read on the evolution of the function b, which is the component over v of the tangent vector x to the curve. Indeed, given $z = \lambda\xi + \mu v + \eta w$, the first variation $z.b$ of the v-component b of $\dot{x}$ reads, see [3], Proposition P3, page 18, also pp20-22:

$$z.b = \dot{\mu} + a\eta\tau + b\eta d\overline{\mu}(\xi)$$

Using $z = Z_0(x)$, we find:

$$\frac{\partial b}{\partial s} = \frac{\ddot{b} + (\dot{\lambda b})}{a} + ab\tau + b^2 d\overline{\mu}(\xi)$$

with $\lambda = -\overline{\mu}b + \int_0^t b^2 - t\int_0^1 b^2$ as above.

The evolution equation, read on b, is a parabolic evolution equation that has two main features [3]: the number of zeros of b never increases along its flow-lines; the L^1-norm of b is bounded on a given flow-line. It has also a third feature: namely, the linking between the two curves under evolution never increases (or decreases, it depends on the definition that one gives to the linking number) when the time increases.

However, this semi-flow has the stringent defect to have " bad" "explosions". These "explosions" are to be expected since we are trying to deform immersed curves and, at the same time, decrease their linking. The v-component b of the tangent vector $\dot{x}$ to the curve under evolution develops Dirac masses, but it has a non-zero weak limit, a fact that is not "natural" since the variations are extremized only when b is zero and the curve is tangent to ξ. The occurrence of Dirac masses is to be expected, for various reasons, almost all of topological origins; but outside of the times when these Dirac masses occur, b should tend to zero.

4 The modified flow of [2], [3]

We have therefore modified the above flow $Z_0(x)$ in [2], [3].

We have built, curve by curve, a suitable function η and a related (semi)-evolution equation on C_β ([2], [3]) that has the following properties:

Proposition 4.1. *There exists a decreasing pseudo-gradient Z for J on C_β such that*

i) The number of zeros of b does not increase on the decreasing flow-lines of Z.

ii) On each flow-line, $\int_0^1 |b|(s) \leq C + \int_0^1 |b|(0)$.

iii) At the blow-up time, $b(s,t)$ converges weakly to $\sum_{i=1}^m c_i\delta_{t_i}$, with $|c_i| \geq c_0 > 0$.

The clear advantage of the semi-flow $Z(x)$ is that it produces a stratified set "at infinity" $\cup\Gamma_{2k}$, where an extended functional J_∞ can be defined.

Γ_{2k} is the set of curves made of $k - \xi$-pieces of orbits, alternating with $k - v$-pieces of $\pm v$-orbits.

J_∞ is then the functional defined as the addition of all the lengths a_i of the various $k - \xi$-pieces.

Proposition 4.1 is the first step in order to build a global deformation of the variational space C_β onto the union of the unstable manifolds of all critical points (periodic orbits) and critical points at infinity.

We need some tools for this global deformation.

5 Transport equations along ξ and along v, ξ-characteristic pieces of orbits, H_0^1-index and index at infinity, H_0^1-space, unstable manifolds at infinity

In order to define a global deformation from C_β onto the this set at infinity (with the addition of the unstable manifolds of the periodic orbits of ξ), we need to understand well the dynamics of v, w in the transport equations along ξ and the dynamics of ξ in the transport equation along v.

If a vector z is transported by the differential of the flow of v, then its components verify (derivatives are taken with respect to the time along v), [1], [2], [3]:

$$\dot{\overline{\lambda + \overline{\mu}\eta}} = \eta$$

$$\dot{\eta} = -\lambda.$$

On the other hand, if a vector v is transported by ξ, then its components satisfy (derivatives are taken along ξ):

$$\dot{\mu} + \eta\,\tau = 0$$

$$\dot{\eta} = \mu.$$

On the stratified space $\cup\Gamma_{2k}$, J_∞ has various critical points "at infinity", x^∞. We need some tools to describe these critical points and their various indexes:

A ξ-piece of a critical point at infinity x_∞ can be either non-degenerate (or free) or it can be characteristic. By definition, when it is characteristic, the v-rotation on this ξ-piece in a ξ-transported frame is a multiple of π, see [2] and [3]. Otherwise, it is non-degenerate.

x_∞ is made of s ξ-pieces, alternating with s $\pm v$-jumps. The space of such curves has been denoted, see above, denoted Γ_{2s}, it is (generically on v) a manifold. x_∞ is a critical point for this functional if either its ξ-pieces are non-degenerate, then its $\pm v$-jumps occur between "conjugate points", [1], [2], [3]; this means that α is mapped onto itself in the v-transport between these points; or (some of) its ξ-pieces are characteristic; then the $\pm v$-jumps have to satisfy more complicated conditions, see [2] and [3].

We then define, given x_∞, i_∞ to be the Morse index of J_∞ at x_∞.

On the other hand, we can define an H_0^1-index for x_∞: given a ξ-piece between two large $\pm v$-jumps, of length a_j, we consider the quadratic form $\int_0^1 \dot\eta^2 - a_j^2\eta^2\tau$ on the space of functions $\eta \in H_0^1[0,1]$. Its index is i_0^j, the H_0^1-index related to the j^{th}-ξ-piece of orbit. The total H_0^1-index is $i_0 = \Sigma i_0^j$.

Each i_0^j therefore corresponds to the (strict if degenerate) Morse index of the functional J when restricted to variations which do not change the $\pm v$-jumps of x_∞, but might change the j^{th} ξ-piece into a nearby curve. This nearby curve starts and ends at the same $\pm v$-jumps. it has in addition a tangent vector $\dot x$ that satisfies $\dot x = a\xi + bv$. The space of such curves, defined between a set of s-$\pm v$-"verticals", or pieces of $\pm v$-orbits, typically the ones produced by a curve x or x_∞ of Γ_{2s} is denoted the H_0^1-space in [2], [3].

When x_∞ has no characteristic ξ-pieces, its index is therefore $i_0 + i_\infty$.

When it has some characteristic ξ-pieces, x_∞ becomes a cluster of several critical points, see [3] and Appendix 2 of [4] for more details.

Indeed, on each characteristic piece, the H_0^1- quadratic form defined above is degenerate, see [3]. Such a ξ-piece has therefore a strict H_0^1-unstable manifold and a (half)-full unstable H_0^1-unstable manifold. Piecing together strict or full (half)-H_0^1-unstable manifolds for the various characteristic pieces -the strict one is always included, the only issue is whether, for a given characteristic piece, it will be extended into the full one or not, in the definition of the cycle corresponding to x_∞-, multiplying by the H_0^1-unstable manifolds of the non-degenerate ξ-pieces and by the unstable manifold in Γ_{2s} corresponding to i_∞, we build cycles. x_∞ corresponds in fact to such a cycle.

Defining then the number ℓ to be the maximal number of full (half)-H_0^1-unstable of characteristic ξ-pieces used together in this cycle, we find that the index of x_∞ (corresponding to the dimension of the cycle as a manifold) is $i_0 + i_\infty + \ell$.

The flow of [2], [3] decreases the number of zeros of the v-component of $\dot x$, b. Thus, it is important to understand what is the maximal number of zeros of b on this cycle. This has been studied in [3], p78 and p139. This number turns out to be

$$i_0 + \gamma + 2\ell,$$

where γ, see [3] p78 , is defined to be $\Sigma\gamma_j$; γ_j is defined as follows: the index j runs over the **non degenerate** ξ-pieces of x_∞. γ_j is 1 if i_0^j is even and the orientations of the incoming $\pm v$-jump and of the outgoing $\pm v$-jump do not coincide, or if i_0^j is odd and these orientations coincide. Otherwise, γ_j is zero.

There is in addition, for each of these cycles x^∞, an unstable manifold $W_u(x^\infty)$, that is built as a product of the unstable manifold at infinity (the unstable manifold in the Γ_{2s}s) with the H_0^1-unstable manifolds of the various ξ-pieces, taken as "full" or "strict" H_0^1-unstable manifolds if these ξ-pieces are characteristic ([2], [3], [4].

6 The Deformation Result

It is the deformation of the curves along the space H_0^1 defined above that we are studying in detail here. The construction of the H_0^1-pseudo-gradient has been carried out on one hand in [3], whereas the flow at infinity, that is the flow in the Γ_{2s} has been introduced and studied in [1], [2] and [3]. We study in the sequel in more details the H_0^1-flow and we show how we can fit it in a global pseudo-gradient for J near infinity in C_β.

Namely, we prove in the next two sections the following result:

Theorem 6.1. *There is a globally defined pseudo-gradient Z for J on the space C_β that does not increase the number of zeros of the v-component of the tangent vector $\dot x = a\xi + bv$ to a curve (including at infinity) and that deforms C_β at infinity on $A \cup B$, where A is the union of the unstable manifolds $\cup W_u(x_\infty)$ of the various cycles at infinity x_∞ and B is the union of the unstable manifolds of the various periodic orbits of ξ.*

The proof of Theorem 6.1 was sketched in [2] and [3]. We provide here the additional details required for the complete proof. Our object is a curve of C_β made of nearly ξ-pieces alternating with nearly $\pm v$-pieces of orbits. The (large) nearly $\pm v$-pieces of orbits are very close (in an L^∞-sense) to corresponding genuine pieces of $\pm v$-orbits. We then apply to them a combination of the H_0^1-flow, see below, with the $(Z_\nu/Z_{\frac{\nu}{2}})$-flow of [3] in order to reach the Γ_{2s} and their H_0^1-unstable manifolds (which we may view as attached to each curve). Once this combination is enacted, the use of a flow at infinity, that is a flow on the $\cup\Gamma_{2s}$s which we assume to exist-this flow is a pseudo-gradient for the functional J_∞ extending J to $\cup\Gamma_{2s}$s-is the final piece, see eg [5], that brings our variations in a neighborhood of the critical points at infinity and their H_0^1-unstable manifolds.

7 The H_0^1-semi-flow

We consider now the H_0^1-flow of [2], [4]; it is a key piece in all the arguments used to deform the curves of C_β onto the set formed by the union of the unstable manifolds of the periodic orbits with the unstable manifolds of the critical points at infinity in $\cup\Gamma_{2k}$, see [2], [4]. We describe in what follows its definition and properties in great detail.

This flow requires the choice of a differentiable family of points that are the starting and ending points of the nearly ξ-pieces, equivalently they are the ending and starting points of the nearly $\pm v$-pieces.

Let us first consider a nearly ξ-piece, defined between two points x_i^-, x_i^+. We imagine the two v-orbits through these two points. Assume that the piece of curve between these two points has $\dot{x} = a\xi + bv$, with the running time t in $[0,1]$ for the sake of simplicity. Assume that $b \in H_0^1(0,1)$ on this interval and consider the differential evolution equation:

$$\frac{\partial b}{\partial s} = \frac{\ddot{b} + a^2 b\tau}{a} + \frac{\left((\int_0^t b^2 - t\int_0^1 b^2)b - \bar{\mu}\dot{b}^2\right)^{\cdot}}{a} - b^2\bar{d}\mu_{(}\xi), b \in H_0^1(0,1)$$

If we denote $b_{|[0,1]}$, b_i, then we can introduce the "tangent vector":

$$Z(x) = (\int_0^t b_i^2 - t\int_0^1 b_i^2)\xi +$$

$$+ \frac{\dot{b}_i + (\int_0^t b_i^2 - t\int_0^1 b_i^2 - \bar{\mu}b_i)b_i}{a}v + b_i w$$

This "tangent" vector is defined only on the portion of curve between x_i^- and x_i^+. At time 0 and at time 1, $Z(x)$ is parallel to v, so that $x_i^\pm$ move along the v-orbits that they (respectively) define.

We have established in [2], pp39-49 and [3], pp123-134 existence, continuity etc for this flow. We refine here the results of [2] and [4]. We consider the operator:

$$A = -(\ddot{\eta} + a^2\eta\tau)$$

under $H_0^1(0,1)$ boundary conditions. The nearly piece of ξ-orbit between x_i^- and x_i^+ is close to a ξ-orbit. Assuming for simplicity (we will discuss the more general case later) that it is "far" from being characteristic, this ξ-piece can be identified as the unique piece of ξ-orbit $\varkappa$ connecting the two v-orbits through x_i^- and x_i^+. It has a Morse index, which we denote i_0^i. Accordingly the operator A defined above, under its boundary conditions, has the same index.

Let A_0 be this operator for the piece of ξ-orbit and let $E^+ \oplus E^-$ be the related decomposition on positive and negative eigenspaces for the L^2-scalar product. b_i can then be decomposed into $b_i^+ + b_i^-$. The following differential inequalities satisfied by b_i^+, b_i^- are not difficult to establish (one uses in particular the equivalence of norms in E^- which is finite dimensional):

$$\frac{\partial \int b_i^{+2}}{\partial s} \leq -c_0 \int \dot{b_i}^{+2} + o(\int b_i^{-2})$$

$$c_2 \int b_i^{-2} + o(\int b_i^{+2}) \leq \frac{\partial \int b_i^{-2}}{\partial s} \leq$$

$$\leq c_1 \int b_i^{-2} + o(\int b_i^{+2})$$

A more difficult estimate reads as follows: We write the evolution equation, under $Z(x)$ defined above, of the v-component of $\dot{x}$, b_i, on the i^{th} ξ-piece of orbit; $\lambda + \bar{\mu}\eta$ is $(\int_0^t b_i^2 - t\int_0^1 b_i^2)b_i$. We have:

$$\frac{\partial b_i}{\partial s} = \frac{\ddot{b}_i + a^2 b_i\tau}{a} + \frac{\left((\int_0^t b_i^2 - t\int_0^1 b_i^2)b_i - \bar{\mu}\dot{b}_i^2\right)^{\cdot}}{a} - b_i^2\bar{\mu}_\xi$$

We project onto the space E^+ of A_0. We find:

$$\frac{\partial b_i^+}{\partial s} = \frac{\ddot{b}_i^+ + a_0^2 b_i^+\tau}{a_0} + (o(b_i) - o(b_i)^-) + (\frac{\left((\int_0^t b_i^2 - t\int_0^1 b_i^2)b_i - \bar{\mu}\dot{b}_i^2\right)^{\cdot}}{a} - b_i^2\bar{\mu}_\xi) - (\frac{\left((\int_0^t b_i^2 - t\int_0^1 b_i^2)b_i - \bar{\mu}\dot{b}_i^2\right)^{\cdot}}{a} - b_i^2\bar{\mu}_\xi)^-$$

We multiply the above equation by $-\ddot{b}_i^+ = -\ddot{b}_i + \ddot{b}_i^-$ and we integrate between 0 and 1.

The most difficult term is

$$\int_0^1 (\bar{\mu}\dot{b}_i^2)\ddot{b}_i$$

This gives rise to terms that are $O(\int_0^1[(b_i^2 + |b_i^3|)|\ddot{b}_i|])$ and $O(\int_0^1[|b_i||\dot{b}_i||\ddot{b}_i|])$. Estimating $\int_0^1(\frac{\left((\int_0^t b_i^2 - t\int_0^1 b_i^2)b_i\right)^{\cdot}}{a})\ddot{b}_i$ requires the addition to these terms of $O(\int_0^1 b_i^2 \times \int_0^1[(|b_i| + |\dot{b}_i|)|\ddot{b}_i|])$.

Observe that:

$$\int_0^1 b_i^2|\ddot{b}_i| \leq (\int_0^1 \ddot{b}_i^2)^{\frac{1}{2}} \times (\int_0^1 \dot{b}_i^2)^{\frac{1}{2}} \times (\int_0^1 b_i^2)^{\frac{1}{2}} = o(\int_0^1 (\ddot{b}_i^2 + \dot{b}_i^2))$$

Also:

$$\int_0^1 [|b_i||\dot{b}_i||\ddot{b}_i|] + \int_0^1 b_i^2 \times \int_0^1 [(|b_i| + |\dot{b}_i|)|\ddot{b}_i|] = o(\int_0^1 (\ddot{b}_i^2 + \dot{b}_i^2))$$

We are left with $\int_0^1 |b_i^3||\ddot{b}_i|$. For this we use the Nash inequality [6], L.Nirenberg [7], in dimension 1:

$$\int_0^1 b_i^6 \leq C(\int_0^1 b_i^2)^2 \int_0^1 \dot{b}_i^2$$

This yields:

$$\int_0^1 |b_i|^3|\ddot{b}_i| \leq C_1(\int_0^1 \ddot{b}_i^2)^{\frac{1}{2}} \times \int_0^1 b_i^2 \times (\int_0^1 \dot{b}_i^2)^{\frac{1}{2}} = o(\int_0^1 (\ddot{b}_i^2 + \dot{b}_i^2))$$

All terms containing a projection u^- onto E^- are in fact combination of a finite set of functions that span E^- with suitable coefficients. With an L^2-orthonormal basis $f_1, ..., f_s$, the coefficients are $\int_0^1 uf_j$. The time-derivatives in all these terms can be switched, after integration by parts, to be taken on u^-; therefore, they are in fact taken on the f_j and $u^-, \dot{u^-}, \ddot{u^-}$ are all L^∞ bounded by $Sup(\int_0^1 uf_j)$. One can also consider $(\dot{u})^-, (\ddot{u})^-, (\dddot{u})^-$. The component of these terms on f_j are equal to as $-\int_0^1 u\dot{f}_j, \int_0^1 \ddot{f}_j, \int_0^1 u\dddot{f}_j$. There are therefore bounded by $|u|_{L^1}$.

In addition,

$$\int u^-\ddot{b}_i^+ = \int u^- A_0(b_i^+) - \int u^- a_0^2 b_i^+ \tau = -\int u^- a_0^2 b_i^+ \tau$$

This implies that:

$$\int_0^1 [(\frac{(\int_0^t b_i^2 - \dot{t}\int_0^1 b_i^2)b_i - \bar{\mu}\dot{b}_i^2}{a} - b_i^2\bar{\mu}_\xi)^- + o(b_i)^-]\ddot{b}_i^+ = o(\int_0^1 b_i^2)$$

On the other hand, using our observations above and the fact that all norms are equivalent on E^-,

$$\int_0^1 (\frac{(\int_0^t b_i^2 - \dot{t}\int_0^1 b_i^2)b_i - \bar{\mu}\dot{b}_i^2}{a} - b_i^2\bar{\mu}_\xi)\ddot{b}_i^- = o(\int_0^1 b_i^2)$$

Using the fact that b_i is in H_0^1 and the equivalence of norms in E^-, we also have:

$$\int_0^1 (a_0\tau b_i^+ + o(b_i))(\ddot{b}_i - \ddot{b}_i^-) = o(\int_0^1 \ddot{b}_i^2) + O(\int_0^1 \dot{b_i^+}^2) + o(\int_0^1 {b_i^-}^2)$$

We thus find, using the equivalence of norms on E^-:

$$\frac{\partial \int_0^1 \dot{b_i^+}^2}{\partial s} \leq -C_2 \int_0^1 \ddot{b_i^+}^2 + O(\int_0^1 \dot{b_i^+}^2) + o(\int_0^1 {b_i^-}^2)\,(1)$$

We recall that we also have:

$$\frac{\partial \int b_i^{+2}}{\partial s} \leq -c_0 \int \dot{b_i}^{+2} + o(\int b_i^{-2})(2)$$

Combining (1) and (2), with the use of a suitable constant $C \geq \frac{2}{c_0}$ (we will use this inequality later), we find:

$$\frac{\partial(\int(\dot{b_i^+}^2 + Cb_i^{+2}))}{\partial s} \leq -C_3(\int \ddot{b_i^+}^2 + \int \dot{b_i}^{+2}) + o(\int b_i^{-2})(3)$$

whereas:

$$c_2 \int b_i^{-2} + o(\int b_i^{+2}) \leq \frac{\partial \int b_i^{-2}}{\partial s} \leq (4)$$
$$\leq c_1 \int b_i^{-2} + o(\int b_i^{+2})(5)$$

We use in the sequel the three differential inequalities (3), 4 and 5. $\delta_1 \ll \delta_0$ are two positive constants. $|b_i(0)|_{L^2}^2$ is assumed to be $\leq \delta_1$. We first claim:

Lemma 7.1. *There exists a positive constant C_1, independent of δ_0, δ_1, such that,*

(i) if $\int_0^1 b_i^{+2}(0) \leq C_1 \int_0^1 b_i^{-2}(0)$, then for any later time s and as long as both quantities are small (measured with a fixed positive constant ϵ_0), $\int_0^1 b_i^{+2}(s) \leq 2C_1 \int_0^1 b_i^{-2}(s)$. $\int_0^1 b_i^{-2}(s)$ increases then exponentially whereas there exists a constant C_2, independent of δ_0, δ_1, such that $\int_0^1 \dot{b}_i^{+2}(s) \leq o(\int_0^1 b_i^{-2}(s)) + C_2 \int_0^1 b_i^{+2}(s-1)$ for $s \geq 1$ as long as $\int_0^1 b_i^{+2}(s)$ and $\int_0^1 b_i^{-2}(s)$ are small (measured as above). Finally, on each $[s-1, s]$, there exists then some time s' such that $\int \ddot{b_i^+}^2(s') \leq o(\int_0^1 b_i^{-2}(s)) + C_2 \int_0^1 b_i^{+2}(s-2)$ for $s \geq 2$.

(ii)If $\int_0^1 b_i^2(0) \leq \delta_1$ and if $\int_0^1 b_i^{+2}(0) \geq \frac{C_1}{2} \int_0^1 b_i^{-2}(0)$, then $(\int \dot{b_i^+}^2 + C \int b_i{}^{+2})(s)$ decreases exponentially or faster as long as $\int_0^1 b_i^{+2}(s) \geq C_1 \int_0^1 b_i^{-2}(s)$ and, with a suitable positive constant c, the following estimate holds: $\int_0^1 b_i^2(s) \leq \delta_1 e^{-cs}(1 + \frac{2}{C_1})$. In particular, either for some time s_0 (which we assume then to be the first time for which this inequality holds), $\int_0^1 b_i^{+2}(s_0) \leq C_1 \int_0^1 b_i^{-2}(s_0)$, with $\int_0^1 b_i^2(s_0) \leq \delta_1 e^{-cs_0}(1 + \frac{2}{C_1})$: (i) then applies; or $\int_0^1 b_i^2(s)$ tends to zero and the flow-line never exits a neighborhood of the rest points through the boundary defined by the inequality $\int_0^1 b_i^2 \leq \delta_0$.

Proof. (ii) follows readily from (3) and (4).

We thus prove (i). Assuming $\int_0^1 b_i^{+2}(0) \geq \frac{C_1}{2} \int_0^1 b_i^{-2}(0)$, then we derive from (5) that $\int_0^1 b_i^{-2}(s)$ increases for s small. As long as $\int_0^1 \dot{b_i^+}^2(s)$ is not $o(\int_0^1 b_i^{-2})(s)$, (1) implies that $\int_0^1 b_i^{+2}(s)$ decreases and therefore $\int_0^1 b_i^{+2}(s) \leq 2C_1 \int_0^1 b_i^{-2}(s)$ holds. However, if $\int_0^1 \dot{b_i^+}^2(s)$ is $o(\int_0^1 b_i^{-2})(s)$, then b_i being in H_0^1, this inequality holds without further argument; $\int_0^1 b_i^{-2}(s)$ goes on as an increasing function of s and the assumptions of (i) at time zero are satisfied at all further time s as long as $\int_0^1 b_i^2(s)$ is small.

(5) implies then that $\int_0^1 b_i^{-2}$ increases exponentially. Starting from δ_1, with $\delta_1 \ll \delta_0$, it takes a long time to reach the level $\frac{2\delta_0}{3}$. Let us consider the time interval $[s, s+1], s \geq 0$.

Either for some $s_1 \in [s, s+1]$, $\int_0^1 \dot{b_i^+}^2(s_1)$ is $o(\int_0^1 b_i^{-2})(s_1) = o(\int_0^1 b_i^{-2})(s+1)$ or (3) implies that:

$$\frac{\partial \int b_i^{+2}}{\partial s} \leq -\frac{c_0}{2} \int \dot{b_i}^{+2}, x \in [s, s+1]$$

Integrating between s and $s+1$, we find:

$$\int b_i^{+2}(s+1) + \frac{c_0}{2} \int_s^{s+1} \int_0^1 \dot{b_i}^{+2}(s,t) dt dx \leq \int b_i^{+2}(s)$$

It follows that, for some time $s_1 \in [s, s+1]$:

$$\int_0^1 \dot{b_i}^{+2}(s_1, t) dt \leq \frac{2}{c_0} \int b_i^{+2}(s)$$

Over both cases, we can claim the existence of $s_1 \in [s, s+1]$ such that:

$$\int_0^1 \dot{b_i}^{+2}(s_1, t) dt \leq \frac{2}{c_0} \int b_i^{+2}(s) + o(\int_0^1 b_i^{-2})(s+1)$$

Integrating then (4) between s_1 and $s+1$ and using the increasing property of $\int_0^1 b_i^{-2}(s)$, we find:

$$\int (\dot{b_i^+}^2 + C b_i^{+2})(s+1) \leq o(\int_0^1 b_i^{-2})(s+1) + \int (\dot{b_i^+}^2 + C b_i^{+2})(s_1)$$

Using then the above inequality, we derive that:

$$\int_0^1 \dot{b}_i^{+2}(s+1) \leq o(\int_0^1 b_i^{-2}(s+1)) + C_2 \int_0^1 b_i^{+2}(s)$$

for $s \geq 0$, or the inequality of (i) at time s for $s \geq 1$ as stated. Having established this inequality, we integrate the second inequality over the interval $[s-1, s]$, with $s \geq 2$. (ii) follows. □

We also have:

Lemma 7.2. *Under the conditions of (i) of Lemma 7.1, the following estimate holds for a suitable fixed positive constant c:*

$$\int_0^1 b_i^{+2}(s) \leq \int_0^1 b_i^{-2}(s)(o(1) + ce^{-cs})$$

Proof. We consider (3). At any time s, either $\int_0^1 b_i^{+2}(s) = o(\int_0^1 b_i^{-2}(s))$ or $\int_0^1 b_i^{+2}(s)$ is exponentially decreasing. Since $\int_0^1 b_i^{-2}$ increases, the claim follows then from the assumption in (i) at the initial time. □

Combining the conclusions of Lemma 7.1 and Lemma 7.2, we see that we can assume, on a flow-line that exits a neighborhood of the rest points at infinity, that from the time $\underline{s}$, which we set for convenience to be 0, at which $\int b_i^2 = \sqrt{\delta_1} \ll \delta_0$, to the exit time $\bar{s}$ at which $\int b_i^2 = \delta_0$, we do have:

$$\int_0^1 b_i^{+2} = o(\int_0^1 b_i^{-2}(s))$$

We then claim:

Lemma 7.3. *At the exit time, the function b_i has at most $(i_0^i - 1)$ interior zeros.*

Proof. Once b_i has $(i_0^i - 1)$ or less at some time s, the non-increasing property of the number of zeros of b_i, a feature of the differential equation that it verifies, implies the result for later times.

On the other hand, (i) of Lemma 7.1 combined with Lemma 7.2 implies that on each $[s-1, s]$, s large enough, there is a time s' such that:

$$|b_i^+|_{C^1} = o(|b_i^-|_{L^2})$$

Consider a non-zero function c_i^- of E^-. Assume now that there exists a fixed positive constant c_3 and another small fixed positive constant ρ such that, near each value t_0 such that $c_i^-(t_0) = o(|c_i^-|_{L^2})$, the following estimate holds: (***)

$$|c_i^{-'}(t)| \geq c_3 |c_i^-|_{L^2}, t \in [t_0 - \rho, t_0 + \rho]$$

It then follows that if we add to c_i^- a function c_i^+ satisfying:

$$|c_i^+|_{C^1} = o(|c_i^-|_{L^2})$$

then the addition c_i of both functions has not more zeros than c_i^-, that is $(i_0^i - 1)$-zeros at most. This follows from a simple application of the mean value theorem to the function c_i: its zeros must be very close to values t_0 of the parameter t for which $c_i^-(t_0) = o(|c_i{}^-|_{L^2})$. The assumption on $c_i^{-'}(t)$ in a uniform neighborhood of t_0 allows then to reach the stated conclusion.

$o(|c_i^-|_{L^2}), o(|b_i^-|_{L^2})$ are here $\leq \delta' |c_i^-|_{L^2}, \delta' |b_i^-|_{L^2}$, where δ' is as small as we please, whereas c_3, ρ are fixed constants, albeit small.

Lemma 7.3 then follows from the claim that $b_i^-(s)$ will satisfy the condition on c_i^- for some $s \in [0, \bar{s}]$.

To see why this claim holds, we come back to the evolution equation satisfied by b_i. $f_1, ..., f_p$ is an orthonormal basis of E^-. E^- is the negative eigenspace of the operator $-(\ddot{\eta} + a_0^2 \eta \tau_0)$ under Dirichlet boundary conditions. We may assume that $f_1, ..., f_p$ are its normalized eigenfunctions. Let $w_1(s), ..., w_p(s)$ be the components of $b_i^-(s)$ along $f_1, ..., f_p$. Multiplying the evolution equation by f_j, integrating between 0 and 1, integrating by parts (all f_j are C^∞), using the fact that $\int_0^1 b_i^{+2} = 0(\int_0^1 b_i^{-2}(s)) = o(\Sigma |w_i|^2)$, we find that:

$$\frac{\partial w_j}{\partial s} = -\mu_j w_j + o(\Sigma |w_i|)$$

The $-\mu_j$s are the negative eigenvalues of the operator $\frac{\ddot{\eta} + a_0^2 \eta \tau_0}{a_0}$. If we remove $o(\Sigma |w_i|)$, this rereads:

$$\frac{\partial b_i^-}{\partial s} = -\frac{A_0}{a_0} b_i^-$$

$u_i = \frac{b_i^-}{|b_i^-|_{L^2}}$ then satisfies:

(1)

$$\frac{\partial u_i}{\partial s} = -\frac{A_0}{a_0} u_i - \lambda u_i$$

where λ is a constant in time, that varies with s, derived from the fact that $|u_i|_{L^2} = 1$. The actual evolution differential equation on $u_i = \frac{b_i^-}{|b_i^-|_{L^2}}$ reads in fact:

(1)'

$$\frac{\partial u_i}{\partial s} = -\frac{A_0}{a_0} u_i - \lambda_1 u_i + o(1)$$

λ_1 behaves as λ does.

Considering (1), we recognize that its rest points are the f_js. We therefore claim that either $u_i(0)$ is in a small neighborhood of one of the $f_j, j = 1, ..., p$, so small that $(***)$ is satisfied at $u_i(0)$. Or $u_i(0)$ is in none of these neighborhoods. Then, $u_i(s)$ has to enter such a neighborhood before some a priori bounded (perhaps large, but a priori bounded) time s_0. The same property holds then for $(1)'$ if $o(1)$ is small enough. We then take δ_1 so small with respect to δ_0 that $\bar{s} \geq s_0$. The conclusion then follows.qed □

We next establish the following Lemma, Lemma 7.4, that holds also under periodic boundary conditions, completely unchanged:

Lemma 7.4. *Let $x(s) = x(s,t)$ be the solution of the differential equation corresponding to the H_0^1-flow $\frac{\partial x}{\partial s} = Z(x)$ and let $b(s,t)$ be the v-component of $\frac{\partial x}{\partial t}$ between two of the v-verticals of $x(s)$. $b(s,t)$ is the solution of the evolution partial differential equation:*

$$\frac{\partial b}{\partial s} = \frac{\ddot{b} + a^2 b\tau}{a} + \frac{(\int_0^t b^2 - \dot{t}\int_0^1 b^2)b - \bar{\mu}\dot{b}^2}{a} - b^2\bar{\mu}_\xi, b \in H_0^1(0,1)$$

Let T be the blow-up time for this equation. There exists a positive constant c_5 such that if $\overline{lim} \int_0^1 |b(s,t)|dt \leq c_5$ as s tends to T from below, then $T = \infty$, $b(s,t) \in H^1(0,1)$ exists for all time s. In addition, $\int_0^1 (b(s,t)^2 + \dot{b}(s,t)^2)dt$ tends to zero as s tends to ∞, $\int_0^\infty \int_0^1 |\ddot{b}|^2 dtds \lneq \infty$ and the end-points $x(s,0)$ and $x(s,1)$ of the piece of curve between the two v-verticals of the curve x converge as s tends to ∞.

Corollary 7.5. *Consider the same evolution equation as in Lemma 7.4. Assume that $\int_0^1 \alpha_x(\dot{x}) \leq a_0$ and assume that $\int_0^1 |b(0,t)|dt \leq \frac{c_5}{2}$. There exists a positive constant c_6, depending only on c_5 and a_0, such that, if the blow-up time T is finite, then $\int_0^1 \alpha_x(\dot{x})dt(T^-) \leq \int_0^1 \alpha(\dot{x})dt(0) - c_6$.*

Proof. (Proof of Lemma 7.4) We are changing under $Z(x)$ portions of a given curve of C_β^+, which has a set of v-verticals under the H_0^1-flow. This flow does not change the v-verticals, but tries to evolve the portions of curves connecting them to pieces of ξ-orbits. $b(s,t)$ designates therefore the v-component of the time-derivative of x between two of these verticals. We take the evolution equation satisfied by b, multiply it by b and integrate between 0 and 1. We find with suitable constants c and C:

$$\frac{\partial \int_0^1 b^2}{\partial s} + c\int_0^1 \dot{b}^2 \leq C(\int_0^1 b^4 + \int_0^1 b^2)$$

Using the Nash inequality for $n = 1$ [10], [11], we bound $\int_0^1 b^4$ by $C(\int_0^1 |b|)^2 \times \int_0^1 \dot{b}^2$. Taking c_5 small enough, we can absorb this term in $c\int_0^1 \dot{b}^2$. Keeping the same notations for the sake of simplicity, we find:

$$\frac{\partial \int_0^1 b^2}{\partial s} + c\int_0^1 \dot{b}^2 \leq C\int_0^1 b^2$$

The basic equation is $\frac{\partial x}{\partial s} = Z(x)$ and it implies that $\frac{\partial \int_0^1 \alpha_x(\dot{x})}{\partial s} \leq -\int_0^1 b^2$ (we do not have equality because we might have several distinct nearly ξ-pieces and we might be using the H_0^1-flow on each of them. We thus know that, whatever the blow-up time T might be, $\int_0^T \int_0^1 b^2 \leq \infty$. Integrating, this implies that:

$$\int_0^1 b^2 dt(T^-) + \int_0^T \int_0^1 \dot{b}^2 dtds \leq \int_0^1 b^2 dt(0) + C\int_0^T \int_0^1 b^2 dtds$$

In fact, $\int_0^1 b^2(s)$ is bounded independently of $s \in [0,T]$.
We now multiply the evolution equation on b by $-\ddot{b}$. We observe that, with C_1 a large constant,

$$\int_0^1 |b|^3|\ddot{b}|dt \leq C_1\int_0^1 b^6 + \frac{1}{C_1}\int_0^1 \ddot{b}^2 \leq C_1'(\int_0^1 b^2)^2 \times \int \dot{b}^2 + \frac{1}{C_1}\int_0^1 \ddot{b}^2$$

Since $\int_0^1 b^2$ is bounded, we derive that

$$\int_0^1 |b|^3|\ddot{b}|dt \leq C_2 \times \int \dot{b}^2 + \frac{1}{C_1}\int_0^1 \ddot{b}^2$$

We also observe that

$$\int_0^1 \bar{\mu}\dot{b}^2\ddot{b}dt \leq O(\int_0^1 (b^2 + |b^3|)|\ddot{b}|dt) + 2\int_0^1 b\bar{\mu}\dot{b}\ddot{b}dt$$

$\int_0^1 |b^3||\ddot{b}|dt$ has been estimated above. On the other hand, using again [3],

$$\int_0^2 b^2|\ddot{b}|dt \le (\int_0^1 b^4)^{\frac{1}{2}} \times (\int_0^1 |\ddot{b}|^2)^{\frac{1}{2}} \le C_2 \int_0^1 \dot{b}^2 + \frac{1}{C_1}\int_0^1 |\ddot{b}|^2$$

We are left with $\int_0^1 b\bar{\mu}\dot{b}\ddot{b}dt$. This is upper-bounded by $C_1\int_0^1 b^2\dot{b}^2 + \frac{1}{C_1}\ddot{b}^2$. Integrating by parts in $\int_0^1 b^2\dot{b}^2$, we find that this rereads $\frac{1}{3}\int_0^1 b^3\ddot{b}$. This has been already estimated. Finally,

$$\int_0^1 \ddot{b}\frac{d(b[\int_0^1 b^2 - t\int_0^1 b^2])}{dt} = O(\int_0^1 |b|^3|\ddot{b}|) + \int_0^1 \ddot{b}\dot{b}(\int_0^1 b^2 - t\int_0^1 b^2)dt = O(\int_0^1 |b|^3|\ddot{b}|) + O(\int_0^1 \dot{b}^2 b^2)$$

All these terms have been estimated above. Taking C_1 large enough, we derive, with suitable constants c and C:

$$\frac{\partial \int_0^1 \dot{b}^2}{\partial s} + c\int_0^1 \ddot{b}^2 \le C\int_0^1 \dot{b}^2$$

Combining this with the estimate on $\frac{\partial \int_0^1 b^2}{\partial s} + c\int_0^1 \dot{b}^2$ above, we derive with the use of another large constant C':

$$\frac{\partial(\int_0^1 \dot{b}^2 + C'\int_0^1 b^2)}{\partial s} + c\int_0^1 \ddot{b}^2 \le C_3\int_0^2 b^2$$

We derive from this inequality that $T = \infty$. Since $\int_0^\infty\int_0^1 b^2 dtds$ is then finite, we derive also that $\int_0^\infty\int_0^1(\dot{b}^2 + \ddot{b}^2)dsdt \le \infty$. It follows that $\int_0^1 b^2 + \int_0^1 \dot{b}^2$ must tend to zero. The piece of curve, which carries a finite amount of "energy" ($\int_0^1 \alpha_x(\dot{x})dt$ is decreasing, positive), must converge to a piece of ξ-orbit connecting the two preassigned verticals. Because these form an isolated set and because $\int_0^1 b^2$ tends to zero $x(s,0$ and $x(s,1)$ must converge as s tends to ∞. Lemma 7.4 is thereby established. □

Proof. (Proof of Corollary 7.5)

We can prove for this evolution equation that, for almost every $\nu \gneq 0$, the following estimate holds, see [3], p33:

$$\frac{\partial \int_0^1 (|b|-\nu)^+}{\partial s} \le \frac{C}{\nu}\int_0^1 b^2 \le -\frac{C}{\nu}\frac{\partial(\int_0^1 \alpha_x(\dot{x})dt)}{\partial s}$$

Taking $\nu \le \frac{c_5}{6}$ and using the above inequality, we derive after integration Corollary 1.qed □

8 Deforming a "nearly" $\pm v$-jump to "infinity" keeping $b\eta \ge 0$

The condition $b\eta \ge 0$, which we have used at each step of the H_0^1-flow and which we have to give up with the flow at infinity has the important consequence that, under the H_0^1-flow, the linking number of the curves under deformation with the periodic orbits of the Reeb vector-field ξ never increases. We want to build a decreasing pseudo-gradient on the $\cup\Gamma_{2s}$s and their unstable manifolds without destroying this property. This property can (and will) be destroyed only by the flow **at infinity**.

We now consider a nearly $\pm v$-jump. It might contain some back and forth nearly runs along v if b has zeros. Let us assume, in a first step, that b has no zero along this nearly $\pm v$-jump and let us assume that it is eg a $+v$-jump.

We are given a constant C_0. This constant will depend on the geometry of the contact form α along v. We divide the nearly v-jump into sub-pieces of length (counted along v) ℓ between $\frac{C_0}{2}$ and C_0.

We consider one of these sub-pieces, between its two extremal points y_i^- and y_i^+. b on this sub-piece, see [3], is very close in the L^1-topology to a very large constant $|b|_\infty$. The time t spanned between these two extremal points is therefore very small.

Let us consider the two v-orbits, through y_i^- and through y_i^+. The sub-piece of curve that we are considering is "small" (depending on the value of C_0) and runs from one v-orbit to the other one. On each v-orbit, a point is "above" another one if the piece of v-orbit between them is along $+v$. It is "below" if the piece of v-orbit between them is along $-v$. We claim:

Lemma 8.1. *(i) There is a unique "small" piece of orbit of ξ running from a point z_i^- on the first v-orbit "above" y_i^- to a point z_i^+ on the second v-orbit "below" y_i^+.*

(ii) Under a J_∞-decreasing, satisfying $b\eta \ge 0$, this sub-piece of curve will converge to the curve made of the v-orbit from y_i^- to z_i^- combined with the piece of ξ-orbit from z_i^- to z_i^+, followed by the piece of v-orbit from z_i^+ to y_i^+.

Proof. We first prove (i). We consider a small section σ to v at y_i^-. The v-orbit through y_i^+ intersects σ at a point u_i. We may assume that ξ is tangent to σ and we may consider coordinates of σ where ξ is constant. Let w_0 be a vector-field in σ independent of ξ. We may assume that w_0 and ξ commute; therefore, we may assume that they are both constant.

We pull back to σ, using the one-parameter group γ_s of v the sub-piece of curve. We find a curve in σ running from y_i^- to u_i. Let $s(t)$ be the time required along $-v$ for the pull-back. $x(t), t \in [0,1]$ denotes the sub-piece of curve. Let us denote x_s the v-orbit through y_i^-. s will be running from 0 to $s_0 = s(1)$. The tangent vector to the curve after pull-back is $d\gamma_{-s(t)}(\xi(x(t))$. Because the sub-piece of curve is a nearly v-piece, we can write:

$$d\gamma_{-s(t)}(\xi(x(t)) = d\gamma_{-s(t)}(\xi(x_{s(t)})) + o(1) = a_1(t)(\xi + c_1(t)w_0)$$

$a_i(t)$ is close to 1 because C_0 is small. We can re-parametrize the curve so that the component of the tangent vector on ξ is now 1:

$$\xi + c_2(t)w_0, t \in [0,\epsilon]$$

On the other hand, ξ can be seen $ker\beta$ transversally to v and β is a contact form with v in its kernel. Therefore, if C_0 is small enough and if the frame (ξ, v, w_0) has the proper orientation (otherwise, change w_0 into $-w_0$):

$$d\gamma_{-s}(\xi(x_s)) = a(s)(\xi + c(s)w_0)$$

with $a(s)$ positive, close to a constant, and $c(s)$ an increasing function of s, for s small in $[0, s_0]$. This follows from the monotone rotation of $ker\beta$, that is of ξ, in a v-transported frame. Replacing ξ by the re-scaled $\frac{\xi(x_s)}{a(s)}$, we find that the pull-back vector is directed by $\xi + c(s)w_0$. If instead of the vector $\frac{\xi(x_s)}{a(s)}$ at x_s, we consider a small piece of curve tangent to $\lambda\xi$, starting at x_s, during the time ϵ, we find after pull-back a piece of curve on σ tangent to $\xi + c(s,t)w_0, t \in [0,\epsilon]$ (the choice λ is embedded in the way the tangent vector reads after pull-back: the component of this vector on ξ is identically 1). The function of t defined by $c(s,t) - c(s)$ is $O(\epsilon)$, uniformly for $s \in [0, s_0]$, in the C^1-sense to the least. We claim that, under our assumptions, there is a positive constant δ which depends only on C_0 such that, if ϵ is small enough:

$$\delta c(s_0) \leq \frac{\int_0^\epsilon c_2(t)dt}{\epsilon} \leq (1-\delta)c(s_0)$$

Indeed, as ϵ tends to zero, this estimate reduces to a "limiting" estimate along a piece of v-orbit through y_i^- of length s_0. s_0 is of the same order than C_0. The function $c(s)$ defined above is strictly monotone increasing, with a derivative bounded away from zero.The estimate follows.

Observe that we also have, after the same arguments:

$$\delta_1 C_0 \leq c(s_0) \leq \delta_2 C_0$$

δ_1, δ_2 are again here positive constants that depend only on C_0.

The function $\theta(s,\epsilon) = \frac{\int_0^\epsilon c(s,t)dt}{\epsilon}$ is a monotone increasing function of s (following the strict monotonicity of $c(s)$, that is the positivity of its derivative). It is equal, uniformly for ϵ small, to $O(\epsilon)$ for $s = 0$ and it is equal to $c(s_0) + O(\epsilon)$ for $s = s_0$, with s_0 of the same order than the fixed constant C_0, whereas ϵ is as small as we please. The equation:

$$\theta(s,\epsilon) = \frac{\int_0^\epsilon c_2(t)dt}{\epsilon}$$

has therefore a unique solution $\bar{s}$ and, using the above estimates on $\frac{\int_0^\epsilon c_2(t)dt}{\epsilon}$, we can assert the existence of a small positive constant δ_3 that depends only on C_0 such that:

$$\delta_3 C_0 \leq \bar{s} \leq (1-\delta_3)C_0$$

(i) then follows. $\square$

Let us solve, under Dirichlet boundary conditions for η on the sub-piece, the following linear differential equation in η on the interval $[0,\epsilon]$:

$$\frac{\ddot{\eta} + a^2\eta\tau - (\bar{}\,\dot{b}\eta\mu)}{a} + \frac{\dot{((\int_0^t b\eta - t\int_0^1 b\eta)b)}}{a} - b\eta\bar{\mu}_\xi = -b$$

We claim that:

Lemma 8.2. *Assume that b is positive and that $\frac{C_0}{2} \leq \epsilon|b|_\infty \leq C_0$. Then, the solution η satisfies $b\eta \geq 0$.qed*

Proof. Assume that η is negative somewhere on $[0,\epsilon]$. Up to a change of notations, we might as well assume that η is negative all over this interval. Multiplying the equation by η and integrating on this interval, we find:

$$-\int_0^\epsilon \dot{\eta}^2 + \int_0^\epsilon b\eta + o(|b|^2_\infty)\int_0^\epsilon \eta^2 + O(|b|_\infty \int_0^\epsilon |\eta\dot{\eta}|) = 0$$

Since η is in $H^1_0(0,\epsilon)$, this implies:

$$-\frac{1}{2}\int_0^\epsilon \dot{\eta}^2 + \int_0^\epsilon b\eta + o(|b|^2_\infty)\int_0^\epsilon \eta^2 \geq 0$$

We know that $\frac{C_0}{2} \leq \epsilon|b|_\infty \leq C_0$. The conclusion follows. □

9 Convex-combination of the semi-flows

We indicate in what follows how to build a global deformation out of the various pieces that we have defined for it.

The construction of our deformation has two essential pieces. One is the Z_ν-semi-flow of [3], the other one is the H^1_0-semi-flow of [2] and [3]. A natural question is to understand how they can be convex-combined into the same global semi-flow.

After having read pp 1-91 of [3], the reader is advised to jump to the pp 184-186 "a direct way to reach the ν or $\tilde{\nu}$-stretched curves. pp 91-183 can be skipped without serious damage to the understanding. This takes care of the Z_ν-semi-flow. [2] and the present paper give all the necessary estimates (they can be improved) for the H^1_0-semi-flow.

The convex-combination of these semi-flows is not obvious because they (a priori) require different spaces for their definition. For the Z_ν-flow, the v-component of $\dot{x}$, $\dot{x}$ being the tangent vector to the curve x of C_β, which we usually denote b needs to be H^1. Using for a short time the regularizing semi-flow that has $\eta = b$, see [3], we can assume that b verifies this assumption. For the H^1_0-semi-flow, we need to have defined nearly large $\pm v$-jumps, and between them nearly ξ-pieces. The H^1_0-semi-flow "slides" then the ends of the nearly ξ-pieces along the nearly $\pm v$-jumps (suitably extended) and seeks to transform the nearly ξ-piece in a genuine ξ-piece. It is called an H^1_0-semi-flow because the w-component of the generalized (H^{-1}) tangent vector that defines it, η is H^1_0, η being zero at both ends of each nearly ξ-piece. Just as for the Z_ν-flow of [3], this H^1_0-semi-flow admits a "compactification", an approximation by a finite dimensional, compact, locally Lipschitz vector-field, see for Z_ν pp 59-70 of [3], the flow Z_ϵ defined using $\eta = \Phi_\epsilon(b)$ in particular. This compactification can be completed for the H^1_0-semi-flow as well, so that one could think that the convex-combination of Z_ν with the finite-dimensional Lipschitz vector-field becomes possible. Only that this approximation lives once these nearly large $\pm v$-pieces are well-defined and extended.

We would hope that the Z_ν-semi-flow would bring us to such curves that would have definite almost large $\pm v$-jumps. It almost does this job, only that b is driven by this semi-flow, see [3], at the blow-up time, to be in the L^1-sense close to a profile where almost Dirac masses in b build (these are "plateaux" where b is almost constant, equal to a very large number $\pm|b|_\infty$) followed by very fast (on sets of support equal in measure to $O(\frac{1}{|b|^N_\infty})$, N as large as we please) decreases to "plateaux" where $b = \pm\nu$, only to fall, again very fast, as fast as above, to 0 or to $-\nu$ and then rise again very fast for the next positive or negative Dirac mass. The difference between b and such a profile is as small as we please in the L^1-sense; certainly we may assume that it is $O(\frac{1}{|b|^N_\infty})$, N as large as we please. It follows that the use of the regularizing semi-flow that has $\eta = b$, of which we spoke above, would, in a very short time, transform the estimate of difference between b and its limit profile from an L^1-estimate into a C^2-estimate. The convex-combination with the H^1_0-flow could then be completed.

However, the semi-flow having $\eta = b$, if we were to use it without further restriction, blows up too often, too fast. Even tamed into $\frac{b}{1+|b|^{1000}_\infty}$, there aren't enough estimates on the curves subject to its associated evolution equation.

For some curves, carrying "enough energy" in their nearly ξ-pieces (derived after the use of the Z_ν-semi-flow) another "flow" can be used, cautiously, and it will provide this regularizing effect whereas it will not move the nearly large $\pm v$-pieces much.

This semi-flow is the same than the Z_ν-flow. It is used on the curves to which the semi-flow Z_ν leads at the blow-up time, that is the curves having b in the L^1-sense close (close as above) to one of the profiles defined above. With respect to the Z_ν-semi-flow, there are two modifications: first ν is replaced by $\frac{\nu}{2}$ and, second, the support of the main part of this semi-flow lies within the nearly ξ-pieces defined by this profile.

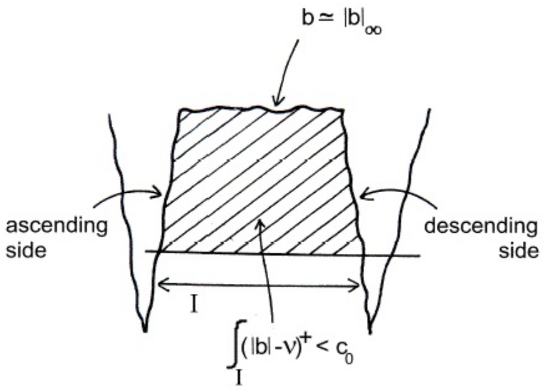
It is not difficult to see then that if b, on these nearly ξ-pieces, is close to a profile containing a $\pm\nu$-"plateau" having a measure that could be $O(\frac{1}{|b|^{N_0}_\infty})$, N_0 large, but would not be $O(\frac{1}{|b|^N_\infty})$, N much larger, as prescribed for an upper-bound between b and its limit profile in the L^1-sense, then the use of the $Z_{\frac{\nu}{2}}$-semi-flow within this "plateau" would provide a rate of decrease in $\int_0^1 \alpha_x(\dot{x})$ that would maybe be $O(\frac{1}{|b|^{N_0}_\infty})$, N_0 large, but would not be $O(\frac{1}{|b|^N_\infty})$. This would allow for a use of a sizable fraction cb of the regularizing flow, that is c would also be maybe $O(\frac{1}{|b|^{N_0+1}_\infty})$, N_0 large, but would not be $O(\frac{1}{|b|^N_\infty})$. In addition, this semi-flow would mainly act **within** the nearly ξ-pieces of the curve. Its action therefore on the nearly v-pieces would be essentially reduced to the action of the generalized tangent vector defined by $\eta = cb$ (there is an additional time translation, required to keep the ξ-component of $\dot{x}$ time (of the curve-independent), that is to the regularizing semi-flow. Skipping details (that require complete proofs), this semi-flow would transform the L^1-estimate on b with respect to its profile on the nearly $\pm v$-pieces into a C^2-estimate and the convex-combination with the H_0^1-semi-flow could be completed.

We are left then with curves that do not have enough "energy" in their nearly ξ-pieces in order to induce a regularizing effect on the large nearly $\pm v$-pieces. Using the same line of thought, we can use the Z_ν-flow on these nearly $\pm v$-pieces exclusively (with the additional, as tiny as we please and need, cb see [3] acting also on the nearly ξ-pieces) so that the curves will enter the set where there is enough "energy" inside the nearly ξ-pieces in order to regularize the large nearly $\pm v$-pieces.

We can also proceed differently: we use the Z_ν-semi-flow with a prescribed large value M for $|b|_\infty$, see [3] this semi-flow controls $|b|_\infty$. The curves reach the set $\mathcal{V}_\epsilon$ where b is L^1-close to one of the profile, $|b|_\infty \leq 2M$. How close is measured by a small constant $\epsilon = O(\frac{1}{|b|_\infty^N})$, N large. In $\mathcal{V}_{\frac{\epsilon}{2}}$ ($|b|_\infty \leq 4M$), we use the tamed regularizing semi-flow that has $\eta = \frac{b}{1+|b|_\infty^{1000}}$. We convex-combine Z_ν and this semi-flow in between $\mathcal{V}_\epsilon$ and $\mathcal{V}_{\frac{\epsilon}{2}}$. Defining a yet smaller $\mathcal{V}_{\frac{\epsilon}{4}}$, $J(x) = \int_0^1 \alpha_x(\dot{x})$ decreases at a rate bounded away from zero over the curves that stay outside of this set,. The (semi)-flow-lines, starting from $\mathcal{V}_\epsilon$, will then not enter $\mathcal{V}_{\frac{\epsilon}{4}}$ unless the v-component, b, of their tangent vector $\dot{x}$, has now been regularized. The convex-combination can be completed now.

There are three additional observations that we wish to make in order to conclude this sub-section:

First, with just the use of the Z_ν-flow of [3], b has "plateaux" where it is essentially equal to $\pm|b|_\infty$. It can "depart" over a "plateau" from this top value and oscillate fast downwards. However, if there are two such oscillations and if in between, the "mass" of b, that is the integral of $|b|$ from the "ascending side"



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of the first oscillation to the descending side of the second one (assuming b is here locally essentially equal to $= +|b|_\infty$) is less than a fixed positive constant c_{10}, see [3] p25, but larger than some $O(\frac{1}{|b|_\infty^{N_0}})$, then the flow Z_ν can still be used, with a sizable decrease for J. Therefore, along the "large" nearly $\pm v$-pieces at the blow-up time, these "sharp downwards" oscillations are "scarce". They are separated by sizable (of length $\geq c_{10}$)nearly $\pm v$-pieces. We cannot state that b is close C^1 on these $\pm v$-pieces to $\pm|b|_\infty$, but it is certainly C^0-close. We can then pick up a "mesh" of points over the curves that are sitting over these (relatively)large nearly $\pm v$-pieces and use these points in order to define (we might need to extend suitably these nearly $\pm v$-pieces beyond the parts defined by the curve itself so that the end-points can move freely, this is not needed **inside** the (large) nearly $\pm v$-pieces, but it is needed near their edge), without further regularization, the H_0^1-semi-flow in a way that can be convex-combined with the Z_ν-semi-flow.

Second, we can use Lemma 7.5, Lemma 7.6 on these large $\pm v$-pieces (maybe interrupted with these "scarce" downwards oscillations), once the "mesh" of points is given. If we take enough of these points so that they are separated by nearly $\pm v$-pieces of length $\ell \leq c_0$ and if b does not change sign in between, then Lemma 7.6 gives us an algorithm, with $b\eta \geq 0$ by which the curve is replaced by two $\pm v$-pieces (of the same orientation then the initial one) separated by a tiny ξ-piece. We thus build a family of tiny ξ-pieces and other large, but not so large $\pm v$-pieces. The only restriction is the restriction over b not to change sign over these intervals. b might have zeros, but they are in finite number and the "mesh" of points can be refined, the length of the nearly $\pm v$-pieces can be decreased as we "approach" a zero of b so that the process will be carried everywhere on the large nearly $\pm v$-pieces except in tiny neighborhoods, as small as we please, of the zeros of b.

Third, the displacement of the nearly $\pm v$-pieces transversally to v is "small" through the $Z_{\frac{\nu}{2}}$-semi-flow when its use is concentrated, as above, "inside" the nearly ξ-pieces. Indeed, then the displacement of these nearly $\pm v$-pieces is due to $\lambda\xi + \eta w$, with $\eta = cb$ and $\lambda + \bar{\mu}\eta = \int_0^t b\eta - t\int_0^1 b\eta = O(\frac{\partial a}{\partial s})$. c is so small that cb is also $O(\frac{\partial a}{\partial s})$. It is in fact, for c small enough, $o(\frac{\partial a}{\partial s})$. λ is not necessarily $o(\frac{\partial a}{\partial s})$, but this is due to the term $t\int_0^1 b\eta$. This term is in fact, see [3] p121 and p124, due to a time re-parametrization required to keep a constant. If we remove this time re-parametrization along the curve, we find the displacement transverse to v to be $o(\frac{\partial a}{\partial s})$. With some further work, this can probably be transformed into an estimate on the transversal displacement of these large nearly $\pm v$-pieces of curves: the ξ-component of $\dot{x}$ along them is $O(\frac{1}{|b|_\infty})$, so that the additional (with respect to the estimates introduced above, transversally to v)displacement transversally to $\dot{x}$ is $O(\frac{c\dot{b}+\lambda b}{a|b|_\infty})$. $\dot{b}$, after regularization, should be $O(|b|_\infty^{N_0})$ and the argument should proceed, yielding a very precise convergence of all pieces of the curves under deformation.

This concludes our observations about the convex-combination of the Z_ν and the H_0^1-semi-flows.

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