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An example of a fully bounded chaotic sea that surrounds an infinite set of nested invariant tori

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1 Abstract

The aim of this letter is to present a fully bounded 2-D model for a chaotic sea that surrounds an infinite set of nested invariant tori. The relevance of this result is that there are some examples of bounded chaotic seas but there is no rigorous proof of this property for specific cases.

2 Introduction

Precise definitions of *elliptic islands* and *chaotic seas* are difficult to give. Models for chaotic seas are rare in theory and practice. A few results on the dynamics in the chaotic region are available [1-2-3-4-6-7-9-15-16]. Some topological characterizations of elliptic islands in a chaotic sea are available in [5-8-11-12-17-18-19]. One of the techniques uses rotation numbers [13-14]. There are many interpretations of the rotation number in real-world models, such as the current in electric oscillators, the frequency of periodically forced pendula, heart rates, the asymptotic firing frequency of pacemaker neurons, and the integrated density of states of certain Schrôdinger operators, and so on.

There are some examples of chaotic seas [5-7-8-10-11-12-17-19-20-21], but there is no rigorous proof of their boundedness for these specific cases. Here we will give a rigorous proof that a chaotic sea presented by a particular 2-D map is fully bounded for all values of its bifurcation parameters.

3 A fully bounded 2-D model for the chaotic sea

In this letter, we propose the following 2-D map as a model for a fully bounded chaotic sea:

$$f:\begin{cases} x_{n+1}=-ax_n^2+y_n \\ y_{n+1}=b-|x_n| \end{cases} \tag{3.1}$$

where $a\geq 0$ and $b\geq 0$ are bifurcation parameters. The importance of this map is that it contains only two nonlinearities with a minimum number of terms and presents a fully bounded 2-D model for a chaotic sea for all values of its bifurcation parameters. The map (3.1) has several important properties including the following: (i) The associated function $f(x,y)$ of map (3.1) is continuous in $\mathbb{R}^2$, but it is not derivable at the point $(x=0,y)$ for all $x\in\mathbb{R}$. (ii) The map (3.1) is a diffeomorphism for all its parameter values since the determinant of its Jacobian is ± 1. (iii) Due to the shape of the vector field f of map (3.1), the plane can be divided into two nonlinear regions denoted by $R_1=\{(x,y)\in$
R $^2/\ x<0$ and $R_2=\left\{(x,y)\in\mathbb{R}^2/\ x>0\right\}$, where in each region the function f is continuous and has continuous derivatives, and the border is given by $B=\left\{(x,y)\in\mathbb{R}^2/\ x=0\right\}$. (iv) If $a>0$ and $b\geq 0$, then there are two fixed points at $P_1=\left(-\frac{\sqrt{ab}}{a},\frac{-\sqrt{ab}+ab}{a}\right)$ and $P_2=\left(\frac{\sqrt{ab+1}-1}{a},\frac{-\sqrt{ab+1}+ab+1}{a}\right)$, respectively. The fixed points P_1 and P_2 exist simultaneously, and they intersect the border $x=0$ at the critical parameter value $b=0$.

Since system (3.1) is conservative, it does not have attractors nor basins of attraction, but it does have a *chaotic sea* that surrounds an infinite set of nested invariant tori. A plot of the solution of the system with $a=3.5$ and $b=0.3$ showing the quasiperiodic and chaotic solutions for a range of initial conditions $x_0=0.123$ and $0.177<y_0<0.277$ in steps of 0.01 is shown in Fig. 1.

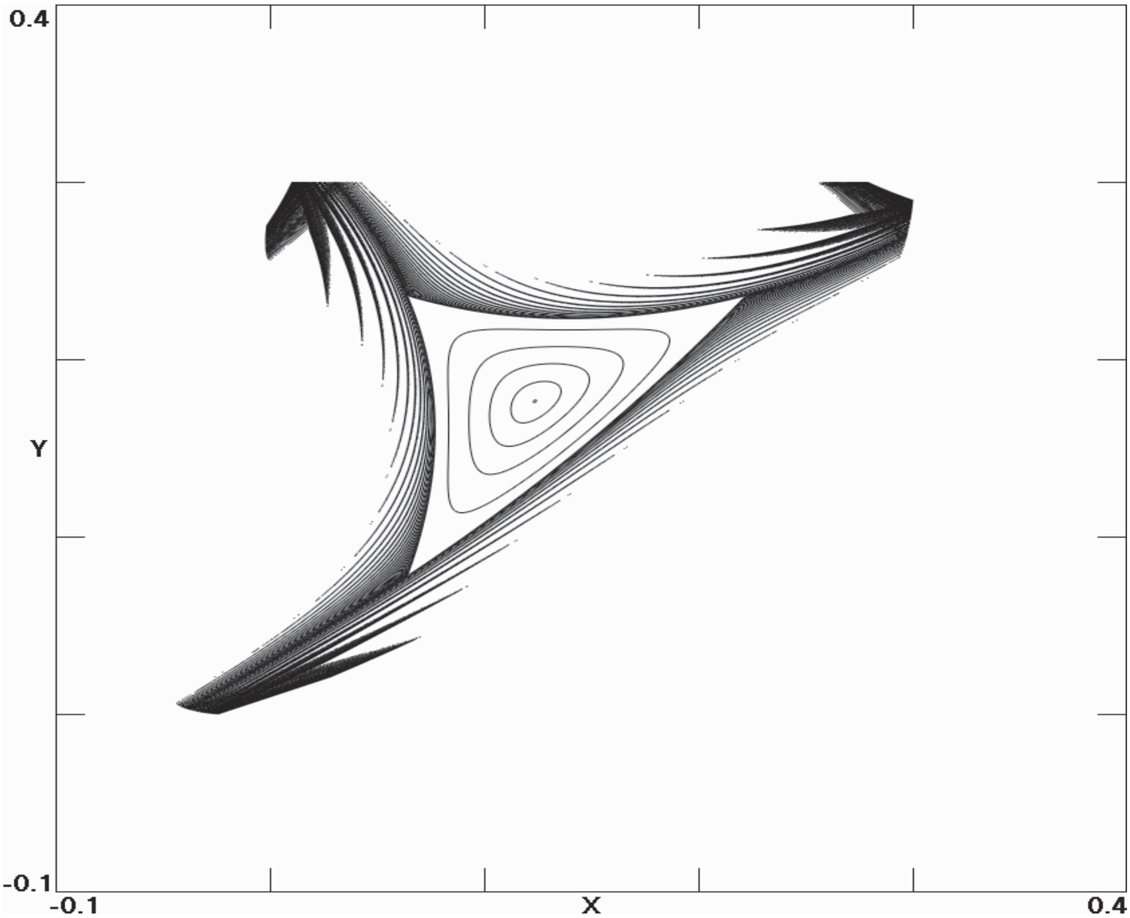

Figure 1: The chaotic sea of the map (3.1) for $a = 3.5, b = 0.3$ showing the quasiperiodic and chaotic solutions for a range of initial conditions $x_0 = 0.123$ and $0.177 < y_0 < 0.277$ in steps of 0.01.

The chaotic sea of the map (3.1) for $a = 3.5$, $b = 0.3$ showing the quasiperiodic and chaotic solutions for a range of initial conditions $x_0 = 0.123$ and $0.177 < y_0 < 0.277$ in steps of 0.01.. This chaotic sea is constructed numerically just like a strange attractor except that one must use a succession of carefully chosen initial conditions. Any initial condition in the sea eventually visits every point in the sea just as with a strange attractor. It seems that this chaotic sea looks qualitatively different from the one displayed by the Gingerbreadman map [20-21] with similar, perhaps even richer, behavior due to the presence of the nonlinear term x^2 in the expression for the map (3.1).

Several interesting features can be seen for the map (3.1). First, it appears that the choice of initial conditions plays a crucial role in the formation of the corresponding chaotic sea. For example, in the region $3.3 < a < 4.8$ with $b = 0.3$, there are two shapes for the chaotic sea. This phenomenon is similar to the case of two coexisting attractors for a dissipative system. Further investigations on the multi-stability of map (3.1) shows that at $a = 3.36$ and $b = 0.3$, there is a *three-lobe chaotic sea* for initial conditions $(0, 0)$ and a *drift ring* (similar to a quasi-periodic orbit) for initial conditions $(0.13, 0.16)$. This multi-stability may persist throughout most of the ab-plane.

The most important property of map (3.1) is that the presented chaotic sea is fully bounded for all $a > 0$ and $b \geq 0$. Indeed, we have $x_{n+1} \leq y_n \leq b$ and $-|x_{n-1}| \leq y_n \leq b$. Furthermore, $y_n = b - |x_{n-1}| \leq b$ and $x_n = -ax_{n-1}^2 + y_{n-1} \leq y_{n-1} \leq b$, also, $x_{n+1} = -ax_n^2 + y_n \leq y_n$ and $y_n = b - |x_{n-1}| \geq -|x_{n-1}|$ since $b \geq 0$. On the other hand, $x_{n+1} + ax_n^2 - b = -|x_{n-1}| \leq 0$. Thus $x_{n+1} \leq b - ax_n^2$ or $|x_n| \leq \sqrt{\frac{b-x_{n+1}}{a}}$ because $b - x_{n+1} \geq 0$. Also, $x_{n+1} + ax_n^2 + |x_{n-1}| = b \geq 0$. Thus we have $x_{n+1} \geq -ax_n^2 - |x_{n-1}| \geq -ax_n^2 - \sqrt{\frac{b-x_n}{a}}$ because $-|x_{n-1}| \geq -\sqrt{\frac{b-x_n}{a}}$. Finally, we have $-ax_n^2 - \sqrt{\frac{b-x_n}{a}} \leq x_{n+1} \leq b - ax_n^2$, which means that if x_n is finite, the next value x_{n+1} is also finite, and all the y_n are also finite for all $n \in \mathbb{N}$ since $x_{n+1} \leq y_n \leq b$. Hence the chaotic sea of map (3.1) is fully bounded for all $a > 0$ and $b \geq 0$ and for all finite initial conditions (x_0, y_0).

4 Conclusion

We have described a fully bounded 2-D model for the chaotic sea that surrounds an infinite set of nested invariant tori. The relevance of this result is that there is no previous rigorous proof of the boundedness of chaotic seas for any specific case in the current literature.

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