Palestine Journal of Mathematics

Vol. 1(2) (2012) , 153–156

Covers for Modules

N. Amiri and M. Ershad and H. Sharif

Communicated by Ayman Badawi

MSC2010 Classifications: 13A10; 13C99; 13E05; 13F05; 13F15.

Keywords: Prime submodule, cover of a module, multiplication module.

Abstract. Let R be a commutative ring with identity and M be an R-module with $Spec(M) \neq \emptyset$. A cover of a submodule K of M is a subset C of $Spec(M)$ satisfying that for any $x \in K$, $x \neq 0$, there is, $N \in C$ such that $ann(x) \subset (N : M)$. If we denote by $J = \bigcap_{N \in C} (N : M)$, and assume that M is finitely generated, then $JM = M$, implies that $M = 0$. We show that if R is a Noetherian ring and M is a finitely generated faithful R-module then M has a finite cover. And we shall see that if R is a Noetherian ring, M a finitely generated R-module, C a cover of M and $I \subset \bigcap_{N \in C} (N : M)$, then $\bigcap_{n=1}^{\infty} I^n M = 0$.

1. Introduction.

Throughout this paper R will be commutative ring with identity and all R-modules are unitary. From now on all modules are finitely generated. A proper submodule P of M is said to be prime if $rm \in P$ implies $m \in P$, or $rM \subset P$, for $r \in R$ and $m \in M$ (see for example [1], [2]). The set of all prime submodules of M is called the Spectrum of M and is denoted by $Spec(M)$. For any finitely generated R-module, it is known that $Spec(M) \neq \phi$. A cover of a submodule N of M is defined to be a subset C of $Spec(M)$ satisfying that for any $0 \neq x \in N$ there exists $P \in C$ such that $ann(x) \subset (P : M)$, where $(P : M) = \{r \in R | rM \subset P\}$. If we denote by J the intersection of all $(P : M), P \in C$ and $M \neq 0$ we have $JM \neq M$. This generalizes the Nakayama's Lemma. In addition if R is Noetherian, then $\bigcap_{n=1}^{\infty} J^n M = 0$. Also let q be a maximal ideal of R and N be a submodule of M, define $Map_q(\frac{M}{N}) = \{x \in M | q^n x \subset N$ for some $n > 0\}$. Then we investigate some of the result between primary submodule N of M and $Map_q(\frac{M}{N}), q \in Max(R) = \{q | q$ is a maximal ideal of $R\}$.

2. Results.

Definition. Let M be a module over a ring R. A proper submodule P of M is a prime submodule, if $rm \in P$ for $r \in R$ and $m \in M$ implies that either $m \in P$ or $rM \subset P$. The set of all prime submodules of M is called the spectrum of M and is denoted by $Spec(M)$.

Definition. Let M be an R-module and K be a submodule of M. A subset C of $Spec(M)$ is a cover of K, if for any $x \in K, x \neq 0$ there is $P \in C$ such that $ann(x) \subset (P : M)$. If C is a finite set, then C is called a finite cover.

Lemma 1. Let C be a cover of M. For any $r \in R - \bigcup_{P \in C} (P : M)$ if $rm = 0$ for some $0 \neq m \in M$, then $r = 0$.

Proof. If $rm = 0$ then $r \in ann(m) \subseteq (P : M)$ for some $P \in C$, which is a contradiction. □

Proposition 2. Let M be a non-zero R-module and C be a cover of M. If $JM = M$, then $M = 0$, where $J = \bigcap_{P \in C} (P : M)$.

Proof. Suppose $M \neq 0$, $JM = M$ there exists $r \in R$ such that $r \equiv 1(modJ)$ and $rM = 0$, $r \in ann(m)$ which is a contradiction.

□

Remark. If M is a multiplication faithful R-module then $Spec(M) \cong Spec(R)$ by [1, Theorem 2], and if $C = Spec(M)$, then $J = \bigcap_{P \in C} (P : M) \subset J(R)$.

Lemma 3. Let M be an R-module, C a cover of M and $I \not\subset ann_R(M)$. Set $J = \bigcap_{P \in C} (P : M)$, then $JM + ann_M(I) \neq M$, where $ann_M(I) = \{x \in M | Ix = 0\}$.

Proof. Since $I \not\subset ann_R(M)$, then $M \neq ann_M(I)$, so $\frac{M}{ann_M(I)} \neq 0$. Let $\bar{X} = x + ann_M(I)$, where $Ix \neq 0$. We have $(ann_M(I) : x) \subset ann(Ix)$, as $Ix \neq 0$ there exists $r \in I$, $rx \neq 0$. Then $ann(\bar{X}) \subset ann(rx) \subset (P : M)$ for some $P \in C$. Hence $J\frac{M}{ann_M(I)} \neq \frac{M}{ann_M(I)}$ and $JM + ann_M(I) \neq M$.

□

Proposition 4. Let R be a Noetherian ring, M an R-module, C a cover of M, $I \subset \bigcap_{P \in C}(P : M)$. Then $\bigcap_{n=1}^{+\infty} I^n M = 0$.

Proof. Let $\bigcap_{n=1}^{+\infty} I^n M = K$. Then by Krull's Theorem $IK = K$. Proposition 2 implies that $K = 0$.

□

Proposition 5. Let C be a finite subset of $spec(M)$ such that $(P : M)$ is maximal for every $P \in C$, and $J = \bigcap_{P \in C}(P : M)$. If $\bigcap_{n=1}^{\infty} J^n M = 0$, then C is a finite cover of M.

Proof. If C is not a cover of M, then there is an element $0 \neq x \in M$ such that $ann_R(x) \not\subset (P : M)$ for every $P \in C$. Hence $ann_R(x) + (P : M) = R$. Let $1 = r + s$ with $s \in (P : M)$ and $r \in ann(x)$. Then for every $n \in \mathbb{N}$, $1^n = (r+s)^n = r' + s'$, $r' \in ann_R(x)$ and $s' \in (P : M)^n$, so $x = r'x + s'x = s'x$. Hence $Rx = (P : M)^n x$ for every $P \in C$, and so $J^n x = Rx$, hence $\bigcap_{n=1}^{\infty} J^n M \neq 0$, which is a contradiction.

□

Theorem 6. Let R be a Noetherian ring and M a faithful R-module. Then M has a finite cover C and $\bigcap_{n=1}^{\infty} J^n M = 0$, where $J = \bigcap_{P \in C}(P : M)$. In particular if $M = R$ then $\bigcap_{n=1}^{\infty} J^n = 0$.

Proof. Since M is a Noetherian R-module, $Ass(M)$ is a finite set, let $Ass(M) = \{q_1, q_2, \ldots, q_m\}$ and for every q_i there exists a maximal ideal q'_i of R such that $q_i \subset q'_i$. By [1, p.3746] there exist maximal submodules P_i of M such that $q'_i = (P_i : M)$. Let $C = \{P_1, \ldots, P_m\}$. For any $0 \neq x \in M$, there is q_i such that $ann(x) \subset q_i \subset q'_i = (P_i : M)$. Hence C is a cover of M. Since for every $P_i \in C$, $(P_i : M)$ is a maximal ideal of R, by Proposition 4 we have $\bigcap_{n=1}^{\infty} J^n M = 0$. If $M = R$, obviously then $\bigcap_{n=1}^{\infty} J^n = 0$.

□

Definition. Let M be an R-module, then we define $Map(M) = \{x \in M |$ every prime ideal containing $ann(x)$ is maximal$\}$.

Lemma 7. $Map(M)$ is a submodule of M.

Proof. Let $x \in Map(M)$ and $r \in R$. Suppose that q is a prime ideal of R such that $ann(rx) \subset q$, since $ann(x) \subset ann(rx) \subset q$ so q is a maximal ideal of R, hence $rx \in Map(M)$. If $x, y \in Map(M)$, and $ann(x+y) \subset q$ so $ann(x) \cap ann(y) \subset q$, this implies that $ann(x) \subset q$ or $ann(y) \subset q$, hence q is a maximal ideal of R, i.e., $Map(M)$ is a submodule of M.

□

Definition. Let q be a maximal ideal of R and N be a submodule of M. Define $Map_q(\frac{M}{N}) = \{x \in M | q^n x \subset N$ for some $n > 0\}$.

Lemma 8. $Map_q(M)$ is a submodule of M, for every maximal ideal q of R.

Proof. Let $x \in Map_q(M)$ and $r \in R$, there exists a positive integer n such that $q^n x = 0$, hence $q^n(rx) = 0$, therefore $q^n \subset ann(rx)$ and $rx \in Map_q(M)$. If $x, y \in Map_q(M)$ there exist $m, n \in \mathbb{N}$ such that $q^m x = 0$ and $q^n y = 0$. Take $k = \max\{m, n\}$ so $q^k(x+y) = 0$, and hence $q^k \subset ann(x+y)$. Therefore $x + y \in Map_q(M)$.

□

Remark. It is clear that $Map(Map(M)) = Map(M)$, $Map_q(Map_q(M)) = Map_q(M)$ for every maximal ideal q of R and if $q' \neq q$ is a maximal ideal of R then $Map_{q'}(Map_q(M)) = Map_q(Map_{q'}(M)) = 0$. Also $Map(Map_q(M)) = Map_q(Map(M)) = Map_q(M)$.

Proposition 9. If N is a proper submodule of an R-module M and $Map_q(\frac{M}{N}) = M$ for some maximal ideal q of R then N is q-primary.

Proof. Let $M =< x_1, x_2, \ldots, x_k >$. Since $Map_q(\frac{M}{N}) = M$, there is $n_i > 0$ such that $q^{n_i}x_i \subset N$ for $i = 1, \ldots, k$. If $n = \max\{n_1, \ldots, n_k\}$ we have $q^n x_i \subset N$, for all $i = 1, 2, \ldots, k$. So $q^n M \subset N$, hence $q \subset \sqrt{(N : M)}$ since q is maximal, $q = \sqrt{(N : M)}$. Now we show that if $r \notin q$ and $x \notin N$ then $rx \notin N$. Suppose $rx \in N$. Since $r \notin q$ and q is a maximal ideal of R, $q+ < r >= R$. Hence $1 = a + rs$, $a \in q$. Also since $q^n x \in N$, we have $(1)^n = (a + rs)^n = a^n + s'r$ where $s' \in q$. Therefore, $x = a^n x + s'rx \in N$, which is a contradiction. Hence by [1, Lemma 1.1] N is q-primary.

□

Proposition 10. If q is a finitely generated maximal ideal of R and N is a q-primary submodule of M, then $Map_q(\frac{M}{N}) = M$.

Proof. Obviously $Map_q(\frac{M}{N}) \subset M$. For the converse, let $q =< r_1, r_2, \ldots, r_k >$, $x \in M$. Since $\sqrt{(N : M)} = q$, there exists $n_i > 0$ such that $r_i^{n_i}x \in N$. Let $n = n_1 + \ldots + n_k$. We have $r_i^n x \in N$. Hence $q^n x \subset N$, so $x \in Map_q(\frac{M}{N})$. So $Map_q(\frac{M}{N}) = M$.

□

Corollary 11. If R is a Noetherian ring, M is an R-module and N is a proper submodule of M, then $Map_q(\frac{M}{N}) = M$ if and only if N is a q-primary submodule of M, for every maximal ideal q of R.

Proof. Obvious.

□

Corollary 12. If R is a Noetherian ring and M is an R-module. Then $Map_q(M) = M$ if and only if 0 is q-primary submodule of M.

Proof. Let $N = 0$ in the above corollary.

□

Lemma 13. Suppose q is a maximal ideal of R and $Map_q(M) = M$. Let $S = R - q$. Then for every $m \in M$ and $s \in S$, there is a unique element $m' \in M$ such that $m = sm'$.

Proof. If $s \notin q$, then $< s > +q = R$. Hence $1 = rs + a$ for some $r \in R$ and $a \in q$ and since $Map_q(M) = M$ there is $n > 0$ such that $q^n m = 0$. So $1^n = (rs + a)^n = r's + a^n$ implies that $m = r'sm + a^n m = s(r'm) = sm'$ for some $m' \in M$. Now we show that m' is unique. If $sm'' = m$, then $sm'' = sm'$ and $s(m' - m'') = 0$. There are $k, k' > 0$ such that $q^k m' = 0$ and $q^{k'}m'' = 0$. Let $t = \max\{k, k'\}$. So $q^t(m' - m'') = 0$, and since $1 = sr' + a^t$, $(m' - m'') = r's(m' - m'') + a^t(m' - m'') = 0$. Therefore $m' = m''$.

□

Proposition 14. Let q be a maximal ideal of R. If $Map_q(M) = M$ then $M \cong M \otimes_R R_q$.

Proof. We show that $M \simeq M_q$. Let $\phi : M \longrightarrow M_q$ be the canonical homomorphism given by $\phi(m) = \frac{m}{1}$. Then $ker\phi = \{m \in M | \frac{m}{1} = 0\}$. If $m \in ker\phi$, then there is $s \in S = R - q$ such that $sm = 0$ and since $s0 = 0$ by Lemma 13, we have $m = 0$. So $ker\phi = \{0\}$. Hence ϕ is one-one. Let $\frac{m}{s} \in M_q$, since $sM = M$ (by Lemma 13) we have $m = sm'$ for some $m' \in M$, so $\phi(m') = \frac{m'}{1} = \frac{sm'}{s} = \frac{m}{s}$ and hence ϕ is an epimorphism. Therefore $M \cong M_q$ and since $M_q \cong M \otimes_R R_q$, $M \cong M \otimes_R R_q$.

□

Theorem 15. Let R be a Noetherian ring and M be an R-module, C be a cover of M such that for every $P \in C$, $(P : M)$ is a maximal ideal of R. Then $Map(M) = \bigcup_{n=1}^{\infty} ann_M(J^n)$, where $J = \bigcap_{P \in C}(P : M)$.

Proof. From [4, Corollary 1] and the fact that $Map(M)$ has a finite length we see that $Map(M) = \bigoplus_{i=1}^{n} Map_{q_i}(M)$ where q_i is a maximal ideal of R.
Let $x \in Map(M)$. Then $x = x_1 + \cdots + x_n$ where $q_i^{k_i}x_i = 0$ for $i = 1, \ldots, n$, $k_i \in N$. Set $k = \max\{k_1, \ldots, k_n\}$. Then $q_i^k x = 0$, so $x \in (0 : J^k)$.
Now let $x \in \bigcup_{n=1}^{\infty}(0 : J^n)$. Then $J^n x = 0$ for some $n \in N$. Let q be a prime ideal of R such that $(0 : x) \subseteq q$. Therefore $J^n = (\bigcap_{P \in C}(P : M))^n \subset q$ and as q is prime there is $P \in C$ such that $(P : M) \subset q$. But this means $q = (P : M)$, therefore q is a maximal ideal, so $x \in Map(M)$.

□

Theorem 16. Let R be a Noetherian ring and M be an R-module, not Artinian. Let C be a finite cover of M and let $J = \bigcap_{P \in C} (P : M)$. Then $(0 :_M J^n)$ is a direct summand of M for some $n \in N$.

Proof. Since M is not Artinian, $Map(M)$ is a proper submodule of M. By [4, Theorem 7], we may assume $Map(M) = (0 : J^k)$ for some $k \in \mathbb{N}$. Now by [3] if we set $A = Mspec(M)$ then ${}^{A}M = Map(M)$ and by [3, Theorem A] $M = Map(M) \oplus K$ for a submodule K of M, and this decomposition is deep, in the sense that if H is a submodule of M then $H = H \cap (Map(M)) \oplus (H \cap K)$. □

REFERENCES

[1] Chin-Pi Lu, *Spectra of Modules*, Communication in Algebra 23(10),3741-3752 (1995).

[2] Chin-pilu, Prime Submodules of Modules Commentarii Mathematici Universitatis Sancti Pauli, 61-69 (1984).

[3] Stephan McAdam, *Deep Decomposition of Modules*, Communication in Algebra, 26(12), 3953-3967 (1998).

[4] E. Matlis, *Modules with descending chain condition*, Trans. Amer. Math. Soc, 97, 495-508 (1960).

[5] Patrick F. Smith, *Primary module over commutative ring*, Glasgow Math. J. 43, 103-111 (2001).

Author information

N. Amiri, Department of Mathematics, Payame Noor University, P.O. Box 19395-3697, Tehran, Iran.
E-mail: n_amiri@pnu.ac.ir

M. Ershad and H. Sharif, Department of Mathematics, Shiraz University, Shiraz, 71454., Iran.
E-mail: ershad@shirazu.ac.ir, sharif@shirazu.ac.ir

Received October 9, 2011

Accepted February 14, 2012