Palestine Journal of Mathematics

Vol. 1(2) (2012) , 104–109

ANALYTICAL SOLUTION OF SCHRÖDINGER EQUATION WITH ECKART POTENTIAL PLUS HULTHEN POTENTIAL VIA NIKIFOROV-UVAROV METHOD

Akaninyene D. Antia, Akpan N. Ikot*, Eno E. Ituen AND Louis E. Akpabbio

Communicated by Jose Luis Lopez-Bonilla

MSC2010 Classifications: 33C45; 34L10 ; 35J10 ; 81Q10.

Keywords: Schrödinger equation, Eckart Potential, Hulthen Potential. Nikiforov-Uvarov method.

This research work was partially supported by NANDY-LEABIO GRANT NO: 432-01-04667.

Abstract. The exact solution of Schrödinger equation for the Eckart potential plus Hulthen potential is obtained. The energy eigenvalues and the corresponding eigenfunctions are obtained via the Nikiforov-Uvarov method. We expressed the wave function in terms of Jacobi polynomials.

1 INTRODUCTION

The exact and analytical solutions of the radial Schrödinger equation is very important in non-relativistic quantum mechanics in recent time [1-5] since the wave function and its associated eigenvalues contain all the necessary information to describe a quantum system fully. Various different methods have been adopted for the solution of the Schrödinger equation with different potentials [6-10]. One of such methods includes the variational method [11], the functional analysis method [12], the supersymetric approach [13], the shiffed $^1/_N$ expansion [14], the asymptotic iteraction method (AIM) [15] and the Nikiforov-Uvarov method (NU) [16]. The NU method is however more suitable for obtaining analytical solutions to such a differential equation. In addition, the Schrödinger equation has been solved exactly or quasi-exactly for $\ell \neq 0$ with a given potential. Recently, the NU method has been used to solve the Schrödinger equation for different potentials such as Woods-Saxon [17], Hulthen potential [18], Pseudo and perturbed Coulomb potential [19], Makorov potential [20], Morse potential [21] and Eckart potential [22]. The purpose of this article is to use the NU method to solve the Schrödinger equation for Eckart potential plus Hulthen potentials.

The paper is arranged as follows. In section two, we review the Nikiforov-Uvarov method. In section three, solution of Schrödinger equation with Eckart potential plus Hulthen potential is presented. Finally, a brief conclusion is given in section four.

2 THE NIKIFOROV-UVAROV (NU) METHOD

The NU method is used to obtain the exact solution of the second-order differential equations such as the Schrödinger, Klein-Gordon and Dirac equations. Therefore, a non-relativistic Schrödinger equation can be solved exactly using this method. In the NU method, second-order differential equation is reduced to hypergeometric type equation. This equation is written as follows

$$\psi''(s) + \frac{\bar{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\bar{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{2.1}$$

Where $\sigma(s)$ and $\bar{\sigma}(s)$ are at most second-degree polynomials, and $\bar{\tau}(s)$ is a first-degree polynomial. In obtaining the exact solution to Eq.(2.1) we set the wave function as

$$\psi(s) = \varphi(s)\chi(s) \tag{2.2}$$

Using Eq.(2.2) in Eq.(2.1), we have the transformation.

$$\sigma(s)\chi''(s) + \tau(s)\chi'(s) + \lambda\chi(s) = 0, \tag{2.3}$$

where the wave function $\psi(s)$ is defined as the logarithmic derivative [16] as

$$\frac{\varphi'(s)}{\varphi(s)} = \frac{\pi(s)}{\sigma(s)} \tag{2.4}$$

Where $\pi(s)$ is at most first degree polynomial and

$$\tau(s) = \bar{\tau}(s) + 2\pi(s) \quad with \quad \tau'(s) < 0 \tag{2.5}$$

Also, the hyper-geometric type function $\chi(s)$ in Eq.(2.3) for a fixed n is given by the Rodriques relation [16] as:

$$\chi_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n} \left[\sigma^n(s)\rho(s)\right], \tag{2.6}$$

where B_n is the normalization constant and the weight function $\rho(s)$ must satisfy the following condition [16,23]

$$\frac{d}{ds}\left(\sigma(s)\rho(s)\right) = \tau(s)\rho(s) \tag{2.7}$$

Therefore, the function $\pi(s)$ in Eq.(2.5) and the parameter λ required for the NU method are defined as follows:

$$\pi(s) = \left(\frac{\sigma'(s) - \bar{\tau}(s)}{2}\right) \pm \sqrt{\left(\frac{\sigma'(s) - \bar{\tau}(s)}{2}\right)^2 - \bar{\sigma}(s) + k\sigma(s)}, \tag{2.8}$$

$$\lambda = k + \pi'(s) \tag{2.9}$$

The k-value in the square root of Eq.(2.8) is possible to evaluate if the expression under the square root must be square of polynomials. This is possible if its descriminant is zero. Thus, the new eigen value equation for the Schrödinger equation becomes

$$\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), n = 0, 1, 2... \tag{2.10}$$

Finally, comparing Eq.(2.9) with Eq.(2.10), we can obtain the energy eigenvalues.

3 EIGENVALUE AND EIGEN FUNCTION OF ECKART POTENTIAL PLUS HULTHEN POTENTIAL

The Eckart potential plus Hulthen potential is given as:

$$V(r) = \cos ech^2 \alpha r + \coth \alpha r + \frac{V_0}{(1 - e^{-2\alpha r})} - \frac{V_1}{(1 - e^{-2\alpha r})^2} \tag{3.1}$$

where V_O and V_1 are the depth of the potential while α is a parameter.
The hyperbolic functions are defined as

$$\sinh \alpha r = \frac{e^{\alpha r} - e^{-\alpha r}}{2}, \quad \cosh \alpha r = \frac{e^{\alpha r} + e^{-\alpha r}}{2}, \quad \tanh \alpha r = \frac{e^{\alpha r} - e^{-\alpha r}}{e^{\alpha r} + e^{-\alpha r}} \tag{3.2}$$

The radial Schrödinger equation is of the form

$$\frac{d^2\psi(r)}{dr^2} + \frac{2m}{\hbar^2}\left[E - V(r)\right]\psi(r) = 0 \tag{3.3}$$

Using Eq. (3.2) in Eq. (3.1), Eq. (3.3) becomes

$$\frac{d^2\psi(r)}{dr^2} + \frac{2m}{\hbar^2}\left[E - \frac{4e^{2\alpha r}}{(1 - e^{-2\alpha r})^2} - \frac{(1 + e^{-2\alpha r})}{(1 - e^{-2\alpha r})} - \frac{V_0}{(1 - e^{-2\alpha r})} + \frac{V_1}{(1 - e^{-2\alpha r})^2}\right]\psi(r) = 0 \tag{3.4}$$

In order to apply NU method to Eq. (3.4), we make a transformation to Eq. (3.4) by applying $s = e^{-2\alpha r}$, thus, we have the hypergeometric equation of the form.

$$\frac{d^2\psi}{ds^2} + \frac{1 - s}{s(1 - s)}\frac{d\psi}{ds} + \frac{1}{s^2(1 - s)^2}\left[(\gamma - \varepsilon)s^2 + (2\varepsilon + \delta - \beta)s + (\xi - \delta - \gamma - \varepsilon)\right] \psi = 0 \tag{3.5}$$

where the following dimensionless quantities have been used.

$$\varepsilon = -\frac{2mE}{\hbar^2\alpha^2}, \beta = \frac{8m}{\hbar^2\alpha^2}, \gamma = \frac{2m}{\hbar^2\alpha^2}, \quad \delta = \frac{2mV_0}{\hbar^2\alpha^2}, \xi = \frac{2mV_1}{\hbar^2\alpha^2} \tag{3.6}$$

Comparing Eq. (3.5) with Eq. (2.1), we have

$$\sigma(s) = s(1 - s), \bar{\tau}(s) = (1 - s), \quad \bar{\sigma}(s) = (\gamma - \varepsilon)s^2 + (2\varepsilon + \delta - \beta)s + (\xi - \delta - \gamma - \varepsilon) \tag{3.7}$$

In the NU method $\pi(s)$in Eq. (2.8) is defined as

$$\pi(s)=\frac{-s}{2}\pm\frac{1}{2}\sqrt{(1-4a-4k)\,s^2+(4k-4b)s-4c} \tag{3.8}$$

where

$$\begin{aligned} a&=\gamma-\varepsilon\\ b&=2\varepsilon+\delta-\beta\\ c&=\xi-\delta-\gamma-\varepsilon \end{aligned} \tag{3.9}$$

The constant parameter k can be found by the condition that the expression under the square root has a double root, that is, its discriminant is zero. So for each k value, we have;

$$\pi(s)=-\frac{s}{2}\pm\frac{1}{2}\begin{cases} \left(2\sqrt{\delta+\gamma+\varepsilon-\xi}-\sqrt{4\beta-4\xi+1}\right)s-2\sqrt{\delta+\gamma+\varepsilon-\xi}\\ for\ \ k=-\delta-\beta+2\xi-2\gamma+\sqrt{(4\beta-4\xi+1)\,(-\xi+\delta+\gamma+\varepsilon)}\\ \\ \left(2\sqrt{\delta+\gamma+\varepsilon-\xi}+\sqrt{4\beta-4\xi+1}\right)s-2\sqrt{\beta+\gamma+\varepsilon-\xi}\\ for\ \ k=-\delta-\beta+2\xi-2\gamma-\sqrt{(4\beta-4\xi+1)\,(-\xi+\delta+\gamma+\varepsilon)} \end{cases} \tag{3.10}$$

In the NU method, $\tau'(s)<0$ must be satisfied in order to obtain a physical eigen function. To do this, we select.

$$k=-\delta-\beta+2\xi-2\gamma-\sqrt{(4\beta-4\xi+1)\,(-\xi+\delta+\gamma+\varepsilon)}$$

and

$$\pi(s)=-\frac{s}{2}-\frac{1}{2}\left[\left(2\sqrt{\delta+\gamma+\varepsilon-\xi}+\sqrt{4\beta-4\xi+1}\right)s-2\sqrt{\delta+\gamma+\varepsilon-\xi}\right]$$

. With this selection, we can obtain $\tau(s)$ as

$$\tau(s)=1-2s-\left[\left(2\sqrt{\delta+\gamma+\varepsilon-\xi}+\sqrt{4\beta-4\xi+1}\right)s-2\sqrt{\delta+\gamma+\varepsilon-\xi}\right] \tag{3.11}$$

$$\tau'(s)=-2-\left(2\sqrt{\delta+\gamma+\varepsilon-\xi}+\sqrt{4\beta-4\xi+1}\right) \tag{3.12}$$

Applying Eqs. (3.7) and (3.11) in Eq. (2.10), we have;

$$\lambda=\lambda_n=n\left(2+2\sqrt{\delta+\gamma+\varepsilon-\xi}+\sqrt{4\beta-4\xi+1}\right)+n(n-1) \tag{3.13}$$

Similarly, from Eq. (2.9), we have;

$$\lambda=-\delta-\beta+2\xi-2\gamma-\sqrt{(4\beta-4\xi+1)\,(-\xi+\delta+\gamma+\varepsilon)}-\frac{1}{2}-\frac{1}{2}\left(2\sqrt{\delta+\gamma+\varepsilon-\xi}+\sqrt{4\beta-4\xi+1}\right) \tag{3.14}$$

Comparing Eq. (3.13) and Eq. (3.14), we have,

$$\begin{aligned} &n\left(2+2\sqrt{\delta+\gamma+\varepsilon-\xi}+\sqrt{4\beta-4\xi+1}\right)+n(n-1)=\\ &-\delta-\beta+2\xi-2\gamma-\sqrt{(4\beta-4\xi+1)\,(-\xi+\delta+\gamma+\varepsilon)}-\frac{1}{2}-\frac{1}{2}\left(2\sqrt{\delta+\gamma+\varepsilon-\xi}+\sqrt{4\beta-4\xi+1}\right)) \end{aligned} \tag{3.15}$$

Solving Eq. (3.15) explicitly for ε and substituting Eq. (3.6), we obtain the energy eigenvalues as,

$$E_n=-\left(V_0+V_1+1\right)-\frac{\hbar^2\alpha^2}{2m}\frac{1}{\left[1+2n+\sqrt{\frac{8m}{\hbar^2\alpha^2}\left(4-V_1\right)+1}\right]^2}$$
$$\left[\frac{2m}{\hbar^2\alpha^2}\left(2V_1-V_0-6\right)-\frac{1}{2}\sqrt{\frac{8m}{\hbar^2\alpha^2}\left(4-V_1\right)+1}-\frac{1}{2}-n\sqrt{\frac{8m}{\hbar^2\alpha^2}\left(4-V_1\right)+1}-\left(n+n^2\right)\right]^2 \tag{3.16}$$

Equation (3.16) is the energy spectrum of the Eckart plus Hulthen potentials in compact form.The corresponding wave function ψ(s) can be found out from φ(s) and χ(s) given in Eq. (2.4) and Eq. (2.6), respectively. Using Eq. (2.4), we have,

$$\varphi(s)=(1-s)^{\frac{1}{2}(1+\mu)}\,s^{v} \tag{3.17}$$

Where $v=\sqrt{\delta+\gamma+\varepsilon-\xi}$ and $\mu=\sqrt{4\beta-4\xi+1}$.

In order to determine the other wave function in the NU method, the hypergeometric function χ_n(s) in Eq. (2.6), we must first determine the weight function ρ(s) from Eq. (2.7) as,

$$\rho(s) = s^{2v}\left(1-s\right)^{\mu} \tag{3.18}$$

Substituting Eq. (3.18) into the Rodriques relation given in Eq. (2.6), we obtain the wave function as,

$$\chi_n(s) = B_n s^{-2v}(1-s)^{-\mu}\frac{d^n}{ds^n}\left[s^{n+2v}(1-s)^{n+\mu}\right] \tag{3.19}$$

Or we express Eq. (3.19) in terms of Jacobi Polynomials as

$$\chi_n(s) = B_n P_n^{(\mu+2v,\mu)}(1-s) \tag{3.20}$$

where B_n is the normalization constant whose numerical value is $\frac{1}{n!}$

Combining Eqs. (3.17) and (3.19), we have the unnormalized wave function for Eckart potential plus Hulthen potential as

$$\psi_n(s) = (1-s)^{1/2^{(1-\mu)}} S^{-v} B_n \frac{d^n}{ds^n}\left[S^{n+2v}(1-s)^{n+\mu}\right] \tag{3.21}$$

In terms of Jacobi polynomials Eq. (3.21) can be expressed as

$$\psi_n(s) = C_n P_n^{(\mu+2v,v)}(1-s) S^v\left(1-s\right)^{\mu-v+1/2} \tag{3.22}$$

where C_n is the new normalization constant.
Equation (3.22) satisfies the normalization condition $\int\left|\psi_n(s)\right|^2 ds = 1$.

4 CONCLUSION

In summary, we have used the NU method to obtain the energy and wave function for Eckart potential plus Hulthen potential. The wave function is expressed in terms of the generalized hypergeometric functions. The obtained exact results are very useful in many fields of physics and quantum chemistry.

References

[1] A. N. Ikot and L. E. Akpabio, International Review of Physics,**4**,224(2010)

[2] A. N. Ikot and L. E. Akpabio, Applied Physics Research,**2**,202(2010).

[3] A.N. Ikot, E. J. Uwah, I.O. Akpan and L. E. Akpabio, *IJPAP*, **6**, 511 (2010).

[4] A. N. Ikot, E.J. Uwah, L. E. Akpabio, and I.O. Akpan, *Afr. J. Math. Phys.* **8,** 43 (2010).

[5] A.D. Antia, A. N. Ikot and L. E. Akpabio, Eur. J. Scien. Res.**46**,107(2010).

[6] O. Bayrak, I. Boztosun and H. Ciffci, *Int. J. Quantum Chem*. **107**, 504 (2007).

[7] C. S. Jia, L. Yun and Y. Sun, *Phys. Lett. A*. **305**, 231 (2002).

[8] C. Berkdemir, A. Berkedemir and R. Sever, *Phys. Rev. C*. **72**, 027001 (2005).

[9] M. Simsek and H. Egrifes, *J. Phys. A; Math. Gen*. **37**, 4379 (2004).

[10] A. R. Akbarieh and H. Motavalli, *Mod. Phys. Lett. A* **23**, 3008 (2008)

[11] B. Batiha, *Aust. J. Basic and Appl. Sci*. **3,** 3826 (2009).

[12] Y. Zhang, *Phys. Scr*. **78,** 015006 (2008).

[13] G. Chen, Z. D. Chen and Z. M. Lou, *Phys. Lett. A* **331**. 374 (2004)

[14] O. Mustafa and R. Sever, *Phys. Rev. A*. **44**, 4142 (1991).

[15] O. Esar, *Chin. Phys. Lett*. **25**, 1939 (2008).

[16] A. F. Nikiforov and V. B. Uvarov, Special Functions of Mathematical Physics, (Birkhauser, Basal, 1988).

[17] B. Gonul, and K. Koksal, *Phys. Scr*. **76**, 565 (2007).

[18] S. Meyur and S. Debnath, *Mod. Phys. Lett. A* **23**, 2077 (2008).

[19] T. Baraket, *Ann. of Phy*. **324**, 725 (2009).

[20] A. A. Makarov et al, *Nuov. Cim. A* **52**, 1061 (1967).

[21] S. Flugge, Practical Quantum Mechanics, Vol. 1, (Springer, Berlin 1994).

[22] W. C. Qiang et al, *Phys. Scr*. **79**, 065011 (2009).

[23] G. Szego, Orthogonal Polynomials (American Mathematical Society, New York, 1959).

Author information

Akaninyene D. Antia, Akpan N. Ikot*, Eno E. Ituen AND Louis E. Akpabbio, Theoretical Physics Group, Department of Physics, University of Uyo, Nigeria.
E-mail: ndemikot2005@yahoo.com

Received: November 10, 2011

Accepted: January 22, 2012