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Some fixed point theorems by altering distance functions

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Abstract. This paper is devoted to obtain fixed point results for generalizations of well known contractive type mappings. To attain our goal we are going to use a control function called altering distance function.

1 Introduction and preliminary facts

The Banach's Contraction Principle (BCP) is one of the most important result in the metric fixed point theory, its significance lies in its vast applicability in many branches of mathematics and other sciences. Since its appearance several generalizations have been appeared.

Theorem 1.1 (BCP). Let (M, d) be a complete metric space and $T : M \longrightarrow M$ a mapping satisfying the following condition

$$d(Tx, Ty) \leq ad(x, y) \tag{1.1}$$

for all $x, y \in M$, where $0 \leq a < 1$. Then T has a unique fixed point $z_0 \in M$ and for each $x \in M$, $\lim\limits_{n\to\infty} T^n x = z_0$.

The purpose of this paper is to extend the BCP by using the generalization introduced in 1962 by E. Rakotch [7] who showed that the BCP holds when the constant a in (1.1) is replaced by a contraction monotonically decreasing function $\alpha : \mathbb{R}_+ \longrightarrow [0, 1)$. Also, we will use the generalization introduced in 1977 by D.S. Jaggi [3] in which T is assumed to be continuous and satisfying the contractive condition

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta m(x, y)$$

for all $x, y \in M$, $x \neq y$ for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and $m(x, y) = \frac{d(x,Tx)d(y,Ty)}{d(x,y)}$. As well as, the generalization introduced in 1984 by M.S. Khan, M. Swalech and S. Sessa [4], where they use a control function which they called an *altering distance function*.

Definition 1.1. A function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called an altering distance function if the following properties are satisfied:

(ψ_1) $\psi(0) = 0$.

(ψ_2) ψ is a monotonically non-decreasing function.

(ψ_3) ψ is a continuous function.

By Ψ we denote the set of all altering distance functions.

Definition 1.2. Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a function satisfying the following conditions:

(φ_1) φ is a Lebesgue integrable function on each compact subset of $\mathbb{R}_+$.

(φ_2) φ is nonnegative.

(φ_3) $\int_0^\varepsilon \varphi(t)dt > 0$ for each $\varepsilon > 0$.

By Φ we denote the set of all functions satisfying the conditions of Definition 1.2. The following lemma shows that using a mapping $\varphi \in \Phi$ we obtain a function $\psi \in \Psi$. This fact will be used in the sequel.

Lemma 1.2 ([1] and [6]). Let $\varphi \in \Phi$. Define $\psi_0 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by

$$\psi_0(s) := \int_0^s \varphi(t)dt \quad \text{for all } t \in \mathbb{R}_+.$$

Then $\psi_0 \in \Psi$.

On the other hand, in 2002, A. Branciari [2] obtained a fixed point theorem for a mapping satisfying an analogue of BCP for an integral type inequality. In 2011, M. Kang *et al* [5] extend and improve such theorem. As a consequence of the results on this paper, we will generalize some of the results given in [5]. (see, Remarks 1, 2 and 3.)

In order to attain our goals, the following lemma is a very important tool.

Lemma 1.3. Let (M,d) be a metric space. Let $(x_n)_n$ be a sequence in M such that $\lim_{n\to\infty} d(x_n,x_{n+1})=0$. If $(x_n)_n$ is not a Cauchy sequence in M, then there exists an $\varepsilon>0$ for which we can find subsequences $(x_{m(k)})_k$ and $(x_{n(k)})_k$ of $(x_n)_n$ with $m(k)>n(k)>k$ such that $d(x_{m(k)},x_{n(k)})\geq\varepsilon$, $d(x_{m(k)-1},x_{n(k)})<\varepsilon$ and

(i) $\lim\limits_{k\to\infty} d(x_{m(k)},x_{n(k)})=\varepsilon$.

(ii) $\lim\limits_{k\to\infty} d(x_{m(k)-1},x_{n(k)})=\varepsilon$.

(iii) $\lim\limits_{k\to\infty} d(x_{m(k)-1},x_{n(k)-1})=\varepsilon$.

(iv) $\lim\limits_{k\to\infty} d(x_{m(k)},x_{n(k)+1})=\varepsilon$.

(v) $\lim\limits_{k\to\infty} d(x_{m(k)+1},x_{n(k)+1})=\varepsilon$.

(vi) $\lim\limits_{k\to\infty} d(x_{m(k)+1},x_{n(k)+2})=\varepsilon$.

2 Main results

In this section, we show the existence, uniqueness and iterative approximations of fixed points for mappings with contractive conditions depending on monotonically deceasing functions and also by using altering distance functions. As consequence of these results, we can extend the fixed point theorems of E. Rakotch [7] and D.S. Jaggi [3], as well as the contractive conditions of integral type given by A. Branciari [2] and M. Kang *et al* [5].

Theorem 2.1. Let (M,d) be a complete metric space and let $T:M\longrightarrow M$ be a mapping satisfying the following condition:

$$\psi(d(Tx,Ty))\leq\alpha(d(x,y))\psi(d(x,y)) \tag{2.1}$$

where $\psi\in\Psi$ and $\alpha:\mathbb{R}_+\longrightarrow[0,1)$ with

$$\limsup_{s\to t}\alpha(s)<1,\quad \text{for all } t>0. \tag{2.2}$$

Then T has a unique fixed point $z_0\in M$ such that for each $x\in M$, $\lim\limits_{n\to\infty}T^nx=z_0$.

Proof. Let x_0 be an arbitrary point. We define the iterate sequence $x_{n+1}=Tx_n=T^nx_0$. It follows from (2.1) and (2.2) that

$$\psi(d(x_n,x_{n+1}))\leq\alpha(d(x_{n-1},x_n))\psi(d(x_{n-1},x_n))<\psi(d(x_{n-1},x_n)). \tag{2.3}$$

Since ψ is a non-decreasing function, then $(d(x_n,x_{n+1}))_n$ is a decreasing sequence, which implies that there exits a constant γ such that $\lim\limits_{n\to\infty}d(x_n,x_{n+1})=\gamma\geq0$. Now, we are going to prove that $\gamma=0$. Suppose that $\gamma>0$, then taking limits $n\to\infty$ in (2.3) and using inequality (2.2) we conclude that

$$\begin{aligned}0<\psi(\gamma)&\leq\limsup_{n\to\infty}\psi(d(x_n,x_{n+1}))\leq\limsup_{n\to\infty}[\alpha(d(x_{n-1},x_n))\psi(d(x_{n-1},x_n)]\\&\leq\limsup_{n\to\infty}\alpha(d(x_{n-1},x_n))\limsup_{n\to\infty}\psi(d(x_{n-1},x_n))\\&\leq\limsup_{s\to\gamma}\alpha(s)\psi(\gamma)<\psi(\gamma)\end{aligned}$$

which is a contradiction. Thus $\gamma=0$, that is

$$\lim_{n\to\infty}d(x_n,x_{n+1})=0.$$

Now, we claim that $(x_n)_n$ is a Cauchy sequence in M. Suppose that $(x_n)_n$ is not a Cauchy sequence, which means that there exists an $\varepsilon_0>0$ such that for each positive integer k, there are positive integers $m(k)$ and $n(k)$ with $m(k)>n(k)>k$ such that $d(x_{m(k)},x_{n(k)})\geq\varepsilon_0$ and $d(x_{m(k)-1},x_{n(k)})<\varepsilon_0$. From Lemma 1.3 we have

$$\varepsilon_0=\lim_{k\to\infty}d(x_{m(k)},x_{n(k)+1})=\lim_{k\to\infty}d(x_{m(k)+1},x_{n(k)+2}). \tag{2.4}$$

In view of (2.1) and (2.2) we deduce that

$$\begin{aligned}\psi(d(x_{m(k)+1},x_{n(k)+2}))\leq&\alpha(d(x_{m(k)},x_{n(k)+1}))\psi(d(x_{m(k)},x_{n(k)+1}))\\<&\psi(d(x_{m(k)},x_{n(k)+1})).\end{aligned} \tag{2.5}$$

Taking limit as $n \to \infty$ in (2.5), and from inequality (2.2) and equation (2.4) we have that

$$\begin{aligned} 0 \leq \psi(\varepsilon_0) =& \limsup_{k\to\infty} \psi(d(x_{m(k)+1}, x_{n(k)+2})) \\ \leq& \limsup_{k\to\infty} \alpha(d(x_{m(k)}, x_{n(k)+1})) \limsup_{k\to\infty} \psi(d(x_{m(k)}, x_{n(k)+1})) < \psi(\varepsilon_0) \end{aligned}$$

which is a contradiction. Thus $(x_n)_n$ is a Cauchy sequence in the complete metric space (M, d), hence there exists $z_0 \in M$ such that $\lim_{n\to\infty} T^n x = z_0$.

Now, we are going to prove that $Tz_0 = z_0$. By using inequalities (2.1) and (2.2) we get

$$\begin{aligned} 0 \leq \psi(d(x_{n+1}, Tz_0)) \leq& \alpha(d(x_n, z_0))\psi(d(x_n, z_0)) \\ <& \psi(d(x_n, z_0)) \end{aligned}$$

where $\psi(d(x_n, z_0)) \to 0$ as $n \to \infty$. I.e., $\lim_{n\to\infty} \psi(d(x_{n+1}, Tz_0)) = 0$. Since $\psi \in \Psi$ we have that $\lim_{n\to\infty} d(x_{n+1}, Tz_0) = 0$. Consequently,

$$d(z_0, Tz_0) \leq d(z_0, x_{n+1}) + d(x_{n+1}, Tz_0) \to 0, \quad \text{as } n \to \infty,$$

which means that $z_0 = Tz_0$. Finally, we will prove that T has a unique fixed point. Suppose that T has another fixed point $y_0 \in M$. Then

$$0 < \psi(d(y_0, z_0)) = \psi(d(Ty_0, Tz_0)) \leq \alpha(d(y_0, z_0))\psi(d(y_0, z_0)) < \psi(d(y_0, z_0)),$$

which is a contradiction, completing in this form the proof. □

Corollary 2.2. Let (M, d) be a complete metric space and let $T : M \longrightarrow M$ a mapping satisfying the inequality,

$$\int_0^{\psi(d(Tx,Ty))} \varphi(t)dt \leq \alpha(d(x, y)) \int_0^{\psi(d(x,y))} \varphi(t)dt, \tag{2.6}$$

where $\psi \in \Psi$, $\varphi \in \Phi$ and $\alpha : \mathbb{R}_+ \longrightarrow [0, 1)$ with $\sup_{s\to t} \alpha(s) < 1$, for all $t > 0$. Then T has a unique fixed point $z_0 \in M$ such that $\lim_{n\to\infty} T^n x = z_0$.

Proof. We define $\psi_0 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by $\psi_0(x) = \int_0^x \varphi(t)dt$ for $\varphi \in \Phi$, then $\psi_0 \in \Psi$ and so inequality (2.6) becomes

$$\psi_0(\psi(d(Tx, Ty))) \leq \alpha(d(x, y))\psi_0(\psi(d(x, y)))$$

which further can be written as

$$\psi_1(d(Tx, Ty)) \leq \alpha(d(x, y))\psi_1(d(x, y))$$

where $\psi_1 = \psi_0 \circ \psi \in \Psi$. Hence, from Theorem 2.1 we conclude that T has a unique fixed point. □

Remark 1. Notice that if we consider $\psi = I_d$ the identity mapping in Corollary 2.2, we obtain the Theorem 3.1 of [5].

Theorem 2.3. Let T a mapping from a complete metric space (M, d) into itself satisfying the following inequality

$$\psi(d(Tx, Ty)) \leq \alpha(d(x, y))\psi(d(x, Tx)) + \beta(d(x, y))\psi(d(y, Ty)) \tag{2.7}$$

where $\psi \in \Psi$ and $\alpha, \beta : \mathbb{R}_+ \longrightarrow [0, 1)$ with

$$\left.\begin{aligned} \alpha(t) + \beta(t) < 1 \ \forall t \in \mathbb{R}_+, \ \limsup_{s\to 0^+} \beta(s) < 1 \\ \limsup_{s\to t^+} \frac{\alpha(s)}{1 - \beta(s)} < 1, \ \forall t > 0 \end{aligned}\right\}. \tag{2.8}$$

Then T has a unique fixed point $z_0 \in M$ such that for each $x \in M$, $\lim_{n\to\infty} T^n x = z_0$.

Proof. Let $x \in M$ be an arbitrary point, and let the sequence $(x_n)_n$ defined as $x_{n+1} = Tx_n$, $n = 1, \ldots$. It follows from (2.7) that

$$\psi(d(x_n, x_{n+1})) \leq \alpha(d(x_{n-1}, x_n))\psi(d(x_{n-1}, x_n)) + \beta(d(x_{n-1}, x_n))\psi(d(x_n, x_{n+1}))$$

or equivalently

$$\psi(d(x_n, x_{n+1})) \leq \frac{\alpha(d(x_{n-1}, x_n))}{1 - \beta(d(x_{n-1,x_n}))}\psi(d(x_{n-1}, x_n)).$$

Now, from (2.8) we obtain
$$\psi(d(x_n, x_{n+1})) \le \psi(d(x_{n-1}, x_n)) \quad \forall n \in \mathbb{Z}_+.$$
As in the proof of Theorem 2.1, we conclude that the sequence $(d(x_n, x_{n+1}))_n$ is non-increasing and converges to 0, that is,
$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.9}$$
We will prove now that $(x_n)_n$ is a Cauchy sequence in M. Suppose that $(x_n)_n$ is not a Cauchy sequence. Then, there exists an $\varepsilon_0 > 0$ and sequences of positive integers $(m(k))_k$ and $(n(k))_k$ such that $m(k) > n(k) > k$ satisfying
$$\left.\begin{array}{l} d(x_{m(k)}, x_{n(k)}) \ge \varepsilon_0 \\ d(x_{m(k)-1}, x_{n(k)}) < \varepsilon_0 \end{array}\right\}.$$
From Lemma 1.3 we have
$$\varepsilon_0 = \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+2})$$
and from (2.7) we get that
$$\begin{aligned}\psi(\varepsilon_0) =& \limsup_{k\to\infty} \psi(d(x_{m(k)+1}, x_{n(k)+2})) \\ \le& \limsup_{k\to\infty} \big[\alpha(d(x_{m(k)}, x_{n(k)+1}))\psi(d(x_{m(k)}, x_{m(k)+1})) \\ &+ \beta(d(x_{m(k)}, x_{n(k)+1}))\psi(d(x_{n(k)+1}, x_{n(k)+2}))\big] \\ &\le \limsup_{k\to\infty} \alpha(d(x_{m(k)}, x_{n(k)+1})) \limsup_{k\to\infty} \psi(d(x_{m(k)}, x_{m(k)+1})) \\ &+ \limsup_{k\to\infty} \beta(d(x_{m(k)}, x_{n(k)+1})) \limsup_{k\to\infty} \psi(d(x_{n(k)+1}, x_{n(k)+2})) \\ <& \limsup_{k\to\infty} \psi(d(x_{m(k)}, x_{m(k)+1})) + \limsup_{k\to\infty} \psi(d(x_{n(k)+1}, x_{n(k)+2})) = 0\end{aligned}$$
which is a contradiction. Hence $(x_n)_n$ is a Cauchy sequence in the complete metric space (M, d), so there exists a $z_0 \in M$ such that $\lim_{n\to\infty} T^n x = z_0$. Now, we are going to prove that z_0 is a fixed point of T. This means that $d(z_0, Tz_0) = 0$. If we suppose that $d(z_0, Tz_0) \ne 0$, from (2.7) and (2.9) we obtain
$$\begin{aligned}0 <& \psi(d(z_0, Tz_0)) = \limsup_{n\to\infty} \psi(d(T^n x, Tz_0)) \\ \le& \limsup_{n\to\infty} \big[\alpha(d(T^{n-1}x, z_0))\psi(d(T^{n-1}x, T^n x))\big] \\ &+ \limsup_{n\to\infty} \big[\beta(d(T^{n-1}x, z_0))\psi(d(z_0, Tz_0))\big] \\ =& \limsup_{s\to 0^+} [\beta(s)\psi(d(z_0, Tz_0))] < \psi(d(z_0, Tz_0))\end{aligned}$$
which is not possible. Thus $d(z_0, Tz_0) = 0$. I.e., $z_0 = Tz_0$. Finally, we prove that T has a unique fixed point. Suppose that T has another fixed point $z_1 \ne z_0$. It follows from (2.7) that
$$\begin{aligned}0 <& \psi(d(z_0, z_1)) = \psi(d(Tz_0, Tz_1)) \\ \le& \alpha(d(z_0, z_1))\psi(d(z_0, Tz_0)) + \beta(d(z_0, z_1))\psi(d(z_1, Tz_1)) = 0\end{aligned}$$
having in this way a contradiction, completing therefore the proof. □

Corollary 2.4. Let (M, d) be a complete metric space and let $T : M \longrightarrow M$ a mapping satisfying the following condition
$$\int_0^{\psi(d(Tx,Ty))} \varphi(t)dt \le \alpha(d(x, y)) \int_0^{\psi(d(x,Tx))} \varphi(t)dt + \beta(d(x, y)) \int_0^{\psi(d(y,Ty))} \varphi(t)dt \tag{2.10}$$
where $\psi \in \Psi$, $\varphi \in \Phi$ and $\alpha, \beta : \mathbb{R}_+ \longrightarrow [0, 1)$ with
$$\left.\begin{array}{c} \alpha(t) + \beta(t) < 1 \ \forall t \in \mathbb{R}_+, \ \limsup_{s\to 0^+} \beta(s) < 1 \\ \limsup_{s\to t^+} \dfrac{\alpha(s)}{1 - \beta(s)} < 1, \ \forall t > 0 \end{array}\right\}.$$
Then T has a unique fixed point $z_0 \in M$ such that for each $x \in M$, $\lim_{n\to\infty} T^n x = z_0$.

Proof. We define $\psi_0 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by $\psi_0(x) = \int_0^x \varphi(t)dt$ for $\varphi \in \Phi$, then $\psi_0 \in \Psi$ and (2.10) becomes,

$$\psi_0(\psi(d(Tx, Ty))) \leq \alpha(d(x, y))\psi_0(\psi(d(x, Tx))) + \beta(d(x, y))\psi_0(\psi(d(y, Ty))).$$

Setting $\psi_1 = \psi_0 \circ \psi \in \Psi$, from Theorem 2.3 we conclude that T has a unique fixed point. □

Remark 2. Notice that if we consider $\psi = I_d$ in Corollary 2.4, we obtain the Theorem 3.2 of [5].

Theorem 2.5. Let (M, d) be a complete metric space and let $T : M \longrightarrow M$ be a mapping satisfying the following condition:

$$\psi(d(Tx, Ty)) \leq \beta(d(x, y))\left(\psi(d(x, Tx)) + \psi(d(y, Ty))\right)$$

where $\psi \in \Psi$ and $\beta : \mathbb{R}_+ \longrightarrow [0, \frac{1}{2})$ is a function such that

$$\lim_{s\to t^+} \sup \frac{\beta(s)}{1 - \beta(s)} < 1, \quad \forall t > 0.$$

Then T has a unique fixed point $z_0 \in M$. Moreover, for each $x \in M$, $\lim_{n\to\infty} T^n x = z_0$.

Proof. The proof is essentially the same as the proof of Theorem 2.3, hence it is omitted. □

Corollary 2.6. Let (M, d) be a complete metric space and let $T : M \longrightarrow M$ be a mapping satisfying the following inequality

$$\int_0^{\psi(d(Tx,Ty))} \varphi(t)dt \leq \beta(d(x, y)) \left(\int_0^{\psi(d(x,Tx))} \varphi(t)dt + \int_0^{\psi(d(y,Ty))} \varphi(t)dt \right)$$

where $\psi \in \Psi$, $\varphi \in \Phi$ and $\beta : \mathbb{R}_+ \longrightarrow [0, \frac{1}{2})$ is a function with

$$\limsup_{s\to t^+} \frac{\beta(s)}{1 - \beta(s)} < 1, \quad \forall t > 0.$$

Then T has a unique fixed point $z_0 \in M$.

Proof. Straightforward as the proof of Corollary 2.4. □

Remark 3. Notice that if we consider $\psi = I_d$ in Corollary 2.6, we obtain the Theorem 3.3 of [5].

Example 1. Let $M = \mathbb{R}_+$ endowed with the euclidean metric $d = |\cdot|$. We define $T : M \longrightarrow M$ by

$$Tx = \frac{x}{1 + x}.$$

Consider $\alpha : \mathbb{R}_+ \longrightarrow [0, 1)$ given by

$$\alpha(t) = \begin{cases} \frac{1}{2}, \text{if } t = 0 \\ \frac{1}{(1+t)^2}, \text{if } t \in (0, \infty). \end{cases}$$

Finally, let $\psi(t) = t^2$ and $\varphi(t) = 2t$ for all $t \in \mathbb{R}_+$. Then is clear that T is not a Banach contraction, thus the BCP cannot be apply. Also, in [5] was proved that the contractive condition of Theorem 3.1 in [5] does not hold, therefore that theorem cannot be applied. Now, notice that

$$\begin{aligned}\psi(d(Tx, Ty)) =& \psi(|Tx - Ty|) = (Tx - Ty)^2 = \left(\frac{x}{1 + x} - \frac{y}{1 + y}\right) \\ =& \frac{(x - y)^2}{(1 + x)^2(1 + y)^2} \leq \frac{(x - y)^2}{(1 + |x - y|)^2} = \alpha(d(x, y))\psi(d(x, y)).\end{aligned}$$

Thus, we have proved that condition (2.1) is satisfied and the conditions of Theorem 2.1 are fulfilled. Therefore, from Theorem 2.1 we guarantee that T has a unique fixed point $z_0 = 0$.

Theorem 2.7. Let (M, d) be a complete metric space and let $T : M \longrightarrow M$ be a continuous mapping. We denote

$$m(x, y) = \max\left\{\frac{d(x, Tx)d(y, Ty)}{d(x, y)}, d(x, y)\right\} \tag{2.11}$$

for all $x, y \in M$, $x \neq y$. Suppose that T satisfies the following condition:

$$\psi(d(Tx, Ty)) \leq \alpha(d(x, y))\psi(m(x, y)) \tag{2.12}$$

for all $x, y \in M$, $\psi \in \Psi$ and $\alpha : \mathbb{R} \longrightarrow [0, 1)$ is a function with

$$\limsup_{s\to t} \alpha(s) < 1, \quad \text{for all } t > 0. \tag{2.13}$$

Then T has a unique fixed point $z_0 \in M$ such that $\lim_{n\to\infty} T^n x = z_0$ for all $x \in M$.

Proof. Let $x_0 \in M$ be an arbitrary point and we define the sequence $x_{n+1} = Tx_n = T^n x_0$, $n = 0, 1, \ldots$.It follows from (2.12) that

$$\psi(d(x_n, x_{n+1})) = \psi(d(Tx_{n-1}, Tx_n)) \leq \alpha(d(x_{n-1}, x_n))\psi(m(x_{n-1}, x_n)).$$

By using (2.11) we get

$$\begin{aligned} m(x_{n-1}, x_n) &= \max\left\{\frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\right\} \\ &= \max\left\{\frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\right\} \\ &= \max\left\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\right\}, \end{aligned}$$

which implies,

$$\psi(d(x_n, x_{n+1})) \leq \alpha(d(x_{n-1}, x_n))\psi(\max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}).$$

Since ψ is monotonically non-decreasing and from (2.13) we have

$$\psi(d(x_n, x_{n+1})) < \psi(\max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}) = \psi(d(x_{n-1}, x_n)).$$

Then, it follows that $(d(x_n, x_{n+1}))_n$ is a monotone decreasing sequence of numbers. Consequently, there exists $\gamma \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = \gamma$. Suppose that $\gamma > 0$, then

$$0 < \psi(\gamma) \leq \psi(d(x_n, x_{n+1})) < \psi(d(x_{n-1}, x_n))$$

taking limits as $n \to \infty$ inequality above yields $\psi(\gamma) < \psi(\gamma)$ which is a contradiction. Therefore $\gamma = 0$, thus

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.14}$$

Now, we prove that $(x_n)_n$ is a Cauchy sequence in M. Suppose that $(x_n)_n$ is not a Cauchy sequence, then there exists a ε_0 and subsequences $(x_{m(k)})_k$, $(x_{n(k)})_k$ of $(x_n)_n$ with $m(k) > n(k) > k$ such that $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon_0$, $d(x_{m(k)-1}, x_{n(k)}) < \varepsilon_0$. From Lemma 1.3 we have

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon_0. \tag{2.15}$$

Inequality (2.12) gives us

$$\begin{aligned} \psi(\varepsilon_0) &\leq \psi(d(x_{m(k)}, x_{n(k)})) = \psi(d(Tx_{m(k)-1}, Tx_{n(k)-1})) \\ &\leq \alpha(d(x_{m(k)-1}, x_{n(k)-1}))\psi(m(x_{m(k)-1}, x_{n(k)-1})). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & m(x_{m(k)-1}, x_{n(k)-1}) = \\ & \max\left\{\tfrac{d(x_{m(k)-1}, x_{m(k)})d(x_{n(k)-1}, x_{n(k)})}{d(x_{m(k)-1}, x_{n(k)-1})}, d(x_{m(k)-1}, x_{n(k)-1})\right\}. \end{aligned}$$

Now, by taking upper limit as $k \to \infty$ and using (2.14) and (2.15), we have

$$\psi(\varepsilon_0) < \psi(\max\{0, \varepsilon_0\}) = \psi(\varepsilon_0)$$

which is a contradiction. Hence, $(x_n)_n$ is a Cauchy sequence in the complete metric space (M, d). Thus, there exists $z_0 \in M$ such that $\lim_{n\to\infty} x_n = z_0$. Furthermore, the continuity of T implies that $z_0 = Tz_0$. Finally, we are going to prove that the fixed point is unique. If there is another fixed point z_1 of T, with $z_0 \neq z_1$, then

$$\psi(d(z_0, z_1)) = \psi(d(Tz_0, Tz_1)) \leq \alpha(d(z_0, z_1))\psi(m(z_0, z_1))$$

where

$$m(z_0, z_1) = \max\left\{\frac{d(z_0, Tz_0)d(z_1, Tz_1)}{d(z_0, z_1)}, d(z_0, z_1)\right\} = \max\{0, d(z_0, z_1)\}.$$

Thus we have that $\psi(d(z_0, z_1)) < \psi(d(z_0, z_1))$, which is a contradiction. Hence z_0 in the unique fixed point of T in M.

□

As a consequence of Theorem 2.7 we have the following result.

Theorem 2.8. Let (M,d) be a complete metric space and let $T : M \longrightarrow M$ be a continuous mapping. If T satisfies any of the following contractive conditions

(i) $\psi(d(Tx,Ty)) \leq \gamma\psi(m(x,y))$.

(ii) $d(Tx,Ty) \leq \alpha(d(x,y))m(x,y)$.

(iii) $d(Tx,Ty) \leq \gamma m(x,y)$.

(iv) $d(Tx,Ty) \leq \delta d(x,y) + \eta\frac{d(x,Tx)+d(y,Ty)}{d(x,y)}$.

(v) $\displaystyle\int_0^{\psi(d(Tx,Ty))} \varphi(t)dt \leq \alpha(d(x,y)) \int_0^{\psi(m(x,y))} \varphi(t)dt.$

(vi) $\displaystyle\int_0^{\psi(d(Tx,Ty))} \varphi(t)dt \leq \gamma \int_0^{\psi(m(x,y))} \varphi(t)dt.$

(vii) $\displaystyle\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \alpha(d(x,y)) \int_0^{m(x,y)} \varphi(t)dt.$

(viii) $\displaystyle\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \gamma \int_0^{m(x,y)} \varphi(t)dt.$

(ix) $\displaystyle\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \delta \int_0^{d(x,y)} \varphi(t)dt + \eta \int_0^{\frac{d(x,Tx)+d(y,Ty)}{d(x,y)}} \varphi(t)dt.$

For all $x, y \in M$, $\gamma, \delta, \eta \in [0,1)$, $\delta + \eta < 1$, $\psi \in \Psi$, $\varphi \in \Phi$, $m(x,y)$ defined as in (2.11) and $\alpha : \mathbb{R}_+ \longrightarrow [0,1)$ a function with $\limsup_{s\to t} \alpha(s) < 1$ for all $t > 0$. Then, T has a unique fixed point $z_0 \in M$ such that for all $x \in M$, $\lim_{n\to\infty} T^n x_= z_0$.

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