Palestine Journal of Mathematics

Vol. 1(2) (2012) , 117–132

Anisotropic Bianchi Type-I Massive String Cosmological Models in General Relativity

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Communicated by Jose Luis Lopez-Bonilla

MSC2010 Classications: 83C15.

PACS number: 98.80.Cq, 04.20.-q.

Keywords: Massive string, Bianchi type I models, Accelerating universe, Anisotropic cosmological models.

Authors would like to thank the Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune, India for providing facility and support where this was carried out. The authors also thank Prof. Jose Luis Lopez-Bonilla for his helpful suggestions.

Abstract. The present study deals with a spatially homogeneous and anisotropic Bianchi type-I cosmological model representing massive string. The energy-momentum tensor for such string as formulated by Letelier (1983) and Stachel (1980) is used to construct massive string cosmological model for which we assume the condition $A = B^m$, where A and B are the metric coefficients and m is constant. Firstly general solution, its physical and kinematic aspects are explored. Two particular cases have also been described which are depending on the vanishing of two constants of integration respectively. It is observed that for $m > \frac{3}{2}$, the universe is dominated by massive strings where as for $m < \frac{3}{2}$, the universe is dominated by string evolution in initial stage but at later stage string dominance disappear, which is in agreement with the current astronomical observations. Our models are found to be in accelerating phase which are consistent to the recent observations of SNe Ia and CMBR. Some physical and geometric aspects of these particular models are also discussed.

1 Introduction

It is a challenging problem to determine the exact physical scenario at very early stages of the formation of the universe. In recent years, there has been considerable interest in string cosmology. Cosmic strings are topologically stable objects which might be found during a phase transition in the early universe (Kibble [1]). Cosmic strings play an important role in the study of the early universe. These arise during the phase transition after the big bang explosion as the temperature goes down below some critical temperature as predicted by grand unified theories (GUTs) (Zel'dovich et al. [2]; Kibble [1, 3]; Everett [4]; Vilenkin [5]). It is believed that cosmic strings give rise to density perturbations which lead to the formation of galaxies (Zel'dovich [6]). Massive closed loops of strings serve as seeds for the formation of large structures like galaxies and cluster of galaxies. While matter is accreted onto loops, they oscillate violently and lose their energy by gravitational radiation and therefore they shrink and disappear. As the stress-energy of a string can be coupled with the gravitational field, it may be interesting to study the gravitational effects that arise from strings. The pioneering work in the formulation of the energy-momentum tensor for classical massive strings was done by Letelier [7] who considered the massive strings to be formed by geometric strings with particle attached along its extension. Letelier [8] first used this idea in obtaining cosmological solutions in Bianchi-I and Kantowski-Sachs space-time. Stachel [9] has also studied massive string as null string and perfect fluid of string in general relativistic set up.

Though, the present day universe is well described by an isotropic and homogeneous FRW space-time, there are serious theoretical arguments about the existence of an anisotropic phase in the evolution of the universe [10]. These ideas were further supported by the observational data from COBE (Cosmic Background Explorer) and WMAP (The Wilkinson Microwave Anisotropy Probe) where small anisotropy in the microwave background radiation was found. Since experimental data favours an anisotropic universe and hence it motivates to study models of the universe with anisotropic background space-time structure. The simplest anisotropic models of the universe are Bianchi type-I homogeneous models whose spatial sections are flat but the expansion or contraction rate are directional dependent. The advantages of these anisotropic models are that they have a significant role in the description of the evolution of the early phase of the universe and they help in finding more general cosmological models than the isotropic FRW models (widely accepted). The homogeneous and anisotropic Bianchi type-I models have been considered by a number of authors [11]−[27] in different contexts.

Bali et al. [28]−[34] have obtained Bianchi types-I, III and IX string cosmological models in general relativity. Yadav et al. [35] have studied some Bianchi type I viscous fluid string cosmological models with magnetic field. Wang [36]−[39] has also discussed LRS Bianchi type-I and Bianchi type-III cosmological models for a cloud of strings with bulk viscosity. Saha et al. [40] and Saha [41] have studied Bianchi type-I cosmological models in presence of magnetic flux in different contexts. Yadav, Pradhan and Rai [42] have obtained the integrability of cosmic string in Bianchi type-III space-time in presence of bulk viscous fluid by applying a new technique. Reddy [43, 44], Reddy et al. [45]−[47], Rao et al. [48]−[51], Pradhan [52, 53], Pradhan et al. [54]− [57], Tripathi et al. [58, 59], Belinchon [60] and Amirhashchi & Zainuddin [61] have studied string cosmological models in different contexts.

Saha and Visinescu [62] and Saha et al. [63] have studied Bianchi type-I models with cosmic string in presence of magnetic flux in two different contexts. Recently, Pradhan had briefly reported first idea about the general solution of anosotropic Bianchi type-I cloud of string cosmological model [64]. Motivated by the above discussions, in this paper, we study all detailed aspects of physical and kinematic properties as well as additional features of Pradhan's study [64]. We present new solutions which are different from the other author's solutions. The detailed study has attractive features for a particular case $k_5 = 0$ in which energy density equation has self consistency built in. The paper is organized as follows. The metric and the field equations are presented in Section 2. In Section 3, we deal with an exact solution of the field equations with cloud of strings. In Section 4, we describe some physical and geometric properties of this general model. In Sections 5 and 6, we have derived two particular models of the universe and also described their physical significances. Finally, in Section 7, concluding remarks are given.

2 The Metric and Field Equations

We consider the spatially homogeneous and anisotropic Bianchi type-I metric in the form

$$ds^2 = -dt^2 + A^2dx^2 + B^2dy^2 + C^2dz^2, \tag{2.1}$$

where A, B and C are the metric functions of cosmic time t only.

The energy-momentum tensor for a cloud of massive strings has the form

$$T_i^j = \rho u_i u^j - \lambda x_i x^j, \tag{2.2}$$

where u_i and x_i satisfy condition

$$u^i u_i = -x^i x_i = -1, \tag{2.3}$$

and

$$u^i x_i = 0, \tag{2.4}$$

where ρ is the rest energy density for a cloud strings with particles attached to them, λ is the string tension density, x^i is a unit space-like vector representing the direction of strings so that $x^1 \neq 0$ and $x^2 = x^3 = x^4$ and u^i is the four velocity vector satisfying the conditions $g_{ij}u^i u^j = -1$. In a co-moving coordinate system, we have

$$u^i = (0, 0, 0, 1), \quad x^i = \left(\frac{1}{A}, 0, 0, 0\right). \tag{2.5}$$

If the particle density of the configuration is denoted by ρ_p, then we have

$$\rho = \rho_p + \lambda. \tag{2.6}$$

The Einstein's field equations (in gravitational units $G = c = 1$) read

$$R_i^j - \frac{1}{2}Rg_i^j = -8\pi T_i^j, \tag{2.7}$$

where R_i^j is the Ricci tensor; $R = g^{ij}R_{ij}$ is the Ricci scalar.

The field equations (2.7) together with (2.2) for the line-element (2.1) subsequently lead to the following system of equations:

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = 8\pi\lambda, \tag{2.8}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} = 0, \tag{2.9}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = 0, \tag{2.10}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} = 8\pi\rho, \tag{2.11}$$

where an over dot stands for the first and double over dot for the second derivative with respect to cosmic time t. As a consequence of the above field equations (2.8)-(2.11), one can get the following relation:

$$\frac{(A\ddot{B}C)}{ABC} = 4\pi(3\rho + \lambda). \tag{2.12}$$

The spatial volume for the model (2.1) is given by

$$V^3 = ABC. \tag{2.13}$$

We define $V = (ABC)^{\frac{1}{3}}$ as the average scale factor so that the Hubble parameter in anisotropic models may be defined as

$$H = \frac{\dot{V}}{V} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right). \tag{2.14}$$

We define the generalized mean Hubble's parameter H as

$$H = \frac{1}{3}(H_1 + H_2 + H_3), \tag{2.15}$$

where $H_1 = \frac{\dot{A}}{A}$, $H_2 = \frac{\dot{B}}{B}$ and $H_3 = \frac{\dot{C}}{C}$ are the directional Hubble's parameters in the directions of x, y and z respectively.

An important observational quantity is the deceleration parameter q, which is defined as

$$q = -\frac{V\ddot{V}}{\dot{V}^2}. \tag{2.16}$$

The physical quantities of observational interest in cosmology i.e. the expansion scalar θ, the shear scalar σ^2 and the average anisotropy parameter A_m are defined as

$$\theta = u^i_{;i} = 3\frac{\dot{V}}{V} = \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right), \tag{2.17}$$

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{2}\left[\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2}\right] - \frac{\theta^2}{6}, \tag{2.18}$$

where

$$\sigma_{ij} = u_{i;j} + \frac{1}{2}(u_{i;k}u^k u_j + u_{j;k}u^k u_i) + \frac{1}{3}\theta(g_{ij} + u_i u_j), \tag{2.19}$$

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\triangle H_i}{H}\right)^2, \tag{2.20}$$

where $\triangle H_i = H_i - H (i = 1, 2, 3)$.

3 Solutions of the Field Equations

The field equations (2.8)-(2.11) are a system of four equations in five unknown parameters A, B, C, ρ and λ. One additional constraint relating these parameters is required to obtain explicit solutions of the system. We assume a relation

$$A = B^m, \tag{3.1}$$

where m is constant. The motive behind assuming this condition is explained with reference to Thorne [65], the observations of the velocity-red-shift relation for extragalactic sources suggest that Hubble expansion of the universe is isotropic today within $\approx$ 30 per cent [66, 67]. To put more precisely, red-shift studies place the limit $\frac{\sigma}{H} \leq 0.3$ on the ratio of shear σ to Hubble constant H in the neighbourhood of our Galaxy today. Collins et al. [68] have pointed out that for spatially homogeneous metric, the normal congruence to the homogeneous expansion satisfies that the condition $\frac{\sigma}{\theta}$ is constant.

Eqs. (2.10) and (3.1) lead to

$$(m+1)\frac{\ddot{B}}{\dot{B}} + m^2\frac{\dot{B}}{B} = 0, \tag{3.2}$$

which on integration reduces to

$$\dot{B}^{m+1}B^{m^2} = k_1, \tag{3.3}$$

where k_1 is a constant of integration. Integrating Eqs. (3.3), we obtain

$$B = \left(\frac{k_2 t + \alpha}{k_3}\right)^{k_3}, \tag{3.4}$$

where α is an integrating constant and $k_2^{m+1} = k_1$ and $k_3 = \frac{m+1}{m^2+m+1}$.

From (3.1) and (3.4), we get

$$A = \left(\frac{k_2 t + \alpha}{k_3}\right)^{mk_3}. \tag{3.5}$$

Now subtracting (2.10) from (2.9), we obtain

$$\frac{\ddot{C}}{C} - \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{C}}{AC} - \frac{\dot{A}\dot{B}}{AB} = 0, \tag{3.6}$$

which after integration reduces to

$$(B\dot{C} - C\dot{B})A = k_4, \tag{3.7}$$

where k_4 is a constant of integration. Again integrating (3.7), we obtain

$$\frac{C}{B} = k_4 \int \frac{dt}{AB^2} + k_5, \tag{3.8}$$

where k_5 is an integrating constant. Using Eqs. (3.4) and (3.5) in to (3.8), we obtain

$$C = \frac{k_4}{1-(m+2)k_3}\left(\frac{k_2 t + \alpha}{k_3}\right)^{1-(m+1)k_3} + k_5\left(\frac{k_2 t + \alpha}{k_3}\right)^{k_3}. \tag{3.9}$$

Hence the model (2.1) is reduced to

$$ds^2 = -dt^2 + \left(\frac{k_2 t + \alpha}{k_3}\right)^{2mk_3} dx^2 + \left(\frac{k_2 t + \alpha}{k_3}\right)^{2k_3} dy^2$$

$$+ \left[\frac{k_4}{1-(m+2)k_3}\left(\frac{k_2 t + \alpha}{k_3}\right)^{1-(m+1)k_3} + k_5\left(\frac{k_2 t + \alpha}{k_3}\right)^{k_3}\right]^2 dz^2. \tag{3.10}$$

After using a suitable transformation of coordinates the model (3.10) reduces to

$$ds^2 = -\left(\frac{k_3}{k_2}\right)^2 dT^2 + T^{2mk_3} dx^2 + T^{2k_3} dy^2$$

$$+ \left[\frac{k_4}{1-(m+1)k_3} T^{1-(m+1)k_3} + k_5 T^{k_3}\right]^2 dz^2. \tag{3.11}$$

4 Some Physical and Geometric Properties of the Model

Here we discuss some physical and geometric properties of string model (3.11). The energy density (ρ), the string tension (λ) and the particle density (ρ_p) for the model (3.11) are given by

$$8\pi\rho = \frac{mk_2^2}{T^2} + (m+1)k_2\left[\frac{MT^{-(m+1)k_3} + k_2 k_5 T^{k_3-1}}{NT^{2-(m+1)k_3} + k_5 T^{k_3+1}}\right], \tag{4.1}$$

$$8\pi\lambda = \frac{(k_3-1)k_2^2}{k_3 T^2} + \frac{LT^{k_3-1} - mk_2 MT^{-(m+1)k_3}}{NT^{2-(m+1)k_3} + k_5 T^{k_3+1}}, \tag{4.2}$$

$$8\pi\rho_p = \frac{(mk_3 - k_3 + 1)k_2^2}{k_3 T^2} + \frac{(2m+1)k_2 MT^{-(m+1)k_3} + PT^{k_3-1}}{NT^{2-(m+1)k_3} + k_5 T^{k_3+1}}, \tag{4.3}$$

where

$$L = \frac{k_2^2 k_5(k_3+1)}{k_3},$$

$$M = \frac{k_2 k_4\{(m+1)k_3 - 1\}}{k_3\{(m+2)k_3 - 1\}},$$

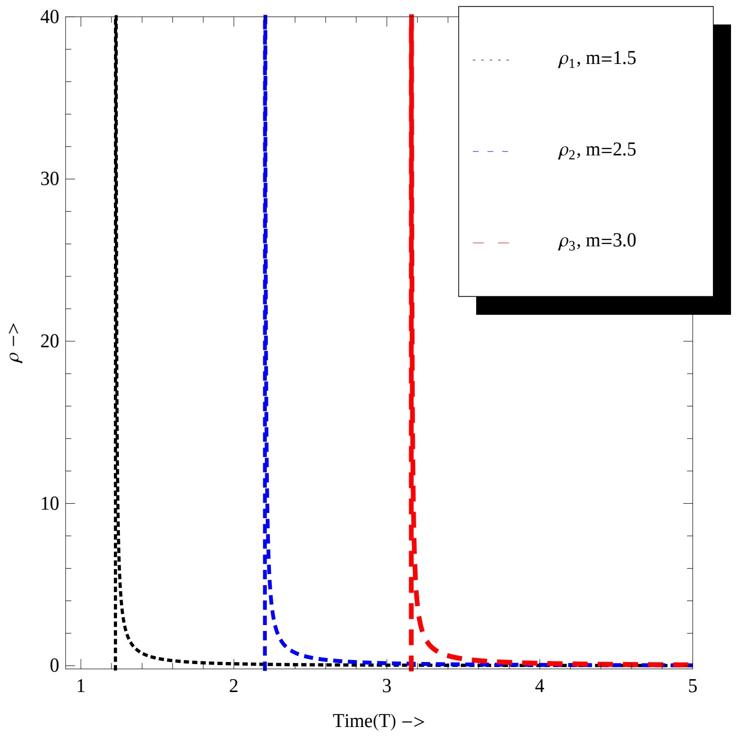

Figure 1: The plot of energy density ρ versus T with $k_2 = 1$, $k_4 = 1$, $k_5 = 1$

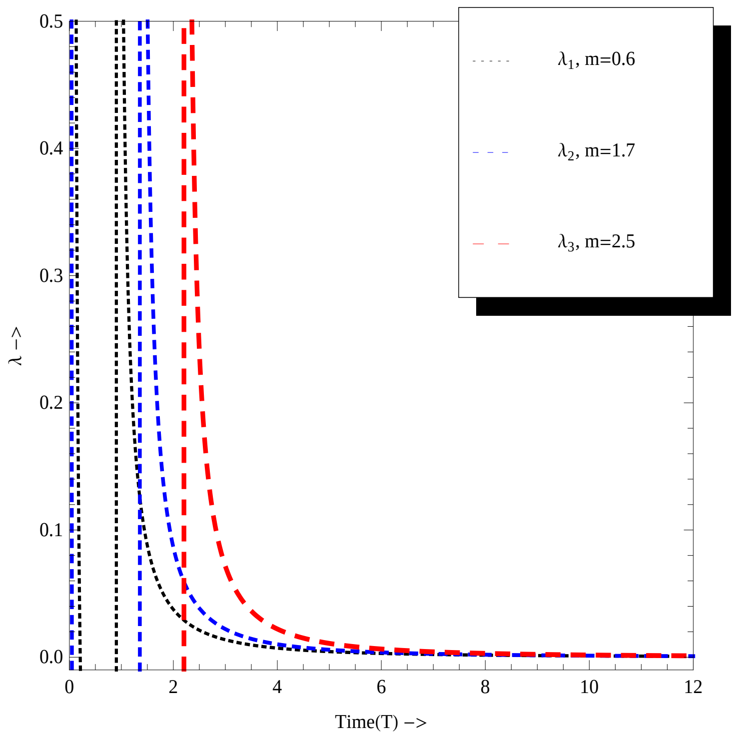

Figure 2: The plot of tension density λ versus T with $k_2 = 1.1$, $k_4 = 1$, $k_5 = 1$

$$N = \frac{k_4}{1-(m+2)k_3},$$

$$P = \frac{k_2^2 k_5 (mk_3 - 1)}{k_3}.$$

The rest energy density $\rho \geq 0$, if

$$\frac{m}{m+1} \geq \frac{\frac{M}{k_2} + k_5 T^{(m+2)k_3 - 1}}{N + k_5 T^{(m+2)k_3 - 1}}, \tag{4.4}$$

with $k_3 \neq \frac{1}{m+2}$.

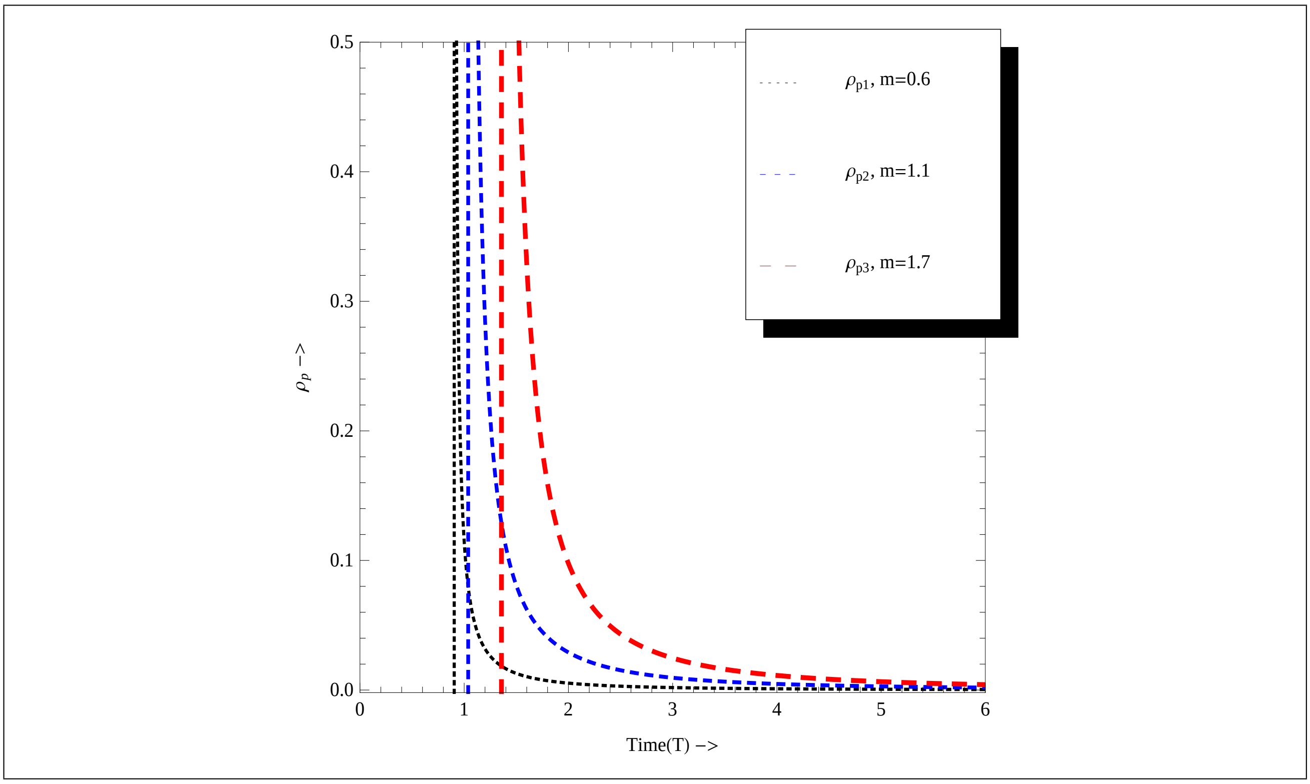

Figure 3: The plot of particle density ρ_p versus T with $k_2 = 1$, $k_4 = 1$, $k_5 = 1$

The string tension density $\lambda \geq 0$, if

$$\frac{k_3 - 1}{T^2} \geq \frac{\frac{mM}{k_2} - \frac{L}{k_2^2} T^{(m+2)k_3 - 1}}{N + k_5 T^{(m+2)k_3 - 1}}, \tag{4.5}$$

with $k_3 \neq \frac{1}{m+2}$.

The particle density $\rho_p \geq 0$, if

$$\frac{k_2 k_3 (k_3 + 1)}{k_3} \geq \frac{k_3(2m+1)M + k_2 k_5 (m k_3 - 1) T^{(m+2)k_3 - 1}}{(1-m)[N + k_5 T^{(m+2)k_3 - 1}]}, \tag{4.6}$$

with $k_3 \neq \frac{1}{m+2}$.

From Eq. (4.1), it is found that the energy density ρ is a decreasing function of time and $\rho > 0$ under condition (4.4). Figure 1 shows plot of energy density ρ versus T for different values of m. It is observed that ρ is a decreasing function of time, near $T = 0$ (very close to singularity), ρ is maximum and then decreases as universe evolves with time T. The appearance of singularity at different time T is an artifact of a coordinate transformation $T \to t$ otherwise ρ is singular very close to origin i.e. $T \approx 0$. The various values of integration constants are just representative case. Also from Eq. (4.4), $\rho \geq 0$ and accordingly integrating constants are chosen.

From Eq. (4.2), it is observed that the tension density is negative at initial time. It is observed that $\lambda \geq 0$ under condition (4.5). Figure 2 depicts a plot of the string tension λ versus T for different values of m. It is observed that λ has large values near $T \sim 0$ and it oscillates initially and then stabilized. After that λ is also decreasing function of time for different values of m and late stage they reach almost constant same value (small positive). It is pointed out by Letelier [8] that λ may be positive or negative. When $\lambda < 0$, the string phase of the universe disappears i.e. we have an anisotropic fluid of the particles.

From Eq. (4.3), it is observed that the particle density ρ_p is also a decreasing function of time and $\rho_p > 0$ under condition (4.6). Figure 3 is a plot of particle density ρ_p versus T. It is clearly seen that ρ_p is a decreasing function of time for all different values of m. Initially ρ_p has large value closure to $T \approx 0$ and the initial singularity in particle density is also closure to T. Later stage ρ_p reaches to some constant value (small positive) for all m. This corresponds

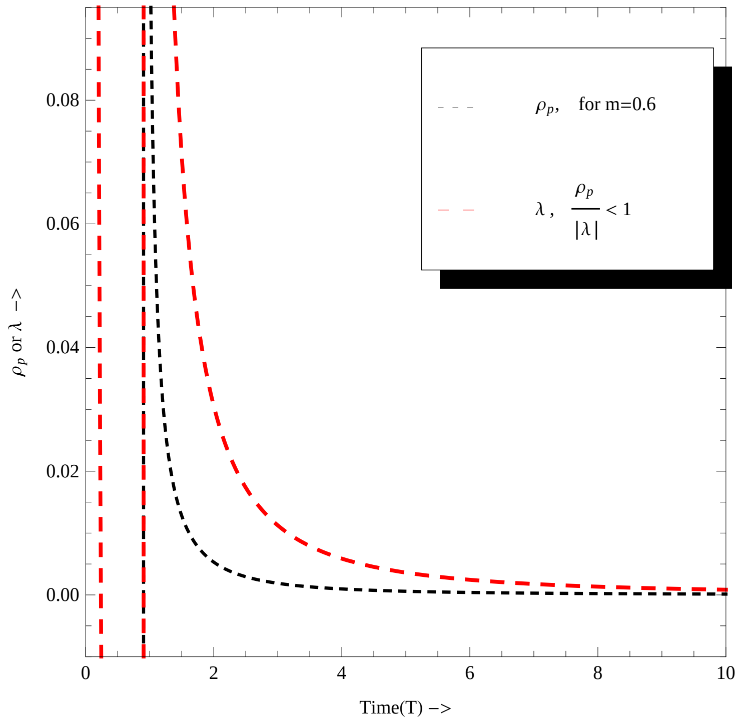

Figure 4: The plot of ρ_p or λ versus T with $k_2 = 1$, $k_4 = 1$, $k_5 = 1$

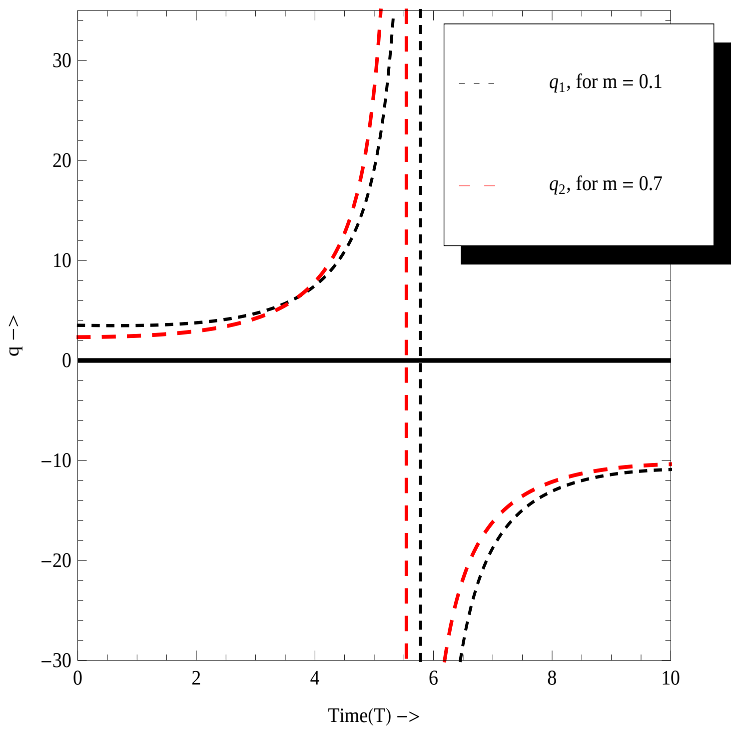

Figure 5: The plot of deceleration parameter q versus T with $k_2 = 1.1$, $k_4 = -3.0$, $k_5 = -0.20$

to total constant number of particles in the universe (may correspond to the decoupling era of radiation and matter density).

Figure 4 depicts a comparative graph of particle density ρ_p and string density λ versus T. It is observed that the massive string dominates the cosmic evolution. Also initially string oscillates and then becomes a positive decreasing function of T. The condition $\rho_p < |\lambda|$ will be valid only in a region where both ρ_p and λ are well behaved decreasing function of time T. Near $T \approx 0$, different dynamics may play role. It may not be possible to address this issue in kinematic set up, even different physical law may be needed to deal with this early phase of evolution.

Both ρ_p and λ tends to infinity at $T = 0$ and 0 at $T = \infty$. The model (3.11) therefore starts with a big-bang at $T = 0$ and it goes on expanding until it comes to rest at $T = \infty$. We also note that $T = 0$ and $T = \infty$ respectively correspond to the proper time $t = t_0$ and $t = \infty$, where $t_0 = -\frac{\alpha}{k_2}$. The initial singularity of the model is of the Point Type. Both ρ_p and λ tend to zero asymptotically.

The expressions for the scalar of expansion θ, magnitude of shear σ^2, the average anisotropy parameter A_m, deceleration parameter q and proper volume V for the model (3.11) are given by

$$\theta = \frac{(m+1)k_2}{T} + \frac{M + k_2k_5T^{(m+2)k_3-1}}{NT + k_5T^{(m+2)k_3}}, \tag{4.7}$$

$$\sigma^2 = \frac{1}{3}\left[\frac{k^2}{T^2} + \left\{\frac{M + k_2k_5T^{(m+2)k_3-1}}{NT + k_5T^{(m+2)k_3}}\right\}^2 - k_2(m+1)\left\{\frac{M + k_2k_5T^{(m+2)k_3-1}}{NT^2 + k_5T^{(m+2)k_3}}\right\}\right], \tag{4.8}$$

$$A_m = -1 + 3\left[\frac{\frac{(m^2+1)k^2}{T^2} + \left\{\frac{M+k_2k_5T^{(m+2)k_3-1}}{NT+k_5T^{(m+2)k_3}}\right\}^2}{\left\{\frac{(m+1)k_2}{T} + \frac{M+k_2k_5T^{(m+2)k_3-1}}{NT^2+k_5T^{(m+2)k_3}}\right\}^2}\right], \tag{4.9}$$

$$q = -1 - \frac{1}{\left[(m+1)k_2 + \frac{M+k_2k_5T^{(m+2)k_3-1}}{N+k_5T^{(m+2)k_3-1}}\right]^2} \times \Bigg[-3(m+1)k_2 + \frac{3k_2k_5\{(m+2)k_3-1\}T^{(m+2)k_3-1}}{N + k_5T^{(m+2)k_3-1}} - \frac{3\left(M + k_2k_5T^{(m+2)k_3-1}\right)\left\{N + k_3k_5(m+2)T^{(m+2)k_3-1}\right\}}{(N + k_5T^{(m+2)k_3-1})^2}\Bigg], \tag{4.10}$$

$$V^3 = \frac{k_4T}{1-(m+2)k_3} + k_5T^{(m+2)k_3}. \tag{4.11}$$

The rate of expansion H_i in the direction of x, y and z are given by

$$H_1 = \frac{mk_2}{T}, \tag{4.12}$$

$$H_2 = \frac{k_2}{T}, \tag{4.13}$$

$$H_3 = \frac{M + k_2k_5T^{(m+2)k_3-1}}{NT + k_5T^{(m+2)k_3}}. \tag{4.14}$$

Hence the average generalized Hubble's parameter is given by

$$H = \frac{1}{3}\left[\frac{(m+1)k_2}{T} + \frac{M + k_2k_5T^{(m+2)k_3-1}}{NT + k_5T^{(m+2)k_3}}\right]. \tag{4.15}$$

From Eq. (4.10), it is observed that the for $k_2 = 0$, the deceleration parameter $q = -1$ as in the case of de Sitter universe. Figure 5 draws the plot of decelerating parameter q versus T for different values of m ($m = 0.1$ and $m = 0.7$ are just representative case. Thus in this case we have two phases of the model i.e. from decelerating to accelerating. It is remarkable to mention here that though the current observations of SNe Ia [69, 70] and CMBR favour accelerating models, but both do not altogether rule out the decelerating ones which are also consistent with these observations (see, Vishwakarma [71]). The large negative value of q in Figure 5 is due to choice of integration constant. With realistic boundary condition q can be made closer to -1. The realistic boundary condition needs further detailed study.

It can be seen that the spatial volume is zero at $T = 0$ and it increases with the increase of T. This shows that the universe starts evolving with zero volume at $T = 0$ and expands with cosmic time T. From equations (4.12) - (4.14), we observe that all the three directional Hubble parameters are zero at $T \to \infty$ and ∞ when $T \to 0$. Here $H_i (i = 1, 2, 3)$ has same functional form of $\frac{1}{T}$. In this case H_3 component represents deviation from average H. So the $\triangle H_i$ may be considered as directional test parameter of Hubble constant. H_i has planner isotropy and third direction has anisotropy. This will also tell us preferential direction in H if any. This would be a test of future observation. In derived model, the energy density tend to infinity at $T = 0$. The model has the point-type singularity at

$T = 0$ (MacCallum [72]). The shear scalar diverse at $T = 0$. As $T \to \infty$, the scale factors $A(t)$, $B(t)$ and $C(t)$ tend to infinity. The energy density becomes zero as $T \to \infty$. The expansion scalar and shear scalar all tend to zero as $T \to \infty$. At the initial stage of expansion, when ρ is large, the Hubble parameter is also large and with the expansion of the universe H, θ decrease as does ρ. Since $\lim_{T\to\infty} \frac{\sigma^2}{\theta^2}$ = constant, the model does not approach isotropy at late time. The dynamics of the mean anisotropy parameter depends on the value of m.

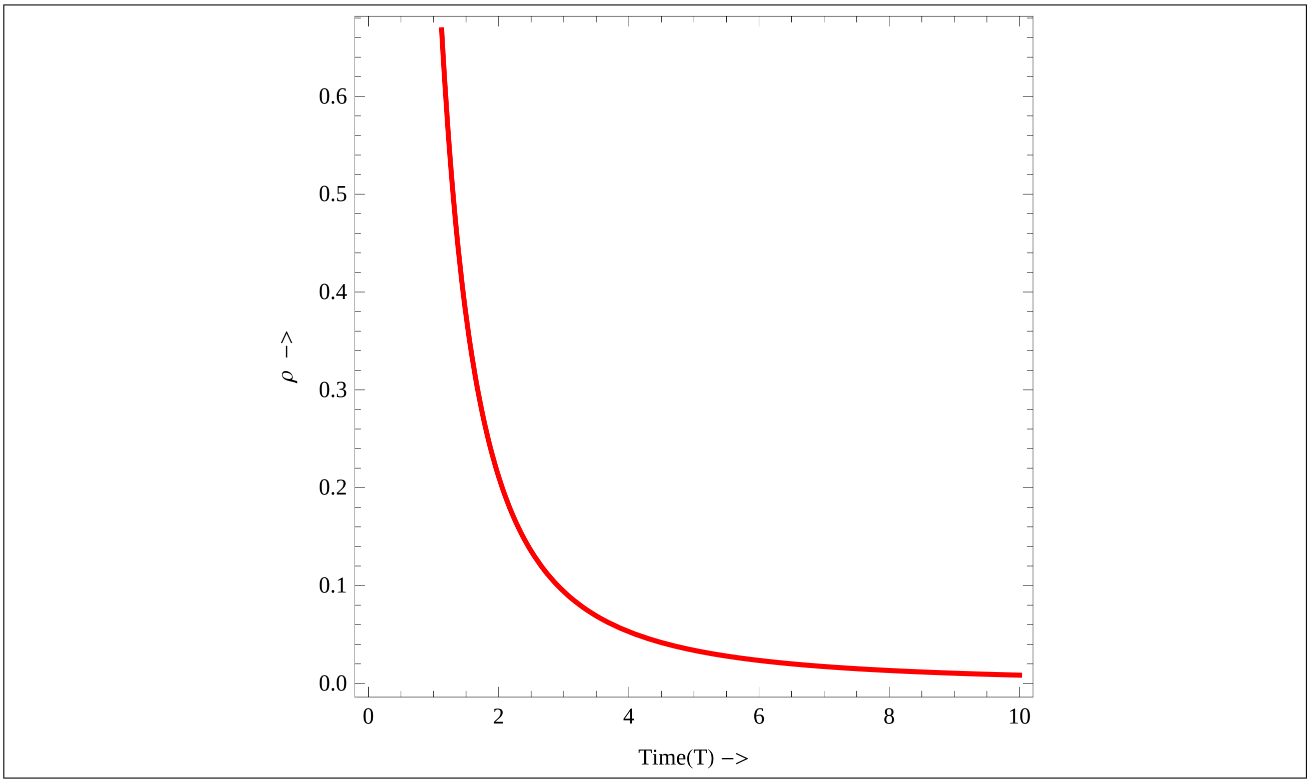

Figure 6: The plot of energy density ρ versus T with $m = 1.2$, $k_2 = 2.5$

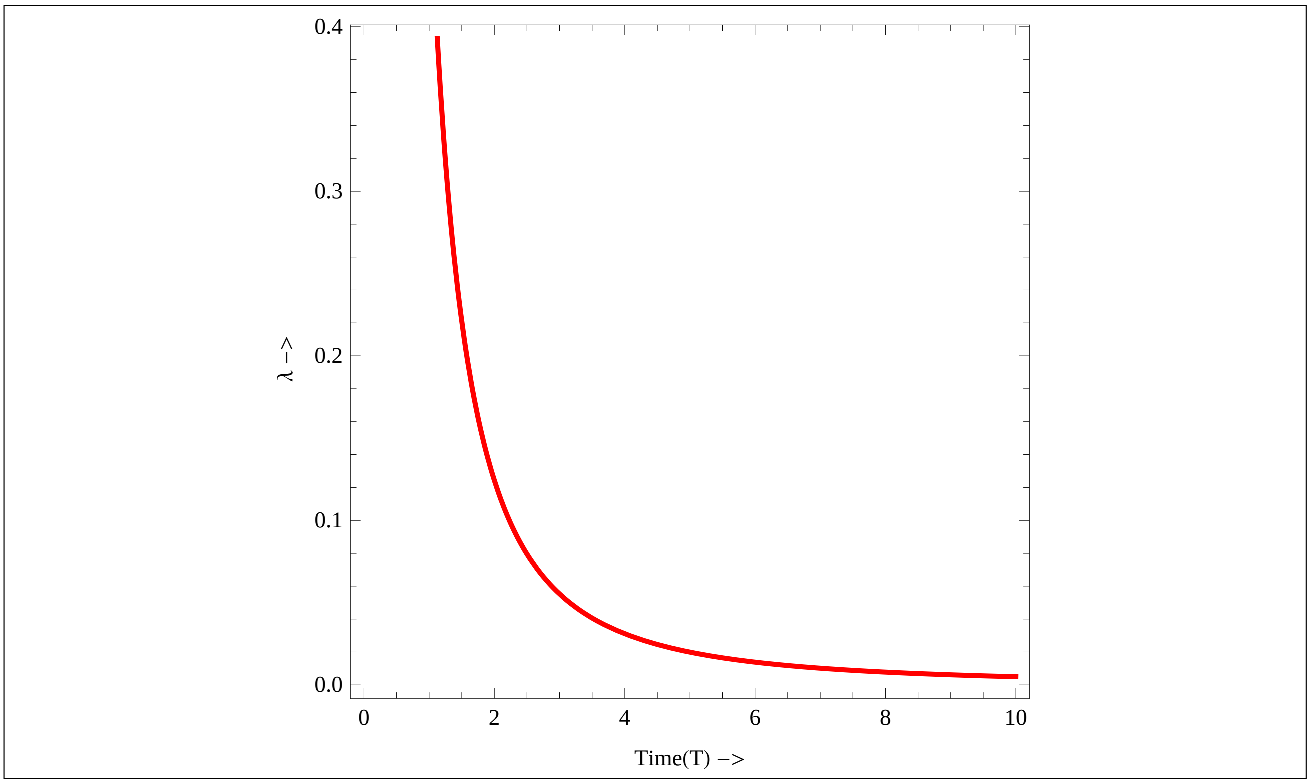

Figure 7: The plot of tension density λ versus T with $m = 1.2$, $k_2 = 2.5$

5 Particular Model I: When $k_4 = 0$

When $k_4 = 0$, the model of the universe (3.11) is reduced to

$$ds^2 = -\left(\frac{k_3}{k_2}\right)^2 dT^2 + T^{2mk_3}dx^2 + T^{2k_3}(dy^2 + k_5^2 dz^2). \tag{5.1}$$

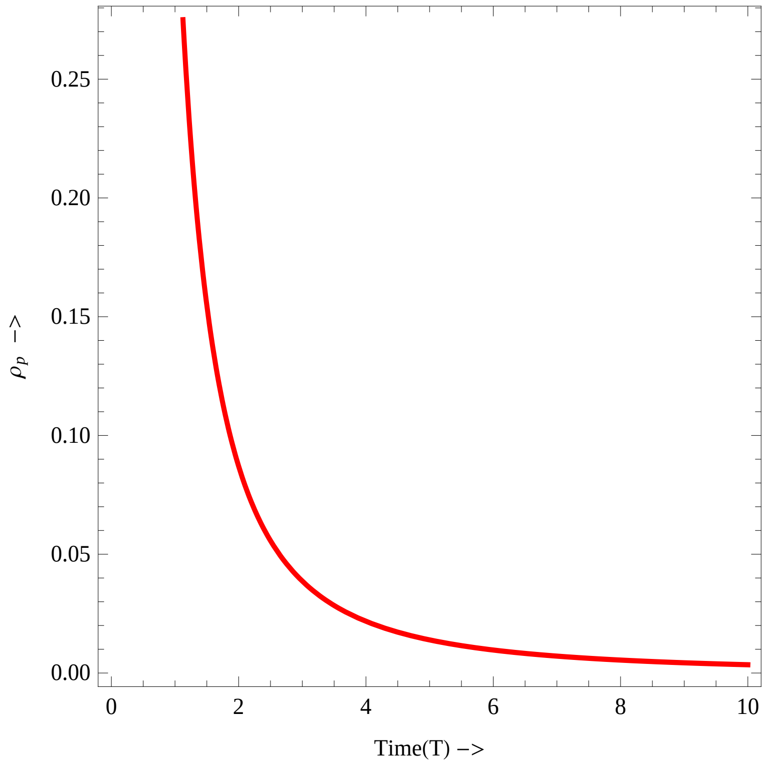

Figure 8: The plot of particle density ρ_p versus $T$with $m = 1.2$, $k_2 = 2.5$

The energy density (ρ), the string tension (λ) and the particle density (ρ_p) for the model (5.1) are given by

$$8\pi\rho = \frac{(2m+1)k_2^2}{T^2}, \tag{5.2}$$

$$8\pi\lambda = \frac{2k_2^2}{T^2}, \tag{5.3}$$

$$8\pi\rho_p = \frac{(2m-1)k_2^2}{T^2}. \tag{5.4}$$

The expressions for the scalar of expansion θ, magnitude of shear σ^2, the average anisotropy parameter A_m, deceleration parameter q and proper volume V for the model (5.1) are given by

$$\theta = \frac{(m+2)k_2}{T}, \tag{5.5}$$

$$\sigma^2 = \frac{1}{3}\left[\frac{(1-m)k_2}{T}\right]^2, \tag{5.6}$$

$$A_m = 2\left(\frac{m-1}{m+2}\right)^2, \tag{5.7}$$

$$q = -1 + \frac{3(m^2+m+1)}{(m+1)(m+2)}. \tag{5.8}$$

$$V^3 = k_5 T^{(m+2)k_3}. \tag{5.9}$$

From Eqs. (5.5) and (5.6) we obtain

$$\frac{\sigma^2}{\theta^2} = \frac{(1-m)^2}{3(m+2)^2} \quad \text{for } m \neq -2. \tag{5.10}$$

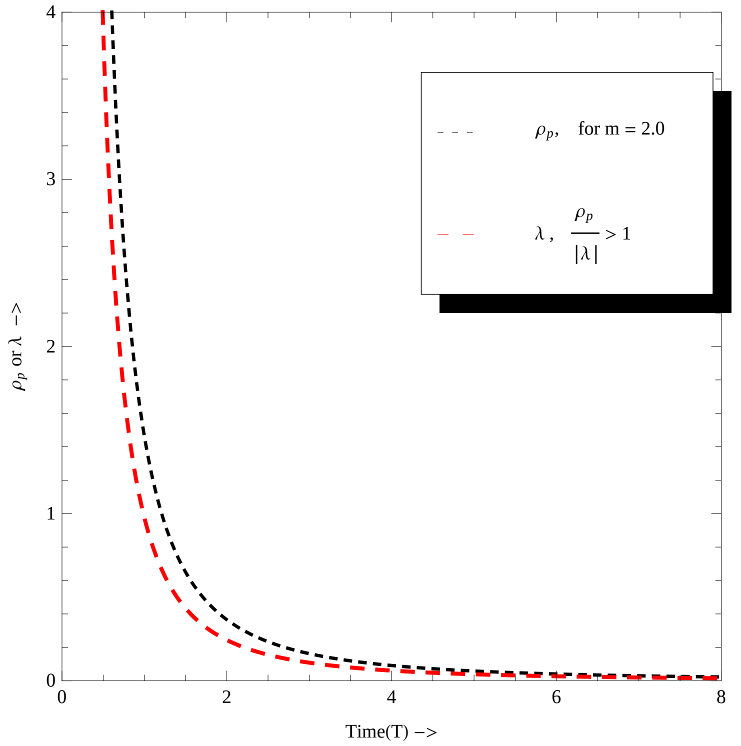

Figure 9: The deceleration parameter q versus T with $m = 2.0$, $k_2 = 3.5$

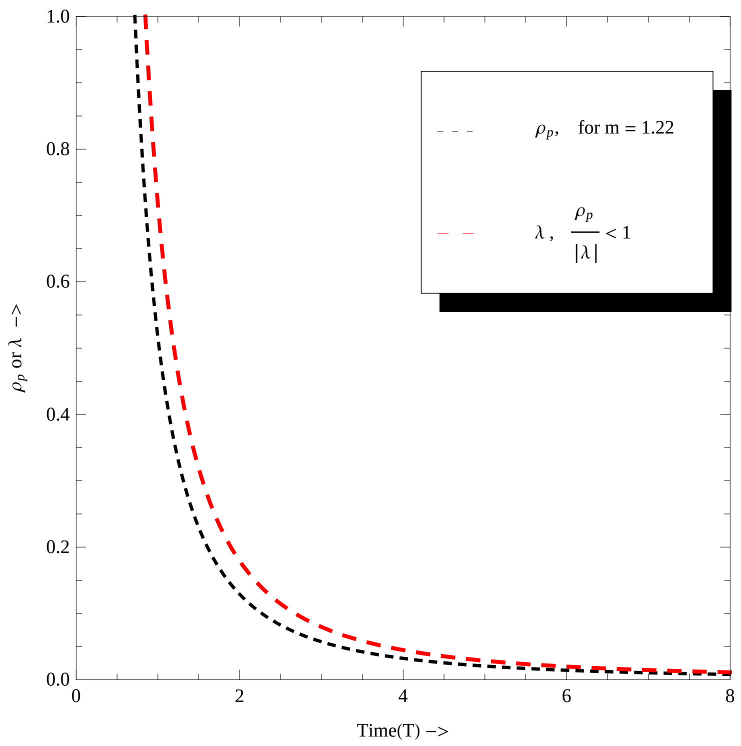

Figure 10: The plot of ρ_p or λ versus T with $m = 1.22$, $k_2 = 3.0$

The rate of expansion H_i in the direction of x, y and z are given by

$$H_1 = \frac{mk_2}{T}, \tag{5.11}$$

$$H_2 = H_3 = \frac{k_2}{T}. \tag{5.12}$$

Hence the average generalized Hubble's parameter is given by

$$H = \frac{(m+2)k_2}{3T}. \tag{5.13}$$

From above Eqs. (5.2) - (5.4), we observe that $\rho \geq 0$ if $m \geq -\frac{1}{2}$; $\lambda > 0$ always and $\rho_p \geq 0$ when $m \geq \frac{1}{2}$. This shows that for suitable values of constant m, we find that the energy conditions $\rho \geq 0$, $\rho_p \geq 0$ are satisfied. Both ρ_p and λ tend to infinity at $T = 0$ and 0 at $T = \infty$. The model (5.1) therefore starts with a big-bang at $T = 0$ and it goes on expanding until it comes to rest at $T = \infty$. We also note that $T = 0$ and $T = \infty$ respectively correspond to the proper time $t = t_0$, where $t_0 = -\frac{\alpha}{k_2}$ and $t = \infty$. The mean anisotropy parameter are uniform throughout whole expansion of the universe. This shows that the universe is expanding with the increase of cosmic time but the rate of expansion and shear scalar decrease to zero and tend to isotropic. The initial singularity of the model is of the Point Type (MacCallum [72]). Both ρ_p and λ tends to zero asymptotically. Figures 6, 7 and 8 represent ρ, λ, ρ_p as decreasing function (well behaved smooth) of T as expected. Since $\frac{\sigma^2}{\theta^2}$ = constant, the model does not approach isotropy at any time.

From (5.8) we observe that

$$(i) \text{ for } \; m \in (-2, +\infty), \;\; q > 0$$

i.e., the model is decelerating (See Figure 5) and

$$(ii) \text{ for } m \in (-\infty, -2), \;\; q < 0$$

i.e., the model is accelerating (See Figure 6). Recent observations of type Ia supernovae [69, 70] reveal that the present universe is in accelerating phase and deceleration parameter lies somewhere in the range $-1 < q \leq 0$. It follows that our model of the universe is consistent with the recent observations.

From (5.3) and (5.4), we find

$$\frac{\rho_p}{\mid \lambda \mid} = \frac{2m-1}{2}. \tag{5.14}$$

Now we find the following three conditions:

$$\frac{\rho_p}{\mid \lambda \mid} > 1, \quad \text{if } m > \tfrac{3}{2}\,, \tag{5.15}$$

$$\frac{\rho_p}{\mid \lambda \mid} < 1, \quad \text{if } m < \tfrac{3}{2}\,, \tag{5.16}$$

and

$$\frac{\rho_p}{\mid \lambda \mid} = 0, \quad \text{when } m = \tfrac{1}{2}\,. \tag{5.17}$$

In theses cases the particle density and the tension density of the string are comparable at the two ends and they fall off asymptotically at similar rate.

According to Refs. [1, 73], when $\rho_p/\mid \lambda \mid> 1$, in the process of evolution, the universe is dominated by massive strings, and when $\rho_p/\mid \lambda \mid< 1$, the universe is dominated by the strings. Thus, in the case of Eq. (5.15), the universe is dominated by massive strings throughout the whole process of evolution. This situation is displayed in Figure 9. It is also noted that in the case of Eq. (5.16), the universe is always dominated by the strings (see Figure 10). Also in the case of Eq. (5.17), the string always dominates over the particle.

6 Particular Model II: When $k_5 = 0$

When $k_5 = 0$, the model of the universe (3.11) is reduced to

$$ds^2 = -\left(\frac{k_3}{k_2}\right)^2 dT^2 + T^{2mk_3}dx^2 + T^{2k_3}dy^2 + \left(\frac{k_4}{1-(m+2)k_3}\right)^2 T^{2[1-(m+1)k_3]}dz^2. \tag{6.1}$$

The energy density (ρ), the string tension (λ) and the particle density (ρ_p) for the model (6.1) are given by

$$\rho = 0, \tag{6.2}$$

$$8\pi\lambda = -\left(\frac{m^2-m-1}{m+1}\right)\frac{k_2^2}{T^2}, \tag{6.3}$$

$$8\pi\rho_p = \left(\frac{m^2-m-1}{m+1}\right)\frac{k_2^2}{T^2}. \tag{6.4}$$

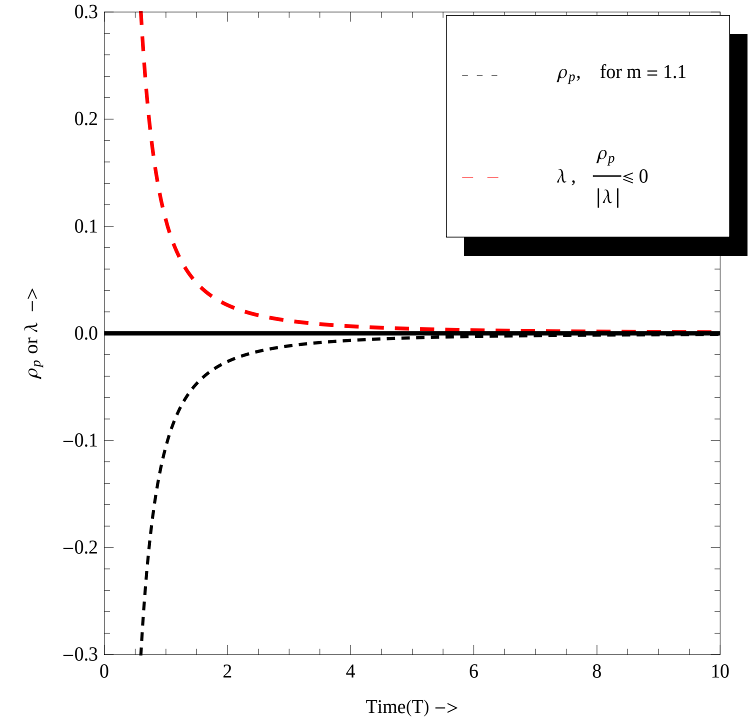

Figure 11: The plot of ρ_p or λ versus T with $m = 1.1$, $k_2 = 2.5$

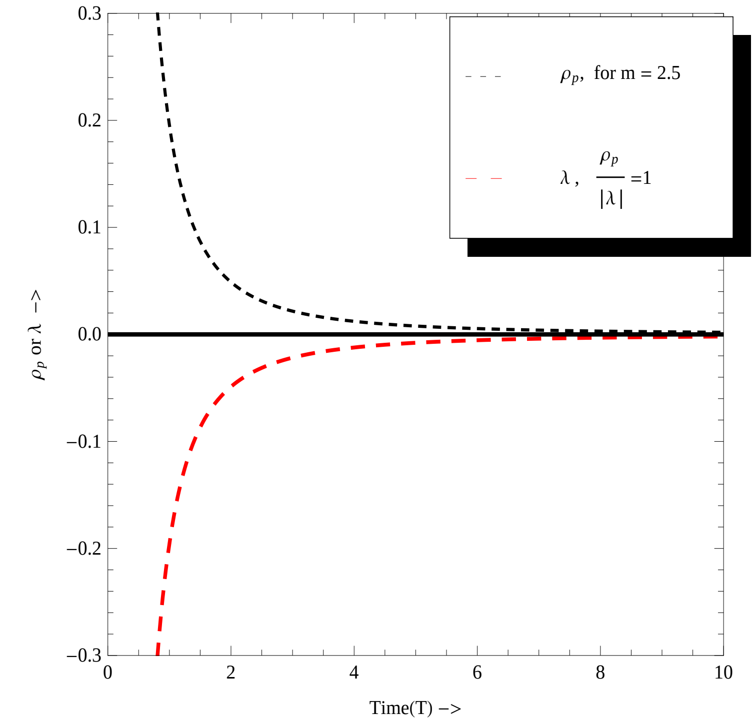

Figure 12: ρ_p or λ versus $T$with $m = 2.5$, $k_2 = 2.5$

The expressions for the scalar of expansion θ, magnitude of shear σ^2, the average anisotropy parameter A_m, deceleration parameter q and proper volume V for the model (6.1) are given by

$$\theta = \frac{k_2}{k_3 T}, \tag{6.5}$$

$$\sigma^2 = \frac{k_2^2[k_3^2 + 2(m+1)k_3 - 1]}{3k_3^2 T^2}, \tag{6.6}$$

$$A_m = -1 + 3[(m^2+1)k_3^2 + \{9(m+1)k_3 - 1\}^2], \tag{6.7}$$

$$q = 2, \tag{6.8}$$

$$V^3 = \frac{k_4}{1-(m+2)k_3}T. \tag{6.9}$$

From Eqs. (6.1) and (6.2) we obtain

$$\frac{\sigma^2}{\theta^2} = \frac{1}{3}[k_3^2 + 2(m+1)k_3 - 1] = \text{constant}. \tag{6.10}$$

The rate of expansion H_i in the direction of x, y and z are given by

$$H_1 = \frac{mk_2}{T}, \tag{6.11}$$

$$H_2 = \frac{k_2}{T}, \tag{6.12}$$

$$H_3 = \frac{k_2\{1-(m+1)k_3\}}{k_3 T}. \tag{6.13}$$

Hence the average generalized Hubble's parameter is given by

$$H = \frac{k_2}{3k_3 T}. \tag{6.14}$$

From above Eqs. (6.3) and (6.4), we observe the following two cases:

$$\lambda \geq 0 \ \text{ and } \rho_p \leq 0 \ \text{ when } m \in \left(-\infty, \frac{1-\sqrt{5}}{2}\right] \cup \left(-1, \frac{1+\sqrt{5}}{2}\right], \tag{6.15}$$

and

$$\lambda \leq 0 \ \text{ and } \rho_p \geq 0 \ \text{ when } m \in \left[\frac{1-\sqrt{5}}{2}, -1\right) \cup \left[\frac{1+\sqrt{5}}{2}, \infty\right). \tag{6.16}$$

Both ρ_p and λ tend to infinity at $T = 0$ and 0 at $T = \infty$. The model (6.1) starts with a big-bang at $T = 0$ and it goes on expanding until it comes to rest at $T = \infty$. We also note that $T = 0$ and $T = \infty$ respectively correspond to the proper time $t = t_0$ for every $t_0 = -\frac{\alpha}{k_2}$ and $t = \infty$. The mean anisotropy parameter are constant throughout whole expansion of the universe. This shows that the universe is expanding with the increase of cosmic time but the rate of expansion and shear scalar decrease to zero and tend to isotropic. The initial singularity of the model is of the Point Type. Both ρ_p and λ tend to zero asymptotically. Since $\frac{\sigma^2}{\theta^2}$ = constant, the model does not approach isotropy at any time. The deceleration parameter is always positive, so in this case the model of the universe is decelerating. Figure 11 is a comparative plot of ρ_p and λ versus T on the same graph plane. It is observed that it follows criteria of Eq. (6.15) i.e. $\lambda \geq 0$ and $\rho_p \leq 0$ hence $\frac{\rho_p}{|\lambda|} \leq 0$. This means the evolution may be string dominated.

From (6.3) and (6.4), we find the relation:

$$\frac{\rho_p}{\mid \lambda \mid} = 1. \tag{6.17}$$

Figure 12 shows a comparison of ρ_p and λ versus T. It follows from Eq. (6.16) that $\frac{\rho_p}{|\lambda|} \geq 0$. Specific choice of constants are made such that Eqs. (6.16) and (6.17) both criteria are satisfied. In this particular case it is to be observed that the particle density and the tension density of the string are comparable and they are equal. Hence the model (6.1) is very interesting to understand the dynamic behaviour of ρ_p and λ which satisfied the condition $\rho_p + \lambda = 0$ as mentioned in Eq. (6.2).

7 Concluding Remarks

In this paper, spatially homogeneous and anisotropic Bianchi type-I models representing massive strings in general relativity have been studied. In general the models are expanding, shearing and non-rotating. The energy conditions $\rho \geq 0$, $\rho_p \geq 0$ are satisfied for appropriate values of constants as discussed in the previous sections. The models start with a big-bang at $T = 0$ and they go on expanding until they come to rest at $T = \infty$. It is worth mentioned here that $T = 0$ and $T = \infty$ correspond to the proper time $t = t_0$, where $t_0 = -\frac{\alpha}{k_2}$ and $t = \infty$ respectively. The initial singularity of the models are of the Point Type (MacCallum [72]). Our universes start evolving with zero volume at $T = 0$ and expand with cosmic time T. Since $\lim_{T\to\infty} \frac{\sigma^2}{\theta^2}$ = constant, the models do not approach isotropy at late time. The particle density and the tension density of the string are comparable at the two ends and they fall off asymptotically at similar rate.

The main features of the models are as follows:

- The models are based on exact solutions of the Einstein's field equations for the anisotropic Bianchi-I space-time filled with massive strings. The literature has hardly witnessed this sort of exact solutions for the anisotropic Bianchi-I space-time. So the derived models add one more feather to the literature.
- The models present the dynamics of strings in the accelerating and decelerating modes of evolution of the universe. It has been found that massive strings dominate in the decelerating universe where as strings dominate in the accelerating universe.
- The strings dominate in the early universe and eventually disappear from the universe for sufficiently large times. This is in good agreement with the current astronomical observations.
- One of the striking feature of the case $k_5 = 0$ is self consistence energy density condition $\rho_p + \lambda$.

References

[1] T. W. B. Kibble, J. Phys. A: Math. Gen. **9** (1976) 1387.

[2] Ya. B. Zel'dovich, I. Yu. Kobzarev and L. B. Okun, Zh. Eksp. Teor. Fiz. **67** (1975) 3; Sov. Phys.-JETP **40** (1975) 1.

[3] T. W. B. Kibble, Phys. Rep. **67** (1980) 183.

[4] A. E. Everett, Phys. Rev. **24** (1981) 858.

[5] A. Vilenkin, Phys. Rev. **D 24** (1981) 2082.

[6] Ya. B. Zel'dovich, Mon. Not. R. Astron. Soc. **192** (1980) 663.

[7] P. S. Letelier, Phys. Rev. **D 20** (1979) 1294.

[8] P. S. Letelier, Phys. Rev. **D 28** (1983) 2414.

[9] J. Stachel, Phys. Rev. **D 21** (1980) 2171.

[10] C. W. Misner, Astrophys. J. **151**, (1968) 431.

[11] V. A. Belinskii and I. M. Khalatnikov, Sov. Phys. JETP **42** (1976) 205.

[12] D. R. K. Reddy, Astrophys. Space Sci. **133** (1987) 389.

[13] D. R. K. Reddy and R. Venkateswarlu, Astrophys. Space Sci. **136** (1987) 17.

[14] D. Lorenz-Petzold, Astrophys. Space Sci. **155** (1989) 335.

[15] R. Venkateswarlu and D. R. K. Reddy, Astrophys. Space Sci. **168** (1990) 193.

[16] T. Singh and A. K. Agrawal, Astrophys. Space Sci. **182** (1991) 289.

[17] E. Gujman, Astrophys. Space Sci. **199** (1993) 289.

[18] L. P. Chimento, A. S. Jacubi, W. Mendez and R. Maartens, Class. Quant.Grav. **14** (1997) 3363.

[19] R. Bali and A. Gokhroo, Astrophys. Space Sci. **278** (2001) 283.

[20] R. Bali and S. Keshav, Astrophys. Space Sci. **283** (2003) 11.

[21] C. B. Kilinc, Astrophys. Space Sci. **289** (2004) 103.

[22] B. Saha, Mod. Phys. Lett. A **20** (2005) 2127.

[23] B. Saha, Astrophys. Space Sci. **302** (2006) 83.

[24] B. Saha, Phys. Rev. D **74** (2006) 124030.

[25] A. Pradhan and P. Pandey, Astrophys. Space Sci. **301** (2006) 221.

[26] S. Kumar and C. P. Singh, Astrophys. Space Sci. **312** (2007) 57.

[27] C. P. Singh and S. Kumar, Int. J. Theor. Phys. **48** (2009) 925.

[28] R. Bali and S. Dave, Pramana-J. Phys. **56** (2001) 513.

[29] R. Bali and R. D. Upadhaya, Astrophys. Space Sci. **283** (2003) 97.

[30] R. Bali and S. Dave, Astrophys. Space Sci. **288** (2003) 503.

[31] R. Bali and D. K. Singh, Astrophys. Space Sci. **300** (2005) 387.

[32] R. Bali and Anjali, Astrophys. Space Sci. **302** (2006) 201.

[33] R. Bali, U. K. Pareek and A. Pradhan, Chin. Phys. Lett. **24** (2007) 2455.

[34] R. Bali and A. Pradhan, Chin. Phys. Lett. **24** (2007) 585.

[35] M. K. Yadav, A. Pradhan and S. K. Singh, Astrophys. Space Sci. **311** (2007) 423.

[36] X. X. Wang, Astrophys. Space Sci. **293** (2004) 433.

[37] X. X. Wang, Chin. Phys. Lett. **21** (2004) 1205.

[38] X. X. Wang, Chin. Phys. Lett. **22** (2005) 29.

[39] X. X. Wang, Chin. Phys. Lett. **23** (2006) 1702.

[40] B. Saha and M. Visinescu, Astrophys. Space Sci. **315** (2008) 99.

[41] B. Saha, Astrophys. Space Sci. **299** (2005) 149.

[42] M. K. Yadav, A. Pradhan and A. Rai, Int. J. Theor. Phys. **46** (2007) 2677.
[43] D. R. K. Reddy, Astrophys. Space Sci. **281** (2003) 365.
[44] D. R. K. Reddy, Astrophys. Space Sci. **300** (2005) 381.
[45] D. R. K. Reddy, Subba Rao and M. V. Rao, Astrophys. Space Sci. **305** (2005) 183.
[46] D. R. K. Reddy and R. L. Naidu, Astrophys. Space Sci. **307** (2007) 395.
[47] D. R. K. Reddy, R. L. Naidu, V. U. M. Rao, Int. J. Theor. Phys. **46** (2007) 1443.
[48] V. U. M. Rao, M. Vijaya Santhi and T. Vinutha, Astrophys. Space Sci. **314** (2008) 73.
[49] V. U. M. Rao, T. Vinutha and M. Vijaya Santhi, Astrophys. Space Sci. **314** (2008) 213.
[50] V. U. M. Rao, T. Vinutha and K. V. S. Sireesha, Astrophys. Space Sci. **323** (2009) 401.
[51] V. U. M. Rao and T. Vinutha, Astrophys. Space Sci. **325** (2010) 59.
[52] A. Pradhan, Fizika B **16** (2007) 205.
[53] A. Pradhan, Commun. Theor. Phys. **51** (2009) 367.
[54] A. Pradhan, K. Jotania and A. Singh, Braz. J. Phys. **38** (2008) 167.
[55] A. Pradhan and P. Mathur, Astrophys. Space Sci. **318** (2008) 255.
[56] A. Pradhan, R. Singh and J. P. Shahi, Elect. J. Theor. Phys. **7** (2010) 197.
[57] A. Pradhan, H. Amirhashchi and M. K. Yadav, Fizika B, **18** (2009) 35.
[58] S. K. Tripathi, S. K. Nayak, S. K. Sahu and T. R. Routray, Astrophys. Space Sci. **323** (2009) 281.
[59] S. K. Tripathi, D. Behera and T. R. Routray, Astrophys. Space Sci. **325** (2010) 93.
[60] J. A. Belinchon, Astrophys. Space Sci. **323** (2009) 307.
[61] H. Amirhashchi and H. Zainuddin, Elect. J. Theor. Phys. **23** (2010) 213.
[62] B. Saha and M. Visinescu, Int. J. Theor. Phys. **49** (2010) 1411.
[63] B. Saha, V. Rikhvitsky and M. Visinescu, Cent. Eur. J. Phys. **8** (2010) 113.
[64] A. Pradhan, Proceeding of Petrov 2010 Anniversary Symposium on General Relativity & Gravitation (2010), p. 212-219.
[65] K. S. Thorne, Astrophys. J. **148** (1967) 51.
[66] R. Kantowski, R. K. Sachs, J. Math. Phys. **7** (1966) 433.
[67] J. Kristian, R. K. Sachs, Astrophys. J. **143** (1966) 379.
[68] C. B. Collins, E. N. Glass, D. A. Wilkinson, Gen. Rel. Grav. **12** (1980) 805.
[69] S. Perlmutter et al., Astrophys. J. **517**, (1999) 565.
[70] R. G. Riess et al., Astron. J. **116** (1998) 1009.
[71] R. G. Vishwakarma, Class. Quant. Grav. **17**, (2000) 3833.
[72] M. A. H. MacCallum, Comm. Math. Phys. **20**, (1971) 57.
[73] K. D. Krori, T. Chaudhuri, C. R. Mahanta and A. Mazumdar, Gen. Rel. Grav. **22** (1990) 123.

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Received: November 12, 2011

Accepted: March 2, 2012