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Some Properties for Certain Subclasses of k-fold Symmetric Functions with Fractional Powers

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Abstract. In the present paper, we discuss some inclusion relations between certain subclasses of k-fold symmetric functions with fractional powers. Their several interesting and important consequences along with some examples are then discussed.

1 Introduction and preliminaries

Let $\mathcal{H} = \mathcal{H}(\mathcal{U})$ represent a space of analytic functions in the unit disk $\mathcal{U} = \{z : z \in \mathcal{C}; |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let

$$\mathcal{H}[a, n] = \left\{f \in \mathcal{H}(\mathcal{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ..., z \in \mathcal{U}\right\}, \tag{1.1}$$

with $\mathcal{H}_0 = \mathcal{H}[0, 1]$.

We recall that a normalized function f analytic in $\mathcal{U}$ is called k-fold symmetric if its power series has the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{nk+1}\, z^{nk+1} \qquad (z \in \mathcal{U}). \tag{1.2}$$

and we denote this class by A_k. Further let $A_{k,\alpha}$ is the class of functions $f(z)$ of the form

$$f(z) = z^{\alpha+1} + \sum_{n=1}^{\infty} a_{nk+\alpha+1} z^{nk+\alpha+1} \qquad (0 \le \alpha < 1), \tag{1.3}$$

which are analytic in the open unit disk $\mathcal{U}^* = \{z : z \in \mathcal{C}; 0 < |z| < 1\}$.

We note that $A_{k,0} = A_k$, the class of normalized k-fold symmetric functions. It is clear that the analytic function $f \in A_{k,\alpha}$ is normalized in the case when $\alpha = 0$. Moreover, we have

$$f \in A_{k,\alpha} \Rightarrow z^{\alpha} f(z) \in A_k \ (0 \le \alpha < 1)$$

Motivated by the works of Irmak et al. [1], Liu [3] and Zhao [5], we define the following new subclasses of $A_{k,\alpha}$ as below

$$\mathrm{Re}\left\{\mu \frac{zf'(z)}{f(z)} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right) + (\mu + \delta)(\alpha + 1)\frac{kz^k}{1 - z^k}\right\} > \beta(z \in \mathcal{U})$$

and

$$\frac{f(z)f'(z)}{z^{2\alpha+1}(1 - z^k)^{2\alpha+2}} \neq 0, \quad \mathrm{Re}\left\{\left(\frac{f(z)}{z^{\alpha+1}(1 - z^k)^{\alpha+1}}\right)^{\mu}\left(\frac{f'(z)}{z^{\alpha}(1 - z^k)^{\alpha+1}}\right)^{\delta}\right\} > \gamma(z \in \mathcal{U})$$

$\mu, \delta \in \mathbb{R}$ and $0 \le \beta < \alpha + 1,\ 0 \le \gamma < (\alpha + 1)^{\delta}\ (0 \le \alpha < 1)$.

We denote these classes by $M_{k,\alpha}(\mu, \delta; \beta)$ and $N_{k,\alpha}(\mu, \delta; \gamma)$ respectively.

We note that, here and throughout this paper, the values of complex powers are taken to be their principal values.

In this investigation, we first focus on certain inequalities consisting of the differential operator:

$$J_{k,\alpha}(\mu, \delta; f)(z) = \mu \frac{zf'(z)}{f(z)} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right) + (\mu + \delta)(\alpha + 1)\frac{kz^k}{1 - z^k}$$

that generalizes the expression used in the definition of the class $M_{k,\alpha}(\mu,\delta;\beta)$ and we then obtain several properties of the expression

$$\left(\frac{f(z)}{z^{\alpha+1}(1-z^k)^{\alpha+1}}\right)^{\mu}\left(\frac{f'(z)}{z^{\alpha}(1-z^k)^{\alpha+1}}\right)^{\delta}$$

including relations between the classes $M_{k,\alpha}(\mu,\delta;\beta)$ and $N_{k,\alpha}(\mu,\delta;\gamma)$.

Following two lemmas (Lemmas 1.1 and 1.2) [4, pp. 33-35] (see also [2]) will be required to prove our main results.

Lemma 1.1. *Let* $\Omega\subset\mathbb{C}$ *and suppose that the function* $\psi:\mathbb{C}^2\times\mathcal{U}\to\mathbb{C}$ *satisfies* $\psi(Me^{i\theta},Ke^{i\theta};z)\notin\Omega$ *for all* $K\geq nM$, $\theta\in\mathbb{R}$ *and* $z\in\mathcal{U}$. *If* $p(z)\in\mathcal{H}[a,n]\}$ *and* $\psi(p(z),zp'(z);z)\in\Omega$ *for all* $z\in\mathcal{U}$, *then* $|p(z)|<M\ (z\in\mathcal{U})$.

Lemma 1.2. *Let* $\Omega\subset\mathbb{C}$ *and suppose that* $\psi:\mathbb{C}^2\times\mathcal{U}\to\mathbb{C}$ *satisfies* $\psi(ix,y;z)\notin\Omega$ *for all* $x\in\mathbb{R}$, $y\leq -n(1+x^2)/2$, *and* $z\in\mathcal{U}$. *If* $p(z)\in H[a,n]$ *and* $\psi(p(z),zp'(z);z)\in\Omega$ *for all* $z\in\mathcal{U}$, *then* $\mathrm{Re}\{p(z)\}>0\ (z\in\mathcal{U})$.

2 Main results and their certain consequences

Throughout the paper, we assume that $\mu,\delta\in\mathbb{R}$, $0\leq\alpha<1$. Now we state and then prove each of our main results given by Theorem 2.1 and Theorem 2.4.

Theorem 2.1. *Let* $f\in A_{k,\alpha}$ *with* $\frac{f(z)f'(z)}{z^{2\alpha+1}(1-z^k)^{2\alpha+2}}\neq 0$ *for all* $z\in\mathcal{U}$, *and also let* $\mu,\delta\in\mathbb{R}$. *If*

$$\mathrm{Re}\left\{J_{k,\alpha}(\mu,\delta;f)(z)\right\}<(\alpha+1)(\mu+\delta)+\frac{kM}{M+(\alpha+1)^{\delta}}\quad(z\in\mathcal{U})$$

where $M\geq(\alpha+1)^{\delta}$ *then*

$$\left|\left(\frac{f(z)}{z^{\alpha+1}(1-z^k)^{\alpha+1}}\right)^{\mu}\left(\frac{f'(z)}{z^{\alpha}(1-z^k)^{\alpha+1}}\right)^{\delta}-(\alpha+1)^{\delta}\right|<M\quad(z\in\mathcal{U}),$$

where the powers are the principal ones.

Proof. Consider $g(z)=z^{\alpha+1}(1+a_{k+\alpha+1}z+a_{2k+\alpha+1}z^2+...)^k\ \ (z\in\mathcal{U}^*)$
then we have $f(z)=\left(g(z^k)\right)^{\frac{1}{k}}$ which gives us

$$\frac{zf'(z)}{f(z)}=\frac{z^kg'(z^k)}{g(z^k)}.\tag{2.1}$$

Differentiate logarithmically (2.1) and simple calculation yields

$$J_{\alpha,k}(\mu,\delta;f)=\mu\frac{zf'(z)}{f(z)}+\delta\left(1+\frac{zf''(z)}{f'(z)}\right)+(\mu+\delta)(\alpha+1)\frac{kz^k}{1-z^k}$$

$$=[\mu+(1-k)\delta]\frac{z^kg'(z^k)}{g(z^k)}+k\delta\left(1+\frac{z^kg''(z^k)}{g'(z^k)}\right)+(\mu+\delta)(\alpha+1)\frac{kz^k}{1-z^k}\tag{2.2}$$

Let us define

$$p(z)=q(z^k)=\left(\frac{f(z)}{z^{\alpha+1}(1-z^k)^{\alpha+1}}\right)^{\mu}\left(\frac{f'(z)}{z^{\alpha}(1-z^k)^{\alpha+1}}\right)^{\delta}-(\alpha+1)^{\delta}.$$

Using $f(z)=\left(g(z^k)\right)^{\frac{1}{k}}$, we easily get

$$q(z^k)+(\alpha+1)^{\delta}=\frac{z^{k\delta}}{z^{(\mu+\delta)(\alpha+1)}}\frac{\left(g'(z^k)\right)^{\delta}}{(1-z^k)^{(\mu+\delta)(\alpha+1)}}\left(g(z^k)\right)^{\frac{\mu+(1-k)\delta}{k}}.\tag{2.3}$$

Differentiating logarithmically (2.3) with respect to z and a simple calculation yields

$$(\mu+\delta)(\alpha+1)+\frac{kz^kq'(z^k)}{q(z^k)+(\alpha+1)^{\delta}}=[\mu+(1-k)\delta]\frac{z^kg'(z^k)}{g(z^k)}+k\delta\left(1+\frac{z^kg''(z^k)}{g'(z^k)}\right)+(\mu+\delta)(\alpha+1)\frac{kz^k}{1-z^k}\tag{2.4}$$

Taking $z^k = \xi$ in (2.4), we have

$$(\mu+\delta)(\alpha+1)+\frac{k\xi q'(\xi)}{q(\xi)+(\alpha+1)^{\delta}}=[\mu+(1-k)\delta]\,\frac{\xi g'(\xi)}{g(\xi)}+k\delta\left(1+\frac{\xi g''(\xi)}{g'(\xi)}\right)+(\mu+\delta)(\alpha+1)\frac{k\xi}{1-\xi}. \quad (2.5)$$

Letting

$$J^*_{k,\alpha}(\mu,\delta;g)(\xi)=[\mu+(1-k)\delta]\,\frac{\xi g'(\xi)}{g(\xi)}+k\delta\left(1+\frac{\xi g''(\xi)}{g'(\xi)}\right)+(\mu+\delta)(\alpha+1)\frac{k\xi}{1-\xi}. \quad (2.6)$$

We can easily observe that $q(\xi)\in H[0,1]$. Using (2.5) in (2.6), we get

$$J^*_{k,\alpha}(\mu,\delta;g)(\xi)=(\alpha+1)(\mu+\delta)+\frac{k\xi q'(\xi)}{q(\xi)+(\alpha+1)^{\delta}}.$$

Letting

$$\psi(r,s;\xi)=(\alpha+1)(\mu+\delta)+\frac{ks}{r+(\alpha+1)^{\delta}}$$

and

$$\Omega=\left\{\omega(\xi)\in C:\mathrm{Re}\{\omega(\xi)\}<(\alpha+1)(\mu+\delta)+\frac{kM}{M+(\alpha+1)^{\delta}}\right\},$$

we then receive for $z^k=\xi$, by (2.2) and (2.6) that $\psi(q(\xi),\xi q'(\xi);\xi)=J^*_{k,\alpha}(\mu,\delta;g)(\xi)\in\Omega$ for all $\xi\in\mathcal{U}$. Further, for any $\theta\in\mathbb{R}$, $K\geq M$ and $\xi\in\mathcal{U}$, since $M\geq(\alpha+1)^{\delta}$, we also have

$$\begin{aligned}\mathrm{Re}\{\psi(Me^{i\theta},Ke^{i\theta};\xi)\}&=(\alpha+1)(\mu+\delta)+K\mathrm{Re}\left(\frac{k}{M+e^{-i\theta}(\alpha+1)^{\delta}}\right)\\&\geq(\alpha+1)(\mu+\delta)+\frac{kM}{M+(\alpha+1)^{\delta}},\end{aligned}$$

i.e. $\mathrm{Re}\{\psi(Me^{i\theta},Ke^{i\theta};\xi)\}\notin\Omega$. Therefore, according Lemma 1.1, we obtain $|q(\xi)|<M$ which gives that $|p(z)|<M$. This completes the proof of Theorem 2.1.

Putting $M=\gamma+1$, the above theorem reduces to the following result.

Corollary 2.2. *Let $f\in A_{k,\alpha}$ with $\frac{f(z)f'(z)}{z^{2\alpha+1}(1-z^k)^{2\alpha+2}}\neq 0$ for all $z\in\mathcal{U}$, and also let $\mu,\delta\in\mathbb{R}$. If*

$$\mathrm{Re}\left\{J_{\alpha,k}(\mu,\delta;f)\right\}<(\alpha+1)(\mu+\delta)+\frac{k(\gamma+1)}{\gamma+1+(\alpha+1)^{\delta}}\quad(z\in\mathcal{U})$$

where $\gamma\geq(\alpha+1)^{\delta}-1$ then

$$\left|\left(\frac{f(z)}{z^{\alpha+1}(1-z^k)^{\alpha+1}}\right)^{\mu}\left(\frac{f'(z)}{z^{\alpha}(1-z^k)^{\alpha+1}}\right)^{\delta}-(\alpha+1)^{\delta}\right|<\gamma+1\quad(z\in\mathcal{U}),$$

where the powers are the principal ones.

Taking $\mu=1-\eta$, $\delta=\eta$, $\eta\in\mathbb{R}$ in the above corollary, we get the following result.

Corollary 2.3. *Let $f\in A_{k,\alpha}$ with $\frac{f(z)f'(z)}{z^{2\alpha+1}(1-z^k)^{2\alpha+2}}\neq 0$ for all $z\in\mathcal{U}$, and also let $\eta\in\mathbb{R}$. If*

$$\mathrm{Re}\left\{J_{\alpha,k}(1-\eta,\eta;f)\right\}<(\alpha+1)+\frac{k(\gamma+1)}{\gamma+1+(\alpha+1)^{\eta}}\quad(z\in\mathcal{U})$$

where $\gamma\geq(\alpha+1)^{\eta}-1$ then

$$\left|\left(\frac{f(z)}{z^{\alpha+1}(1-z^k)^{\alpha+1}}\right)^{1-\eta}\left(\frac{f'(z)}{z^{\alpha}(1-z^k)^{\alpha+1}}\right)^{\eta}-(\alpha+1)^{\eta}\right|<\gamma+1\quad(z\in\mathcal{U}),$$

where the powers are the principal ones.

Theorem 2.4. *Let $f\in A_{k,\alpha}$ with $\frac{f(z)f'(z)}{z^{2\alpha+1}(1-z^k)^{2\alpha+2}}\neq 0$ for all $z\in\mathcal{U}$, and also let $\delta\in\mathbb{R}$ and $\mu\in\mathbb{R}$. If $f(z)\in M_{k,\alpha}(\mu,\delta;\beta_1(\gamma))$ where*

$$\beta_1(\gamma)\equiv\beta(\mu,\delta;\gamma)=\begin{cases}(\alpha+1)(\mu+\delta)-\frac{k\gamma}{2[(\alpha+1)^{\delta}-\gamma]}\ \text{if}\ \gamma\in\left[0,\frac{(\alpha+1)^{\delta}}{2}\right]\\(\alpha+1)(\mu+\delta)-\frac{k[(\alpha+1)^{\delta}-\gamma]}{2\gamma}\ \text{if}\ \gamma\in\left[\frac{(\alpha+1)^{\delta}}{2},(\alpha+1)^{\delta}\right)\end{cases} \quad (2.7)$$

then $f \in N_{k,\alpha}(\mu,\delta;\gamma)$.

Proof. Let

$$p(z) = q(z^k) = \frac{1}{(\alpha+1)^\delta - \gamma}\left\{\left(\frac{f(z)}{z^{\alpha+1}(1-z^k)^{\alpha+1}}\right)^\mu \left(\frac{f'(z)}{z^\alpha(1-z^k)^{\alpha+1}}\right)^\delta - \gamma\right\}.$$

Using the same procedure as in Theorem 2.1, we can easily have

$$(\mu+\delta)(\alpha+1) + \frac{kz^k q'(z^k)[(\alpha+1)^\delta-\gamma]}{[(\alpha+1)^\delta-\gamma]q(z^k)+\gamma} = [\mu+(k-1)\delta]\frac{z^k g'(z^k)}{g(z^k)} + k\delta\left(1+\frac{z^k g''(z^k)}{g'(z^k)}\right)$$
$$+ (\mu+\delta)(\alpha+1)\frac{kz^k}{1-z^k}. \tag{2.8}$$

Putting $z^k = \xi$ in (2.8), we can easily get

$$(\alpha+1)(\mu+\delta) + \frac{k\xi q'(\xi)\,[(\alpha+1)^\delta-\gamma]}{[(\alpha+1)^\delta-\gamma]q(\xi)+\gamma} = J^*_{k,\alpha}(\mu,\delta;g)(\xi).$$

where the powers are the principal ones. We then easily observe that $q(\xi) \in H[1,1]$.
Further, since

$$\psi(r,s;\xi) = (\alpha+1)(\mu+\delta) + \frac{ks\,[(\alpha+1)^\delta-\gamma]}{r[(\alpha+1)^\delta-\gamma]+\gamma}$$

and

$\Omega = \{w(\xi) \in C : \mathrm{Re}\{w(\xi)\} > \beta(\mu,\delta;\gamma)\}$,

we then receive for $z^k = \xi$, by (2.2) and (2.7) that $\psi(q(\xi),\xi q'(\xi);\xi) = J^*_{k,\alpha}(\mu,\delta;g)(\xi) \in \Omega$ for all $\xi \in \mathcal{U}$. Also, for any $x \in \mathbb{R}, y \le -(1+x^2)/2$ and $\xi \in \mathcal{U}$, we have

$$\begin{aligned}\mathrm{Re}\{\psi(ix,y;\xi)\} &= (\alpha+1)(\mu+\delta) + \frac{k\gamma[(\alpha+1)^\delta-\gamma]y}{[(\alpha+1)^\delta-\gamma]^2x^2+\gamma^2}\\
&\le (\alpha+1)(\mu+\delta) - \frac{k\gamma[(\alpha+1)^\delta-\gamma]}{2}\frac{x^2+1}{[(\alpha+1)^\delta-\gamma]^2x^2+\gamma^2} \equiv h(x)\\
&\le \beta(\mu,\delta;\gamma) = \begin{cases}\lim\limits_{x\to+\infty} h(x) & if\,\gamma \in \left[0,\frac{(\alpha+1)^\delta}{2}\right]\\ h(0) & if\gamma \in \left[\frac{(\alpha+1)^\delta}{2},(\alpha+1)^\delta\right)\end{cases},\end{aligned}$$

i.e. $\psi(ix,y;\xi) \notin \Omega$. Finally by Lemma 1.2, we obtain that $\mathrm{Re}\{q(\xi)\} > 0$ $(\xi \in \mathcal{U})$ and hence $\mathrm{Re}\{p(z)\} > 0$ $(z \in \mathcal{U})$ which completes the proof of Theorem 2.4.

Letting $\mu = 1-\eta$, $\delta = \eta$ in Theorem 2.4, we then obtain the following corollary.

Corollary 2.5. *Let* $f(z) \in A_{k,\alpha}$ *with* $\frac{f(z)f'(z)}{z^{2\alpha+1}(1-z^k)^{2\alpha+2}} \neq 0$ *for all* $z \in \mathcal{U}$, *and also let* $\eta \in \mathbb{R}$ *and* $\gamma \in [0,(\alpha+1)^\eta]$. *If* $f \in M_{k,\alpha}(1-\eta,\eta;\beta_2(\gamma))$, *where*

$$\beta_2(\gamma) \equiv \beta(1-\eta,\eta;\gamma) = \begin{cases}(\alpha+1) - \frac{k\gamma}{2[(\alpha+1)^\eta-\gamma]} \text{ if } \gamma \in \left[0,\frac{(\alpha+1)^\eta}{2}\right]\\ (\alpha+1) - \frac{k[(\alpha+1)^\eta-\gamma]}{2\gamma} \text{ if } \gamma \in \left[\frac{(\alpha+1)^\eta}{2},(\alpha+1)^\eta\right).\end{cases}$$

then $f \in N_{k,\alpha}(1-\eta,\eta;\gamma)$.

Putting $\mu = 1$ and $\delta = -1$ in Theorems 2.1 and 2.4, respectively, we next get the following corollaries.

Corollary 2.6. *Let* $f(z) \in A_{k,\alpha}$ *with* $\frac{f(z)f'(z)}{z^{2\alpha+1}(1-z^k)^{2\alpha+2}} \neq 0$ *for all* $z \in \mathcal{U}$, *and also let* $M(\alpha+1) \ge 1$. *Then*

$$\mathrm{Re}\{J_{k,\alpha}(1,-1;f)(z)\} < \frac{k(\alpha+1)M}{M(\alpha+1)+1} \Rightarrow \left|\frac{f(z)}{zf'(z)} - \frac{1}{\alpha+1}\right| < M \quad (z \in \mathcal{U}),$$

$$\begin{cases}\left|\frac{zf'(z)}{f(z)} + \frac{(\alpha+1)}{M^2(\alpha+1)^2-1}\right| > \frac{M(\alpha+1)^2}{M^2(\alpha+1)^2-1} \quad (z \in \mathcal{U}) \text{ when } \mathrm{M}(\alpha+1) > 1\\ \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{\alpha+1}{2} \qquad (z \in \mathcal{U}) \qquad \text{when } \mathrm{M}(\alpha+1) = 1\end{cases}$$

Corollary 2.7. *Let* $f(z) \in A_{k,\alpha}$ *with* $\frac{f(z)f'(z)}{z^{2\alpha+1}(1-z^k)^{2\alpha+2}} \neq 0$ *for all* $z \in \mathcal{U}$, *and also let* $\gamma \in [0, \frac{1}{\alpha+1})$. *If* $\mathrm{Re}\{J_{k,\alpha}(1,-1;f)(z)\} > \beta_3(\gamma) \ (z \in \mathcal{U})$,

$$\beta_3(\gamma) \equiv \beta(1,-1;\gamma) = \begin{cases} -\frac{k\gamma(\alpha+1)}{2[1-\gamma(\alpha+1)]} \text{ if } \gamma \in \left[0, \frac{1}{2(\alpha+1)}\right] \\ -\frac{k[1-\gamma(\alpha+1)]}{2\gamma(\alpha+1)} \text{ if } \gamma \in \left[\frac{1}{2(\alpha+1)}, \frac{1}{\alpha+1}\right), \end{cases}$$

then

$$\mathrm{Re}\left(\frac{f(z)}{zf'(z)}\right) > \gamma \quad (z \in \mathcal{U}),$$

i.e.

$$\begin{cases} \left|\frac{zf'(z)}{f(z)} - \frac{1}{2\gamma}\right| < \frac{1}{2\gamma} \quad (z \in \mathcal{U}) \quad \text{when } \gamma \in (0, \frac{1}{\alpha+1}) \\ \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathcal{U}) \quad \text{when } \gamma = 0. \end{cases}$$

Putting $k = 1,\ \alpha = 0$ in Corollaries 2.6 and 2.7, we get the following results.

Corollary 2.8. *Let* $f(z) \in A_{1,0}$ *with* $\frac{f(z)f'(z)}{z(1-z)^2} \neq 0$ *for all* $z \in \mathcal{U}$, *and also let* $M \geq 1$. *Then*

$$\mathrm{Re}\{J_{1,0}(1,-1;f)(z)\} < \frac{M}{M+1} \Rightarrow \left|\frac{f(z)}{zf'(z)} - 1\right| < M \ \ (z \in \mathcal{U}),$$

$$\begin{cases} \left|\frac{zf'(z)}{f(z)} + \frac{1}{M^2-1}\right| > \frac{M}{M^2-1} \ (z \in \mathcal{U}) \qquad \text{when M} > 1 \\ \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2} \qquad (z \in \mathcal{U}) \qquad \text{when M} = 1 \end{cases}$$

Corollary 2.9. *Let* $f(z) \in A_{1,0}$ *with* $\frac{f(z)f'(z)}{z(1-z)^2} \neq 0$ *for all* $z \in \mathcal{U}$, *and also let* $\gamma \in [0,1)$. *If* $\mathrm{Re}\{J_{1,0}(1,-1;f)(z)\} > \beta_4(\gamma) \ (z \in \mathcal{U})$,

$$\beta_4(\gamma) = \begin{cases} -\frac{\gamma}{2(1-\gamma)} \text{ if } \gamma \in \left[0, \frac{1}{2}\right] \\ -\frac{(1-\gamma)}{2\gamma} \text{ if } \gamma \in \left[\frac{1}{2}, 1\right), \end{cases}$$

then

$$\mathrm{Re}\left(\frac{f(z)}{zf'(z)}\right) > \gamma \quad (z \in \mathcal{U}),$$

i.e.

$$\begin{cases} \left|\frac{zf'(z)}{f(z)} - \frac{1}{2\gamma}\right| < \frac{1}{2\gamma} \quad (z \in \mathcal{U}) \quad \text{when } \gamma \in (0,1) \\ \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathcal{U}) \quad \text{when } \gamma = 0. \end{cases}$$

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