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When does the complement of the zero-divisor graph of a commutative ring admit a cut vertex?

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Abstract. In this article, we determine up to isomorphism of rings, rings R such that R has the following properties: (i) R is a commutative ring with identity which admits at least two nonzero zero-divisors, (ii) the complement of the zero-divisor graph of R is connected and it admits a cut vertex. Indeed, it is proved that there are exactly two such rings up to isomorphism of rings.

1 Introduction

The rings considered in this article are commutative rings with identity which admit at least two nonzero zero-divisors. Let R be a commutative ring with identity which admits at least two non-zero zero-divisors. Recall from [1] that the zero-divisor of R denoted by $\Gamma(R)$ is the graph whose vertices is the set of all nonzero zero-divisors of R and any distinct vertices x, y are joined by an edge in this graph if and only if $xy = 0$. Many researchers have contributed to the area of zero-divisor graphs of commutative rings. For an excellent survey of theorems proved on zero-divisor graphs of commutative rings and for an extensive bibliography on the subject, the reader is referred to [2]. Among the research work done in this area, several researchers concentrate on the problem of investigating the interplay between the ring theoretic properties of R and the graph theoretic properties of $\Gamma(R)$. It was noted in [12, 13] that the complement of the zero-divisor graph of a commutative ring R with identity denoted by $(\Gamma(R))^c$ also has some interesting properties. In [4], the authors studied about the cut vertices of zero-divisor graphs of finite commutative rings with identity. Moreover, in [7], the authors investigated the cut sets of zero-divisor graphs of finite commutative rings with identity. Let R be a commutative ring with identity which admits at least two nonzero zero-divisors such that $(\Gamma(R))^c$ is connected. It is useful to mention here that [12, Theorem 1.1] answers when $(\Gamma(R))^c$ is connected. Motivated by the work on cut vertices of zero-divisor graphs of commutative rings [4,7], in the present article we give a complete answer to the question of when $(\Gamma((R))^c$ admits a cut vertex. In this article, we prove in theorem 5.1 that if $(\Gamma(R))^c$ is connected, then it admits a cut vertex if and only if either R is isomorphic to $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ or R is isomorphic to $(\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z})[x] \times \mathbf{Z}/2\mathbf{Z}$ where $(\mathbf{Z}/2\mathbf{Z})[x]$ is the polynomial ring in one variable over $\mathbf{Z}/2\mathbf{Z}$.

Before we state the results that are proved in this article, it is useful to recall the following definitions from graph theory. Let $G = (V, E)$ be a simple graph. Recall from [5, Definition 1.1.13] that the complement of G denoted by G^c, we mean the graph whose vertex set is V and two distinct vertices x, y are joined by an edge in G^c if and only if there exists no edge joining x and y in G.

Let $G = (V, E)$ be a graph. The induced subgraph of G with vertex set $S \subseteq V$ is called the subgraph of G induced by S and is denoted by $G[S]$ [5, Definition 1.2.1].

Let $G = (V, E)$ be a graph. Let S be a proper subset of V. The subgraph $G[V \backslash S]$ is said to be obtained form G by the deletion of S. This subgraph is denoted by $G - S$. If $S = \{v\}$, then $G - S$ is simply denoted by $G - v$ [5, Definition 1.2.3].

Let $G = (V, E)$ be a connected graph. Recall from [5, Definition 3.1.1 1] that a subset S of V is said to be a vertex cut of G if $G - S$ is disconnected. Let $v \in V$. v is said to be a cut vertex of G if $\{v\}$ is a vertex cut of G. That is, v is a cut vertex of G if $G - v$ is disconnected.

We also recall the following results from commutative ring theory. Let R be a commutative ring with identity. Let I be an ideal of R. Recall from [10] that a prime ideal P of R is said to be a maximal N-prime of I if P is maximal with respect to the property of being contained in $Z_R(R/I) = \{r \in R | rx \in I$ for some $x \in R \setminus I\}$. Let $\{P_\alpha\}_{\alpha \in \Lambda}$ be the set of all maximal N-primes of (0) in R. It is well known that $Z(R) = \bigcup_{\alpha \in \Lambda} P_\alpha$.

Let R be a commutative ring with identity and let I be an ideal of R. A prime ideal P of R is said to be a B-prime of I if $P = (I :_R x)$ for some $x \in R$ [9].

Let R be a commutative ring with identity with at least two nonzero zero-divisors. It was shown in [12, Theorem 1.1] that $(\Gamma(R))^c$ is connected if and only if one of the following condition holds:

(a) R has exactly one maximal N-prime P of (0) and P is not a B-prime of (0).
(b) R has exactly two maximal N-primes P_1, P_2 of (0) and $P_1 \cap P_2 \neq (0)$.
(c) R has more than two maximal N-primes of (0).

Unless otherwise specified we consider commutative rings R with identity which admit at least two nonzero zero-divisors. Moreover, we assume that $(\Gamma(R))^c$ is connected. In section 2 of this article we consider rings which admit exactly one maximal N-prime of (0). That is, equivalently rings R such that $Z(R)$ is an ideal of R. We show in lemma 2.1 that for any nonempty finite subset S of $Z(R)^*$, $(\Gamma(R))^c - S$ is connected.

In section 3, we consider rings R which admit exactly two maximal N-primes of (0). In theorem 3.14 it is shown that $(\Gamma(R))^c$ admits a cut vertex if and only if either R is isomorphic to $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ or R is isomorphic to $(\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z})[x] \times \mathbf{Z}/2\mathbf{Z}$ as rings and this is the main result of this section. Moreover, in this section, we initially prove several results in the form of lemmas that are needed for proving theorem 3.14.

In section 4 we consider rings R which admit more than two maximal N-primes of (0). Let $n \geq 3$. We prove in lemma 4.1 that if R admits at least n maximal N-primes of (0) and if S is any subset of $Z(R)^*$ with $|S| = n-2$, then $(\Gamma(R))^c - S$ is connected. As a consequence, we deduce in corollary 4.2 that $(\Gamma(R))^c$ does not admit any cut vertex and we deduce in corollary 4.3 that if there exists a finite subset S of $Z(R)^*$ such that S is a vertex cut of $(\Gamma(R))^c$, then R can have at most $|S| + 1$ maximal N-primes of (0).

In section 5 of this article, as is already mentioned in this introduction, in theorem 5.1, we state and prove a necessary and sufficient condition (the proof of which follows immediately from the results proved in sections 2, 3, and 4) in order that $(\Gamma(R))^c$ admits a cut vertex.

2 *R* has exactly one maximal *N*-prime of (0)

Let R be a commutative ring with identity which admits at least two nonzero zero-divisors. Suppose that $(\Gamma(R))^c$ is connected. As is mentioned in the introduction, the goal of this article is to characterize rings R such that $(\Gamma(R))^c$ admits a cut vertex. Towards that goal, we first consider rings which admit exactly one maximal N-prime of (0). We begin with the following lemma.

Lemma 2.1. Let R be a commutative ring with identity such that R has only one maximal N-prime of (0). Let P be the unique maximal N-prime of (0) in R. Suppose that $Z(R)^*$ contains at least two elements and $(\Gamma(R))^c$ is connected. Let S be any finite nonempty subset of $P\backslash\{0\}$. Then $(\Gamma(R))^c - S$ is connected. In particular, $(\Gamma(R))^c$ does not admit any cut vertex.

Proof. Observe that $Z(R) = P$ and hence the vertex set of $(\Gamma(R))^c = P\backslash\{0\}$. We are assuming that $|Z(R)^*| \geq 2$ and $(\Gamma(R))^c$ is connected. Hence it follows from [12, Theorem 1.1(a)] that P is not a B-prime of (0) in R. Therefore R must be infinite. Since $Z(R) = P$, it follows from [8, Theorem 1] that P must be infinite. Thus $Z(R)^*\backslash S$ is infinite. We now proceed to prove that $(\Gamma(R))^c - S$ is connected. Let $x, y \in Z(R)^*\backslash S, x \neq y$. We show that there exists a path in $(\Gamma(R))^c - S$ between x and y. If $xy \neq 0$, then $x - y$ is a path in $(\Gamma(R))^c - S$ between x and y. Suppose that $xy = 0$. Since $(\Gamma(R))^c$ is connected, P is not a B-prime of (0) in R and so it follows that $P \not\subseteq ((0) :_R x) \cup ((0) :_R y)$. So there exists $z_1 \in P$ such that $xz_1 \neq 0$ and $yz_1 \neq 0$. Observe that if $z_1 \notin S$, then $x - z_1 - y$ is a path in $(\Gamma(R))^c - S$ between x and y. Suppose that $z_1 \in S$. For convenience, let us denote z_1 by s_1. Since $xs_1 \neq 0$, $ys_1 \neq 0$, and P is not a B-prime of (0) in R, we obtain that $P \not\subseteq ((0) :_R xs_1) \cup ((0) :_R ys_1)$. Hence there exists $z_2 \in P$ such that $xs_1z_2 \neq 0$ and $ys_1z_2 \neq 0$. As $1 - z_2 \notin P = Z(R)$ and $s_1 \neq 0$, it follows that $s_1z_2 \neq s_1$. Hence $x - s_1z_2 - y$ is a path in $(\Gamma(R))^c - s_1$. If $s_1z_2 \notin S$, then $x - s_1z_2 - y$ is indeed a path in $(\Gamma(R))^c - S$ between x and y. Suppose that $s_1z_2 \in S$. Let us denote s_1z_2 by s_2. Again using the fact that P is not a B-prime of (0) in R, it follows that there exists $z_3 \in P$ such that $z_3s_2x \neq 0$ and $z_3s_2y \neq 0$. Observe that $z_3s_2 = z_3z_2s_1 \notin \{s_1, s_2\}$. Thus $x - z_3s_2 - y$ is a path in $(\Gamma(R))^c - \{s_1, s_2\}$ between x and y. It is now clear that on proceeding as above, we obtain in at most $n = |S|$ steps that there exists $w \in Z(R)^*\backslash S$ such that $x - w - y$ is a path in $(\Gamma(R))^c - S$ between x and y. This proves that if S is any finite nonempty subset of $P\backslash\{0\} = Z(R)^*$, then $(\Gamma(R))^c - S$ is connected.

Let $s \in Z(R)^*$ and let $S = \{s\}$. Then it follows from what is shown above that $(\Gamma(R))^c - s$ is connected. This proves that $(\Gamma(R))^c$ does not admit any cut vertex. □

Let R, P be as in the statement of lemma 2.1. Suppose that $|Z(R))^*| \geq 2$ and $(\Gamma(R))^c$ is connected. We verify in the following remark that $\Gamma(R) - S$ is connected for any finite nonempty subset S of $Z(R)^*$.

Remark 2.2. Let R be a commutative ring with identity and suppose that R has exactly one maximal N-prime of (0). Let P be the unique maximal N-prime of (0) in R. Suppose that $|Z(R)^*| \geq 2$ and $(\Gamma(R))^c$ is connected. Then for any finite nonempty subset S of $Z(R)^*$, $\Gamma(R) - S$ is connected. In particular, $\Gamma(R)$ does not admit any cut vertex.

Proof. Note that $Z(R) = P$. We are assuming that $|Z(R)^*| \geq 2$ and $(\Gamma(R))^c$ is connected. Hence as is observed in the proof of lemma 2.1, $Z(R)^*\backslash S$ is infinite. Let $x, y \in Z(R)^*\backslash S$ with $x \neq y$. If $xy = 0$, then $x - y$ is a path in $\Gamma(R) - S$ between x and y. Suppose that $xy \neq 0$. It is well known that $\Gamma(R)$ is connected and $diam(\Gamma(R)) \leq 3$

[1,Theorem 2.3]. Let $x - x_1 - \cdots - x_m - y$ be any path between x and y in $\Gamma(R)$. We want to establish that there exists a path in $\Gamma(R) - S$ between x and y. Note that it is enough to prove the following: if $a - s - b$ is a path in $\Gamma(R)$ with $a \in Z(R)^*\backslash S$, $s \in S$, and b may be possibly an element of S, with $ab \neq 0$, then there exists $c \in Z(R)^*\backslash S$ such that $a - c - b$ is a path in $\Gamma(R)$. Let $s = s_1$. Since P is not a B-prime of (0) in R, there exists $z_1 \in P$ such that $z_1 s_1 \neq 0$. For any $p \in P$, $1 - p \notin P = Z(R)$ and so $(1-p)r \neq 0$ for any $r \in R\backslash\{0\}$. Hence it follows that $z_1 s_1 \neq s_1$. It is now clear that $a - z_1 s_1 - b$ is a path in $\Gamma(R)$ with $z_1 s_1 \neq s_1$. Let $c_1 = z_1 s_1$. If $c_1 \notin S$, then we are done. Suppose that $c_1 \in S$. Let $c_1 = s_2$. Again using the fact that P is not a B-prime of (0) in R, it follows that there exists $z_2 \in P$ such that $z_2 s_2 \neq 0$. That is, $z_2 s_2 = z_2 z_1 s_1 \neq 0$. Note that $z_2 s_2 \notin \{s_1, s_2\}$ and $a - z_2 s_2 - b$ is a path in $\Gamma(R)$. The above argument can be applied repeatedly and in at most $n = |S|$ steps, we arrive at the conclusion that there exists $c \in Z(R)^*\backslash S$ such that $a - c - b$ is a path in $\Gamma(R)$. From this discussion, it is clear that in the path $x - x_1 - \cdots - x_m - y$ those x_i's which are in S can be replaced by $y_i \in Z(R)^*\backslash S$. Hence there exists a path in $\Gamma(R) - S$ between x and y. This proves that $\Gamma(R) - S$ is connected.

For any $s \in Z(R)^*$, $\Gamma(R) - s$ is connected. Hence we obtain that $\Gamma(R)$ does not admit any cut vertex. □

We next provide some examples to illustrate lemma 2.1.

Example 2.3. (i) Let $n \geq 1$. Let V be a valuation domain with $dim V = n$ satisfying the property that the unique maximal ideal M of V is not finitely generated. Let $(0) \subset P_1 \subset \cdots \subset P_n = M$ be the unique chain of prime ideals of V. Let $x \in P_1, x \neq 0$. Let $R = V/xV$. Note that $dim R = n - 1$. We claim that $M/xV = Z(R)$. Since R is quasi-local with M/xV as its unique maximal ideal, it follows that $Z(R) \subseteq M/xV$. Let $y + xV \in M/xV\backslash\{0 + xV\}$. Hence $y \notin xV$. Since V is a valuation domain, we obtain that $x \in yV$. Thus $x = yv$ for some $v \in M$. Note that $(y + xV)(v + xV) = x + xV = 0 + xV$. We assert that $v \notin xV$. For if $v \in xV$, then $v = xw$ for some $w \in V$. Hence we obtain that $x = yv = y(xw)$ and this implies that $x(1 - yw) = 0$. Since $1 - yw$ is a unit in V, it follows that $x = 0$. This is a contradiction. Thus $v \notin xV$ and from $(y + xV)(v + xV) = 0 + xV$, we obtain that $y + xV \in Z(R)$. This proves that $M/xV \subseteq Z(R)$ and so $M/xV = Z(R)$. This shows that M/xV is the unique maximal N-prime of (0) in R. Using the fact that M is not a finitely generated ideal of V, it can be verified as in [12, Example 3.1(ii)] that M/xV is not a B-prime of (0) in R. Hence we obtain from [12, Theorem 1.1(a)] that $(\Gamma(R))^c$ is connected. Now it follows from lemma 2.1 that for any finite nonempty subset S of $Z(R)^*$, $(\Gamma(R))^c - S$ is connected. Moreover, it follows from remark 2.2 that for any finite nonempty subset S of $Z(R)^*$, $\Gamma(R) - S$ is also connected.

(ii) [12, Example 3.4] provides an example of a quasi-local reduced ring R with its unique maximal ideal as its only maximal N- prime of its zero ideal such that $(\Gamma(R))^c$ is connected. Hence it follows from lemma 2.1 and remark 2.2 that for any finite nonempty subset S of $Z(R)^*$, both $(\Gamma(R))^c - S$ and $\Gamma(R) - S$ are connected. □

3 *R* has exactly two maximal *N*-primes of (0)

Let R be a commutative ring with identity. Suppose that R has exactly two maximal N-primes of (0) and $(\Gamma(R))^c$ is connected. The purpose of this section is to determine when $(\Gamma(R))^c$ admits a cut vertex. We first provide an example to illustrate that the complement of zero-divisor graph of a ring with exactly two maximal N-primes of (0) may admit a cut vertex.

Example 3.1. Let $R = \mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ be the direct product of $\mathbf{Z}/4\mathbf{Z}$ and $\mathbf{Z}/2\mathbf{Z}$. For convenience, let us denote $\mathbf{Z}/4\mathbf{Z}$ by R_1 and $\mathbf{Z}/2\mathbf{Z}$ by R_2. Observe that $|R| = 8$ and the set of all prime ideals of R is equal to the set of all maximal N-primes of (0) in R which equals $\{P_1 = (2 + 4\mathbf{Z})R_1 \times R_2, P_2 = R_1 \times \{0 + 2\mathbf{Z}\}\}$. Note that $Z(R)^* = \{(0+4\mathbf{Z}, 1+2\mathbf{Z}), (2+4\mathbf{Z}, 0+2\mathbf{Z}), (2+4\mathbf{Z}, 1+2\mathbf{Z}), (1+4\mathbf{Z}, 0+2\mathbf{Z}), (3+4\mathbf{Z}, 0+2\mathbf{Z})\}$ and moreover, $(2 + 4\mathbf{Z}, 0 + 2\mathbf{Z}) \in P_1 \cap P_2$. Hence $P_1 \cap P_2 \neq \{(0 + 4\mathbf{Z}, 0 + 2\mathbf{Z})\}$. It now follows from [12, Theorem 1.1(b)] that $(\Gamma(R))^c$ is connected. We assert that any path in $(\Gamma(R))^c$ between $(0 + 4\mathbf{Z}, 1 + 2\mathbf{Z})$ and $(1 + 4\mathbf{Z}, 0 + 2\mathbf{Z})$ must pass through $(2 + 4\mathbf{Z}, 1 + 2\mathbf{Z})$. Let $(0 + 4\mathbf{Z}, 1 + 2\mathbf{Z}) - x_1 - \cdots - x_m - (1 + 4\mathbf{Z}, 0 + 2\mathbf{Z})$ be any path in $(\Gamma(R))^c$ between $(0 + 4\mathbf{Z}, 1 + 2\mathbf{Z})$ and $(1 + 4\mathbf{Z}, 0 + 2\mathbf{Z})$. Since $x_1 \in Z(R)^*$ with $(0 + 2\mathbf{Z}, 1 + 2\mathbf{Z})x_1 \neq (0 + 4\mathbf{Z}, 0 + 2\mathbf{Z})$, it follows that $x_1 = (2 + 4\mathbf{Z}, 1 + 2\mathbf{Z})$. This proves that any path in $(\Gamma(R))^c$ between $(0 + 4\mathbf{Z}, 1 + 2\mathbf{Z})$ and $(1 + 4\mathbf{Z}, 0 + 2\mathbf{Z})$ passes through $(2 + 4\mathbf{Z}, 1 + 2\mathbf{Z})$. Hence it follows from [5, Theorem 3.1.6] that $(2 + 4\mathbf{Z}, 1 + 2\mathbf{Z})$ is a cut vertex of $(\Gamma(R))^c$.

Let $R_2 = \mathbf{Z}/2\mathbf{Z}$ be as in the previous paragraph. Let $T = T_1 \times T_2$ be the direct product of the rings $T_1 = R_2[x]/x^2R_2[x]$ and $T_2 = R_2$ where $R_2[x]$ is the polynomial ring in one variable over R_2. Then it can be shown as in the previous paragraph that T has exactly two maximal N-primes of its zero ideal and $(\Gamma(T))^c$ is connected. Moreover, it can be shown that $(x + x^2R_2[x], 1 + 2\mathbf{Z})$ is a cut vertex of $(\Gamma(T))^c$. □

Let R be a commutative ring with identity and suppose that R has exactly two maximal N-primes of (0). Let them be P_1 and P_2. Suppose that $(\Gamma(R))^c$ is connected. If either P_1 or P_2 is not a B-prime of (0) in R, then we prove in lemma 3.3 that $(\Gamma(R))^c$ does not admit any cut vertex. Towards proving lemma 3.3 , we begin with the following.

Lemma 3.2. Let R be a commutative ring with identity and suppose that R has exactly two maximal N-primes of

(0) and let them be P_1 and P_2. Suppose that $(\Gamma(R))^c$ is connected. Let S be any nonempty subset of $(P_1 \cap P_2)\backslash\{0\}$. Then S is not a vertex cut of $(\Gamma(R))^c$. In particular, if $w \in (P_1 \cap P_2)\backslash\{0\}$, then w is not a cut vertex of $(\Gamma(R))^c$.

Proof. Note that $Z(R) = P_1 \cup P_2$. Since $(\Gamma(R))^c$ is connected, we know from [12, Theorem 1.1(b)] that $P_1 \cap P_2 \neq (0)$. We now show that if S is any nonempty subset of $(P_1 \cap P_2)\backslash\{0\}$, then S is not a vertex cut of $(\Gamma(R))^c$. That is, we prove that $(\Gamma(R))^c - S$ is connected. Let $x, y \in Z(R)^*\backslash S, x \neq y$. We prove that there exists a path in $(\Gamma(R))^c - S$ between x and y. This is clear if $xy \neq 0$. So we may assume that $xy = 0$. We know from the proof of [12, Proposition 1.7(i)] that there exist $a \in P_1\backslash P_2$ and $b \in P_2\backslash P_1$ such that $ab \neq 0$. Since $Z(R) = P_1 \cup P_2$, it follows that $a + b \notin Z(R)$. As $x \neq 0$, it follows that $(a + b)x \neq 0$. Hence either $ax \neq 0$ or $bx \neq 0$. Similarly, since $y \neq 0$, it follows that either $ay \neq 0$ or $by \neq 0$. If both ax and ay are nonzero, then it follows that $x - a - y$ is a path in $(\Gamma(R))^c$ and as $a \in P_1\backslash P_2$, whereas $S \subseteq (P_1 \cap P_2)\backslash\{0\}$, it is clear that this path is a path in $(\Gamma(R))^c - S$. If both bx and by are nonzero, then it follows that $x - b - y$ is a path in $(\Gamma(R))^c$ and as $b \in P_2\backslash P_1$, it is clear that this path is a path in $(\Gamma(R))^c - S$.

Suppose that $ax \neq 0$, $ay = 0$, and $bx = 0$, $by \neq 0$. We assert that $x \notin \{a, b\}$ and $y \notin \{a, b\}$. Since $b \in P_2\backslash P_1$, $bx = 0$, and $ab \neq 0$, it is clear that $x \notin \{a, b\}$. Similarly, since $a \in P_1\backslash P_2$, $ay = 0$, and $ab \neq 0$, it is clear that $y \notin \{a, b\}$. Thus both x and y do not belong to $\{a, b\}$. Observe that $x - a - b - y$ is a path in $(\Gamma(R))^c$ between x and y that does not pass through any element of S.

This shows that if S is any nonempty subset of $(P_1 \cap P_2)\backslash\{0\}$, then $(\Gamma(R))^c - S$ is connected and so S is not a vertex cut of $(\Gamma(R))^c$. Now it is clear that if w is any nonzero element of $P_1 \cap P_2$, then $(\Gamma(R))^c - w$ is connected. Hence w is not a cut vertex of $(\Gamma(R))^c$. □

Let R, P_1, P_2 be as in the statement of lemma 3.2. Suppose that one between P_1 and P_2 is not a B-prime of (0) in R. Assuming without loss of generality that P_1 is not a B-prime of (0) in R, we prove in the following lemma that $(\Gamma(R))^c$ does not admit any cut vertex.

Lemma 3.3. Let R, P_1, P_2 be as in the statement of lemma 3.2. Suppose that $(\Gamma(R))^c$ is connected and P_1 is not a B-prime of (0) in R. Then the following hold:
(i) If S is any nonempty subset of $P_2\backslash\{0\}$, then S cannot be a vertex cut of $(\Gamma(R))^c$.
(ii) Let $w \in Z(R)^*$. Then w is not a cut vertex of $(\Gamma(R))^c$.

Proof. (i) Note that $Z(R) = P_1 \cup P_2$. Let S be any nonempty subset of $P_2\backslash\{0\}$. We prove that $(\Gamma(R))^c - S$ is connected. Let $x, y \in Z(R)^*\backslash S, x \neq y$. We prove that there exists a path in $(\Gamma(R))^c - S$ between x and y. If $xy \neq 0$, then $x - y$ is a path in $(\Gamma(R))^c - S$. Suppose that $xy = 0$. By hypothesis, P_1 is not a B-prime of (0) in R. Hence it follows that $P_1 \not\subseteq ((0) :_R x) \cup ((0) :_R y)$. Since $P_1 \not\subseteq P_2$, it follows from the prime avoidance lemma [11, Theorem 81] that $P_1 \not\subseteq P_2 \cup ((0) :_R x) \cup ((0) :_R y)$. Hence there exists $z \in P_1\backslash P_2$ such that $zx \neq 0$ and $zy \neq 0$. Since $z \in P_1\backslash P_2$ whereas $S \subseteq P_2\backslash\{0\}$, it follows that $x - z - y$ is a path in $(\Gamma(R))^c - S$. This proves that if S is any nonempty subset of $P_2\backslash\{0\}$, then $(\Gamma(R))^c - S$ is connected. Hence S is not a vertex cut of $(\Gamma(R))^c$.
(ii) Let $w \in Z(R)^*$. We want to show that w is not a cut vertex of $(\Gamma(R))^c$. Now $w \in Z(R)^* = ((P_1 \cap P_2)\backslash\{0\}) \cup (P_1\backslash P_2) \cup (P_2\backslash P_1)$. If $w \in P_1 \cap P_2$, then it follows from lemma 3.2 that w is not a cut vertex of $(\Gamma(R))^c$. If $w \in P_2$, then it follows from (i) of this lemma that w is not a cut vertex of $(\Gamma(R))^c$. Suppose that $w \in P_1\backslash P_2$. Let $x, y \in Z(R)^*\backslash\{w\}, x \neq y$. We want to show that there exists a path in $(\Gamma(R))^c$ between x and y that does not pass through w. If $xy \neq 0$, then $x - y$ is a path in $(\Gamma(R))^c - w$. Suppose that $xy = 0$. We consider the following cases.
Case(i). $P_2 \not\subseteq ((0) :_R x) \cup ((0) :_R y)$. Then there exists $z \in P_2$ such that $xz \neq 0$ and $yz \neq 0$. Hence $x - z - y$ is a path in $(\Gamma(R))^c$ and as $w \in P_1\backslash P_2$ and $z \in P_2$, it follows that $z \neq w$.
Case(ii). $P_2 \subseteq ((0) :_R x) \cup ((0) :_R y)$. Then either $P_2 \subseteq ((0) :_R x)$ or $P_2 \subseteq ((0) :_R y)$. Without loss of generality, we may assume that $P_2 \subseteq ((0) :_R x)$. Note that $((0) :_R x) \subseteq Z(R) = P_1 \cup P_2$. Hence we obtain that $P_2 \subseteq ((0) :_R x) \subseteq P_1 \cup P_2$, and from this, it follows that $P_2 = ((0) :_R x)$. By hypothesis, P_1 is not a B-prime of (0) in R. If $v \in R\backslash\{0\}$ is such that $vw = 0$, then $Rw \subseteq ((0) :_R v)$. Hence P_1 cannot be a subset of Rw. Observe that $P_1 \not\subseteq P_2 = ((0) :_R x)$. Now it follows from the assumption that P_1 is not a B-prime of (0) in R and from the prime avoidance lemma [11, Theorem 81] that $P_1 \not\subseteq Rw \cup ((0) :_R x) \cup ((0) :_R y)$. Hence there exists $z \in P_1$ such that $z \notin Rw$, and moreover, $xz \neq 0$, and $yz \neq 0$. It is now clear that $x - z - y$ is a path in $(\Gamma(R))^c$ that does not pass through w.

Thus if P_1 is not a B-prime of (0) in R, and if w is any element of $Z(R)^*$, then w is not a cut vertex of $(\Gamma(R))^c$. □

Let R, P_1, P_2 be as in the statement of lemma 3.2. Suppose that $(\Gamma(R))^c$ is connected and if P_1 is not a B-prime of (0) in R, then it is shown in lemma 3.3 that no nonempty subset of $P_2\backslash\{0\}$ is a vertex cut of $(\Gamma(R))^c$. Similarly, if P_2 is not a B-prime of (0) in R, then it can be shown that no nonempty subset of $P_1\backslash\{0\}$ is a vertex cut of $(\Gamma(R))^c$. The following example illustrates that this result may fail to hold if P_2 is a B-prime of (0) in R.

Example 3.4. Let V be a valuation domain with $dim V = 1$ and V is not discrete. Let M denote the unique maximal ideal of V. Let $x \in M, x \neq 0$. Let $R_1 = V/xV$ and let $R_2 = \mathbf{Z}/4\mathbf{Z}$. Let $R = R_1 \times R_2$ be the direct product of the rings R_1 and R_2. Note that R has exactly two maximal N-primes of (0) and they are given by $P_1 = M/xV \times R_2$ and

$P_2 = R_1\times(2+4\mathbf{Z})R_2$ (indeed, P_1 and P_2 are the only prime ideals of R). Note that $P_1\cap P_2 = M/xV\times(2+4\mathbf{Z})R_2 \neq \{(0+xV, 0+4\mathbf{Z})\}$. Hence we obtain from [12, Theorem 1.1(b)] that $(\Gamma(R))^c$ is connected. It is known from [12, Example 3.1(ii)] that M/xV is not a B-prime of (0) in R_1. Hence it follows that P_1 is not a B-prime of the zero ideal in R. Now it follows from lemma 3.3 (i) that if S is any nonempty subset of $P_2\backslash\{0\}$, then $(\Gamma(R))^c - S$ is connected. However we assert that there exists a nonempty subset S of $P_1\backslash\{0\}$ such that S is a vertex cut of $(\Gamma(R))^c$. Let $S = (M/xV\times\{1+4\mathbf{Z}\})\cup(M/xV\times\{3+4\mathbf{Z}\})$. We now show that S is a vertex cut of $(\Gamma(R))^c$. That is, $(\Gamma(R))^c - S$ is not connected. Let $v_1 = (1+xV, 2+4\mathbf{Z})$ and $v_2 = (0+xV, 2+4\mathbf{Z})$. Note that $v_1, v_2\in Z(R)^*\backslash S$. We now verify that there exists no path in $(\Gamma(R))^c - S$ between v_1 and v_2. Since $(\Gamma(R))^c$ is connected, there exists a path in $(\Gamma(R))^c$ between v_1 and v_2. As v_1 and v_2 are not adjacent in $(\Gamma(R))^c$, any path in $(\Gamma(R))^c$ between v_1 and v_2 must be of length at least two. Let $v_1 - w_1 - \cdots - w_m - v_2$ be any path in $(\Gamma(R))^c$ between v_1 and v_2. Let $w_m = (a_m, b_m)$ for some $a_m\in R_1$ and $b_m\in R_2$. Since $w_m v_2 = (a_m, b_m)(0+xV, 2+4\mathbf{Z}) \neq (0+xV, 0+4\mathbf{Z})$, it follows that $b_m(2+4\mathbf{Z})\neq 0+4\mathbf{Z}$. Therefore $b_m\in\{1+4\mathbf{Z}, 3+4\mathbf{Z}\}$. Since $w_m\in Z(R)^*$, it follows that $a_m\in M/xV$. Thus $w_m = (a_m, b_m)\in S$. This shows that any path in $(\Gamma(R))^c$ between v_1 and v_2 must pass through at least one element of S and so there exists no path in $(\Gamma(R))^c - S$ between v_1 and v_2. Hence $S = (M/xV\times\{1+4\mathbf{Z}\})\cup(M/xV\times\{3+4\mathbf{Z}\})$ is a vertex cut of $(\Gamma(R))^c$.

Remark 3.5. Let R, P_1, P_2 be as in the statement of lemma 3.2. Suppose that $(\Gamma(R))^c$ is connected. It follows from lemma 3.3 that if at least one between P_1 and P_2 is not a B-prime of (0) in R, then $(\Gamma(R))^c$ does not admit any cut vertex. Moreover, example 3.1 illustrates that this result may fail to hold if both P_1 and P_2 are B-primes of (0) in R. We now assume that both P_1 and P_2 are B-primes of (0) in R (note that this assumption holds if R is Noetherian [3, Proposition 7.17]). Under this assumption, we now proceed to give some sufficient condition in order that $(\Gamma(R))^c$ does not admit any cut vertex. □

We begin with the following lemma.

Lemma 3.6. Let R be a commutative ring with identity and let $\{P_1, P_2\}$ be the set of all maximal N-primes of (0) in R. Suppose that $(\Gamma(R))^c$ is connected. Let $u, v\in R$ be such that $P_1 = ((0):_R u)$ and $P_2 = ((0):_R v)$. Then the following hold:
(i) $uv = 0$.
(ii) Either $u\in P_1\cap P_2$ or $v\in P_1\cap P_2$.

Proof. (i) This is well known [6 , Lemma 3.6].
(ii) Let $x\in P_1\backslash P_2$. Note that $xu = 0\in P_2$ and as $x\notin P_2$, it follows that $u\in P_2$. Let $y\in P_2\backslash P_1$. We have $yv = 0\in P_1$. As $y\notin P_1$, it follows that $v\in P_1$. Suppose that $u\notin P_1\cap P_2$ and $v\notin P_1\cap P_2$. Then we obtain that $u\in P_2\backslash P_1$ and $v\in P_1\backslash P_2$. Since $Z(R) = P_1\cup P_2$, it follows that $u+v\notin Z(R)$. Now by hypothesis, $(\Gamma(R))^c$ is connected. Hence we obtain from [12, Theorem 1.1(b)] that $P_1\cap P_2\neq(0)$. Let $z\in(P_1\cap P_2)\backslash\{0\}$. Note that $zu = 0$ and $zv = 0$. Hence $(u+v)z = 0$. Since $u+v\notin Z(R)$, it follows that $z = 0$. This is a contradiction. Hence either $u\in P_1\cap P_2$ or $v\in P_1\cap P_2$.

□

Let R, P_1, P_2 be as in the statement of lemma 3.2. Suppose that both P_1 and P_2 are B-primes of (0) in R. The following lemma gives a sufficient condition under which $(\Gamma(R))^c$ does not admit any cut vertex.

Lemma 3.7. Let R, P_1, P_2 be as in the statement of lemma 3.2. Assume that both P_1 and P_2 are B-primes of (0) in R. Suppose that $(\Gamma(R))^c$ is connected. If $|P_1\cap P_2|\geq 3$, then $(\Gamma(R))^c$ does not admit any cut vertex.

Proof. Let $w\in Z(R)^*$. Note that $Z(R) = P_1\cup P_2 = (P_1\cap P_2)\cup(P_1\backslash P_2)\cup(P_2\backslash P_1)$. If $w\in P_1\cap P_2$, then we know from lemma 3.2 that w is not a cut vertex of $(\Gamma(R))^c$. So we may assume that $w\notin P_1\cap P_2$. Hence either $w\in P_1\backslash P_2$ or $w\in P_2\backslash P_1$. Suppose that $w\in P_1\backslash P_2$. We verify that w is not a cut vertex of $(\Gamma(R))^c$. Since $(\Gamma(R))^c$ is connected, to prove that w is a cut vertex of $(\Gamma(R))^c$, it is enough to prove the following: if $x - w - y$ is a path in $(\Gamma(R))^c$ with $xy = 0$, then there exists a path in $(\Gamma(R))^c$ between x and y that does not pass through w. Since P_1 and P_2 are both B-primes of (0) in R, there exist $u, v\in R$ such that $P_1 = ((0):_R u)$ and $P_2 = ((0):_R v)$.
If $P_2\not\subseteq((0):_R x)\cup((0):_R y)$, then there exists $z\in P_2$ such that $zx\neq 0$, and $zy\neq 0$. Since $w\in P_1\backslash P_2$, it follows that $z\neq w$. Note that $x - z - y$ is a path in $(\Gamma(R))^c$ that does not pass through w. Suppose that $P_2\subseteq((0):_R x)\cup((0):_R y)$. Then it follows that either $P_2 = ((0):_R x)$ or $P_2 = ((0):_R y)$. Without loss of generality we may assume that $P_2 = ((0):_R x)$. We consider the following cases.
Case(i). Both x and y belong to P_1
Since $P_1 = ((0):_R u)$, it follows that $xu = yu = 0$. We know from lemma 3.6(ii) that either $u\in P_1\cap P_2$ or $v\in P_1\cap P_2$. Suppose that $u\in P_1\cap P_2$. Then, observe that $u+w\in P_1\backslash P_2, u+w\neq w$, and $x(u+w) = xw\neq 0$, $y(u+w) = yw\neq 0$. Hence $x - (u+w) - y$ is a path in $(\Gamma(R))^c$ that does not pass through w.
Suppose that $v\in P_1\cap P_2$. Note that $v+w\in P_1\backslash P_2$. By assumption, $P_2 = ((0):_R x)$. Hence we obtain that

$x(v+w) = xv + xw = xw \neq 0$. Since $xy = 0$, it follows that $y \in P_2$. Then $yv = 0$, and so $(v+w)y = yw \neq 0$. Thus we obtain that $x - (v+w) - y$ is a path in $(\Gamma(R))^c$ that does not pass through w.

Case(ii)**.** Both x and y belong to P_2

Note that $v \in P_1$ and $w \in P_1$. Hence $v + w \in P_1$ and it is clear that $v + w \neq w$. Since $P_2 = ((0) :_R v)$, it follows that $x(v+w) = xw \neq 0$, and $y(v+w) = yw \neq 0$. Observe that $x - (v+w) - y$ is a path in $(\Gamma(R))^c$ that does not pass through w.

Case(iii)**:** $x \in P_1 \backslash P_2$ and $y \in P_2 \backslash P_1$

By assumption $P_1 = ((0) :_R u)$ and $P_2 = ((0) :_R v) = ((0) :_R x)$. Since $x \in P_1 \backslash P_2$, it follows that $xv \neq 0$ and so $v \notin P_2$. Hence it follows from lemma 3.6(ii) that $u \in P_1 \cap P_2$. Note that $u + v \in P_1 \backslash P_2, x(u+v) = xv \neq 0$, and since $y \in P_2 \backslash P_1, y(u+v) = yu \neq 0$. Thus if $u + v \neq w$, then $x - (u+v) - y$ is a path in $(\Gamma(R))^c$ that does not pass through w.

Suppose that $u+v = w$. We are assuming that $|P_1 \cap P_2| \geq 3$. Hence there exists $z \in P_1 \cap P_2$ such that $z \notin \{0, u\}$. We claim that $x - (z+v) - y$ is a path in $(\Gamma(R))^c$ that does not pass through w. Since $z \neq u$ and $w = u+v$, it is clear that $z + v \neq w$. As $z \in P_1 \cap P_2$ and $v \in P_1 \backslash P_2$, it follows that $z + v \in P_1 \backslash P_2$. Now it is clear that $x(z+v) \in P_1 \backslash P_2$ and hence it is nonzero. As $x + y \notin Z(R)$, it follows that $(x+y)z \neq 0$. Since $P_2 = ((0)) :_R x)$, we obtain that $z(x+y) = zy \neq 0$. Now on using the facts that $P_2 = ((0) :_R v)$ and $y \in P_2$, we obtain that $(z+v)y = zy \neq 0$. This shows that $x - (z+v) - y$ is a path in $(\Gamma(R))^c$ that does not pass through w.

This proves that if $w \in P_1 \backslash P_2$, then w is not a cut vertex of $(\Gamma(R))^c$. Similarly, it follows that if $w \in P_2 \backslash P_1$, then w is not a cut vertex of $(\Gamma(R))^c$.

Thus if $(\Gamma(R))^c$ is connected, both P_1 and P_2 are B-primes of (0) in R, and if $|P_1 \cap P_2| \geq 3$, then $(\Gamma(R))^c$ does not admit any cut vertex. □

Let R, P_1, P_2 be as in the statement of lemma 3.2. Suppose that $(\Gamma(R))^c$ is connected. If both P_1 and P_2 are B-primes of (0) in R, then the following lemma also gives a sufficient condition under which $(\Gamma(R))^c$ does not admit any cut vertex.

Lemma 3.8. Let R, P_1, P_2, be as in the statement of lemma 3.2. Suppose that $(\Gamma(R))^c$ is connected and both P_1 and P_2 are B-primes of (0) in R. Then the following hold:
(i) If $|P_1 \backslash P_2| \geq 3$, then no nonzero element of P_1 is a cut vertex of $(\Gamma(R))^c$.
(ii) If $|P_2 \backslash P_1| \geq 3$, then no nonzero element of P_2 is a cut vertex of $(\Gamma(R))^c$.
(iii) If $|P_1 \backslash P_2| \geq 3$ and $|P_2 \backslash P_1| \geq 3$, then $(\Gamma(R))^c$ does not admit any cut vertex.

Proof. (i) Let $w \in P_1 \backslash \{0\}$. We know from lemma 3.2 that no nonzero element of $P_1 \cap P_2$ is a cut vertex of $(\Gamma(R))^c$. Hence we may assume that $w \in P_1 \backslash P_2$. In order to prove that w is not a cut vertex of $(\Gamma(R))^c$, it is enough to prove the following: if $x - w - y$ is a path in $(\Gamma(R))^c$ with $xy = 0$, then there exists a path in $(\Gamma(R))^c$ between x and y that does not pass through w. Let $u, v \in Z(R)^*$ be such that $P_1 = ((0) :_R u)$ and $P_2 = ((0) :_R v)$. Proceeding as in the proof of lemma 3.7, we may assume that $P_2 = ((0) :_R x)$. If both x and y are in P_1, or both x and y are in P_2 then it follows as in the proof of lemma 3.7 that there exists a path in $(\Gamma(R))^c$ between x and y that does not pass through w. Hence we may assume that $x \in P_1 \backslash P_2$ and $y \in P_2 \backslash P_1$. If $u + v \neq w$, then it follows as in the proof of lemma 3.7 that $x - (u+v) - y$ is a path in $(\Gamma(R))^c$ that does not pass through w. So we may assume that $u + v = w$. Now $xw = x(u+v) = xv \neq 0$. Hence $v \notin P_2$ and so we obtain from lemma 3.6(ii) that $u \in P_1 \cap P_2$. If $|P_1 \cap P_2| \geq 3$, then it is shown in the proof of lemma 3.7 that there exists a path in $(\Gamma(R))^c$ between x and y that does not pass through w. Suppose that $|P_1 \cap P_2| = 2$. Now $w \in P_1 \backslash P_2$ and by hypothesis $|P_1 \backslash P_2| \geq 3$. Hence there exist distinct elements $w_1, w_2 \in P_1 \backslash P_2$ such that $w_i \neq w$ for $i = 1, 2$. Note that $xw_i \in P_1 \backslash P_2$ for $i = 1, 2$. Hence $xw_i \neq 0$ for $i = 1, 2$. If $yw_1 \neq 0$, then $x - w_1 - y$ is a path in $(\Gamma(R))^c$ that does not pass through w. Similarly, if $yw_2 \neq 0$, then $x - w_2 - y$ is a path in $(\Gamma(R))^c$ that does not contain w as a vertex. So we may assume that $yw_i = 0$ for $i = 1, 2$. Since $|P_1 \cap P_2| = 2$, either $w - w_1 \notin P_1 \cap P_2$ or $w - w_2 \notin P_1 \cap P_2$. Without loss of generality we may assume that $w - w_1 \notin P_1 \cap P_2$. Thus $w - w_1 \in P_1 \backslash P_2$. Hence $x(w - w_1) \in P_1 \backslash P_2$ and so it is nonzero. Moreover, $(w - w_1)y = wy \neq 0$. Observe that $x - (w - w_1) - y$ is a path in $(\Gamma(R))^c$ between x and y that does not pass through w.
(ii) This can be proved as in (i).
(iii) Since $Z(R) = P_1 \cup P_2$, (iii) follows immediately from (i) and (ii). □

We next proceed to determine commutative rings R with identity such that R has the following properties: (i) R has exactly two maximal N-primes of (0), (ii) $(\Gamma(R))^c$ is connected, and (iii) $(\Gamma(R))^c$ admits a cut vertex. We begin with the following lemma.

Lemma 3.9. Let R, P_1, P_2 be as in the statement of lemma 3.2. Suppose that $(\Gamma(R))^c$ is connected. If $(\Gamma(R))^c$ admits a cut vertex, then the following hold:
(i) $R \cong R/P_1^2 \times R/P_2^2$ as rings.
(ii) Exactly one between R/P_1^2 and R/P_2^2 is an integral domain and the other is either isomorphic to $\mathbf{Z}/4\mathbf{Z}$ or is isomorphic to $(\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z}))[x]$ as rings where $(\mathbf{Z}/2\mathbf{Z}))[x]$ is the polynomial ring in one variable over $\mathbf{Z}/2\mathbf{Z}$.

:**Proof.**(i) Note that $Z(R) = P_1 \cup P_2$. Assume that $(\Gamma(R))^c$ is connected and it admits a cut vertex. We know from lemma 3.3 that both P_1 and P_2 are B-primes of (0) in R. Moreover, since $P_1 \cap P_2 \neq (0)$, it follows from lemma 3.7 that $|P_1 \cap P_2| = 2$. Let $P_1 \cap P_2 = \{0, z\}$. As $z^2 \in P_1 \cap P_2$, either $z^2 = z$ or $z^2 = 0$. If $z^2 = z$, then $z(1 - z) = 0$. This implies that $z = 0$ since $1 - z \notin P_1 \cup P_2 = Z(R)$. This contradicts the fact that $z \neq 0$. Hence $z^2 = 0$. Thus $(P_1 \cap P_2)^2 = (0)$. We next verify that $P_1 + P_2 = R$. Suppose that this does not hold. Then there exists a maximal ideal M of R such that $P_1 + P_2 \subseteq M$. Let $a \in P_1 \backslash P_2$ and $b \in P_2 \backslash P_1$. Let $m = a + b$. Observe that $m \in M$ and since $Z(R) = P_1 \cup P_2$, it is clear from the choice of the elements a and b that $m = a + b \notin Z(R)$. As $1 - m \notin M$, it follows from $P_1 + P_2 \subseteq M$ that $1 - m \notin P_1 \cup P_2 = Z(R)$. Now $mz \in (P_1 \cap P_2)\backslash\{0\}$. Hence $mz = z$. This implies that $z(1 - m) = 0$. Hence $z = 0$. This is a contradiction. This shows that $P_1 + P_2 = R$. Hence P_1^2 and P_2^2 are also comaximal and so $(0) = (P_1 \cap P_2)^2 = P_1^2P_2^2 = P_1^2 \cap P_2^2$. Now it follows from the Chinese remainder theorem [3, Proposition 1.10(ii)] that $R \cong R/P_1^2 \times R/P_2^2$ as rings.

(ii) We next prove (ii). It is shown in (i) above with the help of the Chinese remainder theorem that the mapping $f : R \to R/P_1^2 \times R/P_2^2$ given by $f(r) = (r + P_1^2, r + P_2^2)$ for any $r \in R$ is an isomorphism of rings. Note that under the isomorphism f, $Z(R) = P_1 \cup P_2$ is mapped onto $(P_1/P_1^2 \times R/P_2^2) \cup (R/P_1^2 \times P_2/P_2^2) = Z((R/P_1^2 \times R/P_2^2)) = (Z(R/P_1^2) \times R/P_2^2) \cup (R/P_1^2 \times Z(R/P_2^2))$. Hence we obtain that $Z(R/P_1^2) = P_1/P_1^2$ and $Z(R/P_2^2) = P_2/P_2^2$. Moreover, $f(P_1 \cap P_2) = P_1/P_1^2 \times P_2/P_2^2$. Hence $2 = |P_1 \cap P_2| = |P_1/P_1^2 \times P_2/P_2^2|$. Therefore, either $|P_1/P_1^2| = 2$ and $|P_2/P_2^2| = 1$ or $|P_1/P_1^2| = 1$ and $|P_2/P_2^2| = 2$. Without loss of generality we may assume that $|P_1/P_1^2| = 2$ and $|P_2/P_2^2| = 1$. This implies that $|Z(R/P_1^2)| = |P_1/P_1^2| = 2$ and $P_2 = P_2^2$. Hence $\Gamma(R/P_1^2)$ is a graph with a single vertex and so R/P_1^2 is either isomorphic to $\mathbf{Z}/4\mathbf{Z}$ or is isomorphic to $(\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z})[x]$ as rings [1, Example 2.1]. As $P_2 = P_2^2$, it follows that R/P_2^2 is an integral domain. This proves (ii). □

Let R, P_1, P_2 be as in the statement of lemma 3.2. Suppose that $(\Gamma(R))^c$ is connected. Theorem 3.12 provides a necessary and sufficient condition in order that $(\Gamma(R))^c$ admits a cut vertex. We next have the following lemma which is used in the proof of theorem 3.12.

Lemma 3.10. Let R, P_1, P_2 be as in the statement of lemma 3.2. Suppose that $(\Gamma(R))^c$ is connected. If $|P_2\backslash P_1| = 2$ and $|P_1\backslash P_2| \geq 3$, then $(\Gamma(R))^c$ admits a cut vertex if and only if $P_1 = ((0) :_R b)$ for some $b \in P_2\backslash P_1$.

Proof. By hypothesis, $(\Gamma(R))^c$ is connected, $|P_2\backslash P_1| = 2$, and $|P_1\backslash P_2| \geq 3$. Suppose that $(\Gamma(R))^c$ admits a cut vertex. We know from lemma 3.3 that both P_1 and P_2 are B-primes of (0) in R. Let $u, v \in Z(R)^*$ be such that $P_1 = ((0) :_R u)$ and $P_2 = ((0) :_R v)$. Let $w \in Z(R)^*$ be a cut vertex of $(\Gamma(R))^c$. It follows from lemma 3.2 and lemma 3.8(ii) that $w \in P_2\backslash P_1$. We want to prove that $P_1 = ((0) :_R b)$ for some $b \in P_2\backslash P_1$. This is immediate if $u \in P_2\backslash P_1$. Suppose that $u \in P_1 \cap P_2$. Since w is a cut vertex of $(\Gamma(R))^c$, it follows from [5, Theorem 3.1.6] that there exist $x, y \in Z(R)^*\backslash\{w\}, x \neq y$ such that every path in $(\Gamma(R))^c$ between x and y passes through w. Let $x - \cdots - x_1 - w - y_1 - \cdots - y$ be any path in $(\Gamma(R))^c$ between x and y. Observe that $x_1y_1 = 0$. We claim that either $P_1 = ((0) :_R x_1)$ or $P_1 = ((0) :_R y_1)$. Suppose that this does not hold. Then $P_1 \not\subseteq ((0) :_R x_1) \cup ((0) :_R y_1)$. Hence there exists $z \in P_1$ such that $zx_1 \neq 0$ and $zy_1 \neq 0$. Now on replacing w by z in the above path, we obtain a path in $(\Gamma(R))^c$ between x and y that does not pass through w. This is impossible. Thus either $P_1 = ((0) :_R x_1)$ or $P_1 = ((0) :_R y_1)$. Without loss of generality we may assume that $P_1 = ((0) :_R x_1)$. Since $x_1y_1 = 0$, it follows that $y_1 \in P_1$. If $x_1 \in P_1$, then on replacing w by $u + w$ in the above path we obtain a path in $(\Gamma(R))^c$ between x and y that does not pass through w. This is a contradiction. Thus $x_1 \notin P_1$. Thus $P_1 = ((0) :_R x_1)$ and $x_1 \in P_2\backslash P_1$.

Conversely assume that $P_1 = ((0) :_R b)$ for some $b \in P_2\backslash P_1$. Let $a \in P_1\backslash\{0\}$. Note that $ab = 0$. Let $P_2\backslash P_1 = \{b, c\}$. We now show that c is a cut vertex of $(\Gamma(R))^c$ by verifying that any path in $(\Gamma(R))^c$ between a and b passes through c. Let $a - y_1 - \cdots - y_m - b$ be any path in $(\Gamma(R))^c$ between a and b. Since $P_1 = ((0) :_R b)$ and $y_mb \neq 0$, it follows that $y_m \notin P_1$. Hence $y_m \in P_2\backslash P_1 = \{b, c\}$. As $y_m \neq b$, it follows that $y_m = c$. This proves that $(\Gamma(R))^c$ admits a cut vertex. □

The following lemma is also needed for proving theorem 3.12.

Lemma 3.11. Let $T = T_1 \times T_2$ be the direct product of the rings T_1 and T_2 where either $T_1 = \mathbf{Z}/4\mathbf{Z}$ or $T_1 = (\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z})[x]$ and T_2 is an integral domain. Then the following hold:
(i) $(\Gamma(T))^c$ is connected.
(ii) $(\Gamma(T))^c$ admits a cut vertex if and only if $T_2 \cong \mathbf{Z}/2\mathbf{Z}$ as rings.

Proof. Suppose that $T_1 = \mathbf{Z}/4\mathbf{Z}$.

(i) Note that T has exactly two maximal N-primes of the zero ideal of T and they are given by $P_1 = (2+4\mathbf{Z})T_1 \times T_2$ and $P_2 = T_1 \times \{0\}$. Note that $(2 + 4\mathbf{Z}, 0) \in P_1 \cap P_2$ and hence $P_1 \cap P_2$ is nonzero. Therefore, we obtain from [12, Theorem 1.1(b)] that $(\Gamma(T))^c$ is connected.

(ii) Suppose that $(\Gamma(T))^c$ admits a cut vertex. Let P_1, P_2 be as in the proof of (i). Note that $P_2\backslash P_1 = \{(1 + 4\mathbf{Z}, 0), (3+4\mathbf{Z}, 0)\}$ and $P_1\backslash P_2 = (2+4\mathbf{Z})T_1 \times T_2\backslash\{0\}$. Thus $|P_2\backslash P_1| = 2$. We want to establish that $|T_2| = 2$. Suppose that $|T_2| \geq 3$. Then it follows that $|P_1\backslash P_2| \geq 3$. Now we obtain from lemma 3.10 that $P_1 = ((0+4\mathbf{Z}, 0) :_T (t_1, t_2))$ for some $(t_1, t_2) \in P_2\backslash P_1$. Note that either $(t_1, t_2) = (1 + 4\mathbf{Z}, 0)$ or $(t_1, t_2) = (3 + 4\mathbf{Z}, 0)$. Observe that $(2 + 4\mathbf{Z}, 0) \in P_1$

and is such that $(2+4\mathbf{Z}.0)(t_1, t_2) = (2+4\mathbf{Z}, 0)$. This is in contradiction to the fact that $P_1 = ((0+4\mathbf{Z}, 0) :_T (t_1, t_2))$. Hence $|T_2| = 2$ and so $T_2 \cong \mathbf{Z}/2\mathbf{Z}$ as rings.

It is verified in example 3.1 that the ring $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ admits a cut vertex.

The proof of (i) and (ii) in the case $T_1 = (\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z})[x]$ is similar. □

The following theorem determines commutative rings R with identity such that R has exactly two maximal N-primes of (0), $((\Gamma(R))^c$ is connected, and $(\Gamma(R))^c$ admits a cut vertex.

Theorem 3.12. Let R be a commutative ring with identity. Suppose that R has exactly two maximal N-primes of (0). Let them be P_1 and P_2. Suppose that $(\Gamma(R))^c$ is connected. Then $(\Gamma(R))^c$ admits a cut vertex if and only if either R is isomorphic to $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ or R is isomorphic to $(\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z})[x] \times \mathbf{Z}/2\mathbf{Z}$ as rings where $(\mathbf{Z}/2\mathbf{Z})[x]$ is the polynomial ring in one variable over $\mathbf{Z}/2\mathbf{Z}$.

Proof. Suppose that $(\Gamma(R))^c$ admits a cut vertex. Then it follows from lemma 3.9(i) and (ii) that R is isomorphic to the direct product of rings T_1 and T_2 where either $T_1 = \mathbf{Z}/4\mathbf{Z}$ or $T_1 = (\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z})[x]$ and T_2 is an integral domain. Hence $(\Gamma(R))^c$ admits a cut vertex if and only if $(\Gamma(T_1 \times T_2))^c$ admits a cut vertex. We know from lemma 3.11(ii) that $(\Gamma(T_1 \times T_2))^c$ admits a cut vertex if and only if T_2 is isomorphic to $\mathbf{Z}/2\mathbf{Z}$ as rings. This proves that $(\Gamma(R))^c$ admits a cut vertex if and only if either R is isomorphic to $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ or R is isomorphic to $(\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z})[x] \times \mathbf{Z}/2\mathbf{Z}$ as rings. □

4 *R* has more than two maximal *N*-primes of (0)

Let R be a commutative ring with identity with at least three maximal N-primes of (0). If there exists a nonempty finite subset S of $Z(R)^*$ such that S is a vertex cut of $(\Gamma(R))^c$, then we prove in proposition 4.3 that R can admit only a finite number of maximal N-primes of (0). We make use of the following lemma in the proof of proposition 4.3 and moreover, we deduce as a corollary to this lemma that $(\Gamma(R))^c$ does not admit any cut vertex.

Lemma 4.1. Let R be a commutative ring with identity. Let $n \geq 3$. Suppose that R has at least n maximal N-primes of (0). Let S be any subset of $Z(R)^*$ such that $|S| = n-2$. Then $(\Gamma(R))^c - S$ is connected.

Proof. Let $\mathcal{P}$ denote the set of all maximal N-primes of (0) in R. By hypothesis $|\mathcal{P}| \geq n$ where $n \geq 3$. Let S be a subset of $Z(R)^*$ such that $|S| = n-2$. We want to establish that $(\Gamma(R))^c - S$ is connected. We know from [12, Theorem 1.1 (c)] that $(\Gamma(R))^c$ is connected. Hence in order to show that $(\Gamma(R))^c - S$ is connected, it is enough to prove the following: if $x - w - y$ is a path in $(\Gamma(R))^c$ with $x \in Z(R)^*\backslash S$, $w \in S$, and $y \in Z(R)^*$(y may be possibly in S) satisfying the further condition that $xy = 0$, then there exists a path in $(\Gamma(R))^c$ between x and y whose vertices except possibly y, do not belong to S. Let $S = \{w_1, w_2, \ldots, w_{n-2}\}$. Note that $w = w_i$ for some $i \in \{1, 2, \ldots, n-2\}$. Let $P_j \in \mathcal{P}$ for $j = 1, 2, \ldots, n-2$ be such that $w_j \in P_j$. It may happen that for some distinct $s, t \in \{1, 2, \ldots, n-2\}$, $P_s = P_t$. Since the number of maximal N-primes of (0) in R is at least n, it is possible to find distinct elements $P_{n-1}, P_n \in \mathcal{P} \backslash \{P_j | j = 1, 2, \ldots n-2\}$. Let $k \in \{n-1, n\}$. If $P_k \not\subseteq ((0) :_R x) \cup ((0)) :_R y)$, then it follows from the prime avoidance lemma [11, Theorem 81] that there exists $z_k \in P_k$ such that $z_k \notin \cup_{j=1}^{n-2} P_j$, $z_k x \neq 0$, and $z_k y \neq 0$. Note that $z_k \notin S$ and $x - z_k - y$ is a path in $(\Gamma(R))^c$.

Suppose that $P_k \subseteq ((0) :_R x) \cup ((0) :_R y)$ for each $k \in \{n-1, n\}$. (1)

Since $((0) :_R x) \cap (R\backslash Z(R)) = \emptyset$ and $((0) :_R y) \cap (R\backslash Z(R)) = \emptyset$, it follows from Zorn's lemma and [11, Theorem 1] that there exist $P, Q \in \mathcal{P}$ such that $((0) :_R x) \subseteq P$ (2)

and $((0) :_R y) \subseteq Q$. (3)

Now it follows from $(1), (2)$, and (3) that $\{P_{n-1}, P_n\} = \{((0) :_R x), ((0) :_R y)\} = \{P, Q\}$. Without loss of generality we may assume that $P_{n-1} = ((0) :_R x)$ and $P_n = ((0) :_R y)$. It follows from the prime avoidance lemma that there exist $x_1 \in P_n\backslash(\cup_{j=1}^{n-1} P_j)$ and $y_1 \in P_{n-1}\backslash((\cup_{j=1}^{n-2} P_j) \cup P_n)$. Now it is clear from the choice of the elements x_1, y_1 that $x_1, y_1 \notin S, x_1 x \neq 0, y_1 y \neq 0$, and $x_1 \neq y_1$. Since both x_1 and y_1 are not in P_1, it follows that $x_1 y_1 \notin P_1$ and so $x_1 y_1 \neq 0$.

If both x_1 and y_1 are not in $\{x, y\}$, then it is clear that $x - x_1 - y_1 - y$ is a path in $(\Gamma(R))^c$ between x and y with $x_1, y_1 \notin S$. Suppose that $x_1 \in \{x, y\}$. Since $xy = 0$ whereas $xx_1 \neq 0$, it follows that $x_1 = x$. Again since $xy = 0$ but $x_1 y_1 \neq 0$, it follows that $y_1 \neq y$. Observe that $x = x_1 - y_1 - y$ is a path in $(\Gamma(R))^c$ between x and y with $y_1 \notin S$. Suppose that $y_1 \in \{x, y\}$. We have $xy = 0$ but $yy_1 \neq 0$, it follows that $y_1 = y$. Since $xy = 0$, whereas $x_1 y_1 \neq 0$, we obtain that $x_1 \neq x$. Note that $x - x_1 - y_1 = y$ is a path in $(\Gamma(R))^c$ between x and y with both x_1 and $y_1 = y$ are not in S.

This shows that if R has at least n maximal N-primes of (0) with $n \geq 3$, then for any subset S of $Z(R)^*$ with $|S| = n-2$, $(\Gamma(R))^c - S$ is connected. □

We next have the following corollary.

Corollary 4.2. Let R be as in the statement of lemma 4.1. Then $(\Gamma(R))^c$ does not admit any cut vertex.

Proof. On applying lemma 4.1 with $n = 3$, it follows that $(\Gamma(R))^c - w$ is connected for any $w \in Z(R)^*$. This proves that $(\Gamma(R))^c$ does not admit any cut vertex. □

The following result is also a consequence of lemma 4.1.

Proposition 4.3. Let R be a commutative ring with identity such that R has at least three maximal N-primes of (0). Let S be a finite nonempty subset of $Z(R)^*$ such that $(\Gamma(R))^c - S$ is not connected. Then R can have at most $|S| + 1$ maximal N-primes of (0).

Proof. Suppose that R has at least $n = |S| + 2$ maximal N- primes of (0). Note that $|S| = n - 2$. Now it follows from lemma 4.1 that $(\Gamma(R))^c - S$ is connected. This is in contradiction to the hypothesis that $(\Gamma(R))^c - S$ is not connected. Hence R can have at most $|S| + 1$ maximal N-primes of (0). □

5 Conclusion

Let R be a commutative ring with identity such that R contains at least two nonzero zero-divisors. Suppose that $(\Gamma(R))^c$ is connected. The following theorem characterizes rings R such that $(\Gamma(R))^c$ admits a cut vertex.

Theorem 5.1. Let R be a commutative ring with identity admitting at least two nonzero zero-divisors. Suppose that $(\Gamma(R))^c$ is connected. Then $(\Gamma(R))^c$ admits a cut vertex if and only if either R is isomorphic to $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ or R is isomorphic to $(\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z})[x] \times \mathbf{Z}/2\mathbf{Z}$ as rings where $(\mathbf{Z}/2\mathbf{Z})[x]$ is the polynomial ring in one variable over $\mathbf{Z}/2\mathbf{Z}$.

Proof. Suppose that $(\Gamma(R))^c$ is connected and admits a cut vertex. Then it follows from lemma 2.1 and corollary 4.2 that R must have exactly two maximal N-primes of (0). Now it follows from theorem 3.12 that $(\Gamma(R))^c$ admits a cut vertex if and only if either R is isomorphic to $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ or R is isomorphic to $(\mathbf{Z}/2\mathbf{Z})[x]/x^2(\mathbf{Z}/2\mathbf{Z})[x] \times \mathbf{Z}/2\mathbf{Z}$ as rings. □

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