MEAN VARIANCE OPTIMIZATION OF PORTFOLIOS
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Abstract. The Mean-Variance Portfolio Theory continues to be the cardinal tool for much of portfolio management. Traditional concerted literature on the Mean-Variance theory can be segmented almost exclusively into (i) chapters in books that provide simply a write up on the theory and (ii) books that contain a purely mathematical analysis without emphasizing the financial implications and interpretations. The fallout of this mutually exclusive segmentation is that both segments cover Mean- Variance portfolio theory only marginally i.e. in the asymptote rather than as a mainstream course. The coverage is, nowhere near adequate for a student to get acquainted with the intricacies of the theory and hence, appreciate its nuances. This article fights that trend by covering in detail the topics that are thrown by the wayside in the traditional coverage. We look at a comprehensive mathematical analysis of the two security problem in risk return space, obtain several interesting mathematical results and follow up each of them with their interpretation and explanations in financial markets. We, then, extend the framework to three security dynamics and again elucidate some intriguing mathematical inferences.

1 Introduction

The Mean-Variance portfolio theory continues to emerge as the cornerstone of modern portfolio management. The vital feature of the theory is its robustness just like the Economic Order Quantity (EOQ) model in Operational Research. The objective of portfolio management viz. the optimal allocation of the investments between available spectrum of assets is evaluated in a two dimensional risk-return framework. The “efficient frontier” in the Mean-Variance framework enables us to identify the optimal portfolio given a particular level of risk tolerance. It is emphasized, however, that the level of risk tolerance is singular to the investor’s risk profile and, as such, does not, strictly speaking, come within the domain of Mean-Variance portfolio theory – it may be handled by, for example, utility theory or the indifference map etc.

2 Concept of Risk & Return in the Mean-Variance Framework

In the Mean – Variance framework of portfolio management, we represent and evaluate securities in a two dimensional framework (i.e. risk and return) with, conventionally, the risk, being expressed along the abscissa ( axis) and expected return along the ordinate ( axis). In this context, instantaneous return is usually measured in terms of the accretion in the value of the security over an infinitesimal time period i.e.

\[ dR(t) = \frac{dS(t)}{S(t)} \quad (2.1) \]

Correspondingly, the return over a finite time span \( t_2 - t_1 \) is given by

\[ R(t_2, t_1) = \frac{S(t_2) - S(t_1)}{S(t_1)} \quad (2.2) \]

Eq. (2.2) shall, obviously need to be adjusted for any intermediate cash flows emanating from the security during the period \( t_2 - t_1 \). While the measure of return (2.2) is very convenient for measuring single period returns, extension to multi-period cases results in a very serious problem viz. the formula (2.2) is not additive. In other words,

\[ R_{\text{average}}(t_2, t_0) = \frac{S(t_2) - S(t_0)}{2S(t_0)} \neq \frac{R(t_2, t_1) + R(t_1, t_0)}{2} \quad (2.3) \]

As a remedy to the problem of non-additivity, the concept of “logarithmic return” finds its way into the literature. It is defined by integrating eq. (2.1) and we obtain

\[ R_{\text{ln}}(t_2, t_1) = \log_e \frac{S(t_2)}{S(t_1)} \quad (2.4) \]

whence

\[ R_{\text{ln,average}}(t_2, t_0) = \frac{1}{2} \log_e \frac{S(t_2)}{S(t_0)} = \frac{1}{2} \left[ \log_e \frac{S(t_2)}{S(t_1)} + \log_e \frac{S(t_1)}{S(t_0)} \right] = \frac{R_{\text{ln}}(t_2, t_1) + R_{\text{ln}}(t_1, t_0)}{2} \quad (2.5) \]

For small returns, the two measures are equivalent for
log_e \left( \frac{S(t_2)}{S(t_1)} \right) = \log_e \left[ 1 + \frac{S(t_2) - S(t_1)}{S(t_1)} \right] \approx \frac{S(t_2) - S(t_1)}{S(t_1)} \tag{2.6}

“Risk” is, usually, interpreted as the “uncertainty” associated with an experiment in achieving its desired outcome i.e. the probability of the outcome of the experiment not being able to attain the target. Consequently, while evaluating risk, we are concerned with “downside” aspect of the return patterns i.e. the probability of actual returns falling short of targeted returns rather than “upside” or the probability of returns exceeding targets. However, in “Mean – Variance” Portfolio Theory, we measure risk in terms of the standard deviation of the security returns. There is some rationale behind this e.g.

(a) The “uncertainty” is directly related to the level of fluctuations or “dispersion” about the mean value i.e. higher the amplitude of swing about the mean value, higher is the uncertainty of achieving the targeted return;

(b) The return structure of securities is assumed symmetric so that the level of downside fluctuations equals the level of upside fluctuations;

(c) The stock process or the logarithm thereof is assumed to follow a normal distribution which is completely parameterized by the mean/expected value and standard deviation.

3 Return and Variance of a Portfolio of Securities

The instantaneous, expected returns and variances of a portfolio of N securities with composition vector $X = \{X_i, i = 1, 2, 3, \ldots, N\}$, $\sum_{i=1}^{N} X_i = 1$ are given respectively by:

$$R_P = \sum_{i=1}^{N} X_i R_i \tag{3.1}$$

$$E(R_P) = \sum_{i=1}^{N} X_i E(R_i) \tag{3.2}$$

$$\sigma_P^2 = E[R_P - E(R_P)]^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \sigma_{ij} = \sum_{i=1}^{N} x_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \rho_{ij} \sigma_i \sigma_j$$

$$\sigma_P^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + (1 - X_1)^2 \sigma_2^2 + 2X_1 (1 - X_1) \rho \sigma_1 \sigma_2 \tag{3.3}$$

4 The Portfolio Possibilities Curve (PPC) for Two Security Portfolio

We define the portfolio possibilities curve (PPC) as the locus of a point in risk-return space that identifies an admissible portfolio. For a two security portfolio with composition vector $X = \{X_1, 1 - X_1\}$, we have, from eqs. (3.2), (3.3) with $\rho = \sigma_1 \sigma_2^{-1}$

$$E(R_P) = X_1 E(R_1) + (1 - X_1) E(R_2) \tag{4.1}$$

$$\sigma_P^2 = X_1^2 \sigma_1^2 + (1 - X_1)^2 \sigma_2^2 + 2X_1 (1 - X_1) \rho \sigma_1 \sigma_2 \tag{4.2}$$

Eliminating $X_1$ between eqs. (4.1) & (4.2), we obtain the equation for the PPC for the two security case as:

$$x^2 - y^2 \frac{(\sigma_1^2 + \sigma_2^2 - 2\rho \sigma_1 \sigma_2)}{(R_1 - R_2)^2} + 2y \frac{[R_2 \sigma_1^2 + R_1 \sigma_2^2 - (R_1 + R_2) \rho \sigma_1 \sigma_2]}{(R_1 - R_2)^2} - \frac{(R_2^2 \sigma_1^2 + R_2^2 \sigma_2^2 - 2R_1 R_2 \rho \sigma_1 \sigma_2)}{(R_1 - R_2)^2} = 0 \tag{4.3}$$

where we have abbreviated $E(R_P) \equiv y, \sigma_P = x, E(R_1) \equiv R_1, E(R_2) \equiv R_2$.

Comparing eq. (4.3) with the general equation of a conic viz.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{4.4}$$

we obtain $a = 1, h = 0, b = -\frac{(\sigma_1^2 + \sigma_2^2 - 2\rho \sigma_1 \sigma_2)}{(R_1 - R_2)^2}$ whence $h^2 - ab > 0$ so that the PPC represents a hyperbola in shape.

The equation of the PPC can be written as

$$x^2 - by^2 + 2fy - c = 0 \text{ or } \frac{x^2}{E} - \frac{y^2}{\frac{f}{E}} = 1 \tag{4.5}$$

where

$$b = \frac{(\sigma_1^2 + \sigma_2^2 - 2\rho \sigma_1 \sigma_2)}{(R_1 - R_2)^2}, f = \frac{[R_2 \sigma_1^2 + R_1 \sigma_2^2 - (R_1 + R_2) \rho \sigma_1 \sigma_2]}{(R_1 - R_2)^2}, c = \frac{(R_2^2 \sigma_1^2 + R_2^2 \sigma_2^2 - 2R_1 R_2 \rho \sigma_1 \sigma_2)}{(R_1 - R_2)^2} \tag{4.6}$$
It is important at this point to note the following:
(a) \( x \equiv \sigma \) represents standard deviation of a random variable (security returns) and hence, can never be negative; 
(b) Assuming no short sales, the portfolio return \( y \equiv R_P \) must necessarily lie between \( R_1 \) & \( R_2 \) so that no point of the PPC can lie outside the region bounded by the abscissa through \( R_1 \) & \( R_2 \); 
(c) We must also have \(-1 \leq \rho \leq 1\). Let us, now, examine these two extremal cases: 

For perfectly correlated assets, \( \rho = +1 \), eq. (4.3) becomes 

\[
y = \frac{(R_1 - R_2)}{(\sigma_1 - \sigma_2)} x + \frac{(R_2 \sigma_1 - R_1 \sigma_2)}{(\sigma_1 - \sigma_2)} 
\]

which is a straight line with gradient \((\frac{R_1 - R_2}{\sigma_1 - \sigma_2})\), intercept on the ordinate axis \((\frac{R_2 \sigma_1 - R_1 \sigma_2}{\sigma_1 - \sigma_2})\) and passing through the points \( A(\sigma_1, R_1) \) & \( B(\sigma_2, R_2) \) representing the two securities in risk-return space. Hence, any portfolio of two perfectly correlated securities will lie on the straight line joining the two securities in risk-return space and the PPC, in this case, is the straight line joining these two points. The case of anti-correlated assets (\( \rho = -1 \)) is relatively more involved. The eq. of the PPC becomes 

\[
(R_1 - R_2) x = \pm [(\sigma_1 + \sigma_2) y - (R_1 \sigma_2 + R_2 \sigma_1)]
\]

Since \( x \) being standard deviation must necessarily be positive, the sign of the RHS is dictated by the sign of \((R_1 - R_2)\) so that we shall have two scenarios and hence, a pair of straight lines

(i) in the region where \( sgn (R_1 - R_2) = sgn [(\sigma_1 + \sigma_2) y - (R_1 \sigma_2 + R_2 \sigma_1)] \) the positive sign outside the square bracket will hold and the equation of the PPC in this region will be 

\[
y = \frac{(R_1 - R_2)}{(\sigma_1 + \sigma_2)} x + \frac{(R_2 \sigma_2 + R_1 \sigma_1)}{(\sigma_1 + \sigma_2)} \quad \text{(4.9)}
\]

(ii) in the region where \( sgn (R_1 - R_2) \neq sgn [(\sigma_1 + \sigma_2) y - (R_1 \sigma_2 + R_2 \sigma_1)] \), the negative sign outside the square bracket will hold and the equation of the PPC in this region will be 

\[
y = -\frac{(R_1 - R_2)}{(\sigma_1 + \sigma_2)} x + \frac{(R_2 \sigma_2 + R_1 \sigma_1)}{(\sigma_1 + \sigma_2)} \quad \text{(4.10)}
\]

In fact, by an appropriate numbering of the two securities, we can always ensure that \((R_1 - R_2) \geq 0\) whence eq. (4.9) will operate in the region where \( y \geq \frac{(R_1 \sigma_2 + R_2 \sigma_1)}{(\sigma_1 + \sigma_2)} \) or equivalently \( X_1 \geq \frac{\sigma_2}{\sigma_1 + \sigma_2} \) and eq. (4.10) in the region where \( X_1 \leq \frac{\sigma_2}{\sigma_1 + \sigma_2} \). It is pertinent to mention that the straight lines (4.9) & (4.10) intersect each other and the ordinate axis at the point \( F \left(0, \frac{(R_1 \sigma_2 + R_2 \sigma_1)}{(\sigma_1 + \sigma_2)} \right) \) which identifies the risk free rate of return. Further, eq. (4.9) is the join of the point \( F \) & \( A \) while (4.10) is the join of \( F \) & \( B \) so that for the entire range of values \( 0 \leq X_1 \leq 1 \), the risk free ordinate \( F \left(0, \frac{(R_1 \sigma_2 + R_2 \sigma_1)}{(\sigma_1 + \sigma_2)} \right) \) is unique. Needless to add, the return \( R_F = \frac{(R_1 \sigma_2 + R_2 \sigma_1)}{(\sigma_1 + \sigma_2)} \) lies between \( R_1 \), \( R_2 \).

The fallout of the observations above is that the PPC shall be confined to the section of the hyperbola lying in the first quadrant between the lines given by eqs. (4.7), (4.9) & (4.10) that, incidentally form a triangle with the vertices \( A(\sigma_1, R_1), B(\sigma_2, R_2) \) and \( F \left(0, \frac{(R_1 \sigma_2 + R_2 \sigma_1)}{(\sigma_1 + \sigma_2)} \right) \). The exact shape of the hyperbola is parameterized by the correlation coefficient between the two given securities, \( \rho \).
It is pertinent to note that for the two extremal cases \( \rho = \pm 1 \), the PPC coincides with its asymptotes.

(b) **Axes:** The axes of the PPC are respectively

\[
x = 0, y = \frac{f}{b} = \frac{R_2 \sigma_1^2 + R_1 \sigma_2^2 - (R_1 + R_2) \rho \sigma_1 \sigma_2}{\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2}
\]

(c) **Coordinates of the Centre:**

\[
x = 0, y = \frac{f}{b} = \frac{R_2 \sigma_1^2 + R_1 \sigma_2^2 - (R_1 + R_2) \rho \sigma_1 \sigma_2}{\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2}
\]

(d) **Point of Inflexion:** From eq. (4.5), we have \( \frac{dx}{dy} = \frac{by - f}{x} \). For the point of inflexion, we set \( \frac{dx}{dy} = 0 \) whence

\[
y_{\text{inflexion}} = \frac{f}{b} = \frac{R_2 \sigma_1^2 + R_1 \sigma_2^2 - (R_1 + R_2) \rho \sigma_1 \sigma_2}{\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2}
\]

The corresponding abscissa is

\[
x_{\text{inflexion}} = \pm \sqrt{(by^2 - 2fy + c)} = \pm \sqrt{b \left( \frac{f}{b} \right)^2 - 2f \left( \frac{f}{b} \right) + c} = \pm \sqrt{c - \frac{f^2}{b}}
\]

\[
= \pm \left[ \frac{(1 - \rho^2) \sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2} \right]^{1/2}
\]

We shall show in the sequel that this point of inflexion corresponds to point of minimum variance.
\[ \sigma_M = \left[ \frac{(1 - \rho^2) \sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2} \right]^{1/2} \]  
\[ R_M = \frac{R_2 \sigma_1^2 + R_1 \sigma_2^2 - (R_1 + R_2) \rho \sigma_1 \sigma_2}{\sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2} \]  
showing that the point of inflexion of the PPC coincides with \( M \). The locus of \( M \) is found by eliminating \( \rho \) between eqs. (6.2) & (6.3) and we have

\[ \frac{\left( R_1 \sigma_2^2 + R_2 \sigma_1^2 \right) - y \left( \sigma_1^2 + \sigma_2^2 \right)}{R_1 + R_2 - 2y} - x^2 = \left( x - \sigma_1^2 \right) \left( x - \sigma_2^2 \right) \]

where \( \sigma_M \equiv x, R_M = y \).

7 The PPC with Short Sales Permitted

In the event when short sales are permitted, the components of the composition vector become unbounded \(- \infty < X_1 < \infty\) with the only constraint \( X_1 + X_2 = 1 \). Hence, we can create portfolios with unbounded positive as well as negative returns (hypothetically) by short selling one or the other asset and investing the proceeds on the second asset. In such a case, the PPC gets extended beyond \( A, B \) along the same hyperbola i.e. the PPC consists of the entire section of the hyperbola that lies in the right hand side half plane bounded by the \( Y \) axis.

8 The PPC with one of Securities being Riskfree

Let the asset \( A \), renamed \( F \) be a riskfree asset so that \( \sigma_1 = \rho = 0, R_1 = R_F \) so that eq. (4.3) for the PPC becomes

\[ y = \pm \frac{R_2 - R_F}{\sigma_2} x + R_F \]  

which is a pair of straight lines that intersect each other and the \( Y \) axis at the point \( F \) \((0, R_F)\). Since \( x \), being standard deviation must necessarily be positive, we have the following situation:

(a) If \( R_2 - R_F > 0 \), then the positive sign holds in eq. (8.1) in the region where \( y - R_F > 0 \) which corresponds to \( X_F < 1 \) i.e. no short sales of the risky security \( B \). The PPC is the line segment \( FB \) terminating at the point \( B (\sigma_2, R_2) \). The negative sign shall hold in the region where \( y - R_F < 0 \) corresponding to \( X_F > 1 \) that represents short sales of \( B \) and investment of the proceeds in the riskfree asset. With the possibility of unlimited short selling of \( B \) and investment of proceeds in \( F \), the PPC in this case is the ray originating from \( F \) and extending to infinity with a slope that is the mirror image of \( FB \). If \( R_2 - R_F < 0 \), the converse will hold i.e. the negative sign holds in eq. (8.1) in the region where \( y - R_F > 0 \) and vice versa.

(b) Let short sales of the riskfree asset i.e. riskfree borrowing be permitted, so that \( X_F < 0 \) becomes admissible. With the potential possibility of unlimited riskless borrowing and investing in the risky asset, the PPC, in this case does not terminate at the point \( B (\sigma_2, R_2) \) but extends beyond \( B \) indefinitely along the straight line \( FB \).

9 The PPC with two risky securities and a riskfree security

Let \( A (\sigma_1, R_1) \) & \( B (\sigma_2, R_2) \) be two risky securities and \( F \) \((0, R_F)\) be a riskfree security.

(a) In the case when short sales is not permitted in either of the two risky securities and riskless borrowing is also not allowed, the PPC takes the form of a surface bounded by the straight line segments \( AF, BF \) and the arc of the hyperbola \( ACB \). The line segment \( AF \) will represent combinations of \( A \) and \( F \) with \( B \) being absent and \( BF \) will represent combinations of \( B \) and \( F \) with \( A \) being absent. The arc of the hyperbola \( ACB \) will represent combinations of \( A \) and \( B \) exclusively. Any line segment \( CF \) will be a combination of all the three securities \( A, B \) & \( F \) where the relative proportion of \( A \) and \( B \) shall be determined by the location of \( C \) on \( ACB \) and that of \( F \) on the position of the portfolio point on \( CF \).

It is pertinent to mention here that both straight line segments \( AF, BF \) shall intersect the closed arc \( ACB \) at no points other than \( A \) and \( B \). This follows from (i) the point \( A \) must lie on the arc \( ACB \) since this arc represents portfolios of \( A \) and \( B \) that includes the portfolio of \( A \) alone and (ii) let, if possible, \( AF \) intersect \( ACB \) at another point \( D \). Now, all points on the line segment \( AF \) must necessarily consist of only \( A \) and \( F \). However, the security represented by the point \( D \), that is common to \( ACB \) and \( AF \) shall consist of all the three securities, which is a contradiction.

(b) When short sales is permitted in \( A \) and \( B \) and riskless borrowing is not allowed, the PPC is determined as follows. We construct tangents from the point \( F \) to the arc of the hyperbola \( ACB \) extended beyond \( A \) and \( B \), if required. Let these tangents meet the extended arc of the hyperbola \( ACB \) at the points \( P \) and \( Q \). The PPC, then consists of (i) the region \( PFQ \) being bounded by the straight line segments \( PF, QF \) and the arc \( PCQ \) (ii) the points on the arc of the hyperbola beyond \( CP, CQ \) extended indefinitely. The region \( PFA \) will represent combinations of the riskfree
asset (long) with the asset A (long) and B (short) and similarly QFB will include A (short), B (long) together with the riskfree asset (long). Points within the region AFB will consist of combinations that are long in all the three securities. Points on CP, CQ extended beyond A and B respectively shall represent combinations of only A (long), B (short) and vice versa.
(c) When short sales is permitted in A and B and riskless borrowing is also allowed, the PPC is determined as in (b) above by constructing tangents from the point F to the arc of the hyperbola ACB, extended beyond A and B, if required intersecting ACB at the points P and Q. The PPC, then consists of the entire region of the positive X half plane that lies within the straight lines PF, QF extended indefinitely. In addition to the combinations explained in (b) above, points to the right of the arc PCQ shall contain riskless borrowing in addition to A and B while points lying in the region between the arc CP (extended) and FA (extended) beyond A represent combinations of riskless borrowing together with A (long) and B (short).

The coordinates of P and Q can be obtained in any of the following two ways:
(i) Let \( y = mx + R_F \) be tangent to the hyperbola (4.5) so that it intersects the hyperbola at two coincident points, the condition for which is that the quadratic equation \( x^2 - b(mx + R_F)^2 + 2f(mx + R_F) - c = 0 \) must have equal roots which gives \( m = \pm \sqrt{\frac{bR_F^2 - 2fR_F + c}{bc - f^2}} \) whence the equation of the two tangents is
\[
y = \pm x \sqrt{\frac{bR_F^2 - 2fR_F + c}{bc - f^2}} + R_F
\] (9.1)
and the coordinates of P and Q are respectively given by
\[
\left( \frac{m(bR_F - f)}{1 - bm^2}, \frac{m^2(bR_F - f)}{1 - bm^2} + R_F \right)
\] (9.2)
(ii) The second method makes use of the fact that the tangents PF, QF maximize \( \tan \theta = \frac{R_P - R_F}{\sigma_P} \). Making use of eqs. (4.1) & (4.2), we obtain
\[
\tan \theta = \frac{X_1(R_1 - R_F) + X_2(R_2 - R_F)}{[X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\rho_1\sigma_1\sigma_2]^{1/2}}
\] (9.3)
Taking partial derivatives, with respect to \( X_1, X_2 \) and equating them to zero, writing \( \frac{R_P - R_F}{\sigma_P} = \lambda, Z_k = \lambda X_k, Z_1 + Z_2 = \lambda \), we obtain the following eqs. for the composition vector:
\[
R_1 - R_F = Z_1\sigma_1^2 + Z_2\rho_1\sigma_2
\] (9.4)
\[
R_2 - R_F = Z_2\sigma_1^2 + Z_1\rho_1\sigma_2
\] (9.5)
which can be solved to obtain the composition vector \( X \) whence we can obtain the coordinates of P and Q.

10 Concept of “Efficient Frontier”
To introduce the concept, we consider, first, the case of “no” short sales. Let \( x = k \) be any line \( \parallel Y \)-axis. Its intercepts with the PPC of eq. (4.3) are obtained by solving...
\[ k^2 - \frac{y^2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} + 2y \frac{R_2\sigma_1^2 + R_1\sigma_2^2 - (R_1 + R_2)\rho\sigma_1\sigma_2}{(R_1 - R_2)^2} \]

This is a quadratic in \( y \). For equal roots, we must have,

\[ k^2 = \frac{(1 - \rho^2)\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} = \sigma_M^2 \]

showing that there is only one point such that the straight line \( Y \)-axis is tangent to the PPC. Incidentally, this point coincides with the minimum variance point \( M \) and \( A \) (assuming \( R_1 > R_2 \)) superordinates over the segment of the PPC lying between \( M \) and \( B \) in the sense that corresponding to every point on \( MB \) there exists a point on \( M A \) that provides a higher return for the same level of risk. Thus, the portion of the arc \( MA \) dominates over the portion \( MB \) and, hence, is called the “efficient frontier”. The “efficient frontier” corresponding to various scenarios discussed above is tabulated here:

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Efficient Frontier</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two risky assets, no short sales, no riskfree asset</td>
<td>The arc of the hyperbola lying between the minimum variance point ( M ) and ( A ) (assuming ( R_1 &gt; R_2 ))</td>
</tr>
<tr>
<td>Short sales allowed, no risky asset</td>
<td>The arc of the hyperbola from the minimum variance point ( M ) and extending through ( A ) (assuming ( R_1 &gt; R_2 )) indefinitely</td>
</tr>
<tr>
<td>One risky asset with riskless lending</td>
<td>The straight line joining the riskfree asset and the risky asset in risk-return space</td>
</tr>
<tr>
<td>One risky asset and riskless lending &amp; borrowing</td>
<td>The straight line extending from the riskfree asset through the risky asset to infinity in risk-return space</td>
</tr>
<tr>
<td>Two risky assets, no short sales, riskfree lending</td>
<td>The straight line segment joining the riskfree asset to the risky asset with higher return in risk-return space</td>
</tr>
<tr>
<td>Two risky assets, short sales allowed, riskless lending</td>
<td>The straight line segment from the riskfree asset and tangent, with positive slope, to the arc of the hyperbola up to the point of contact. From that point on, the arc of the hyperbola that represents combinations of the two risky assets only.</td>
</tr>
<tr>
<td>Two risky assets, short sales allowed, riskless lending &amp; borrowing allowed</td>
<td>The straight line segment from the riskfree asset and tangent, with positive slope, to the arc of the hyperbola extended indefinitely.</td>
</tr>
</tbody>
</table>

11 The Case of Three Risky Securities

In the case of two risky securities, the problem of tracing out the PPC is relatively simple because of its immediate compatibility with the two dimensional framework. However, an analysis of the three securities PPC elucidates some intriguing features of the portfolio optimization problem. We shall illustrate these features by means of an example to avoid getting lost in a plethora of calculations.

For the purpose, we consider three risky securities \( A \) (6, 14), \( B \) (3, 8) and \( C \) (15, 20) with \( \rho_{AB} = 0.50, \rho_{BC} = 0.40 \) and \( \rho_{CA} = 0.20 \) with the composition vector \( X = \{X_1, X_2, X_3\} \equiv \{1 - X_2 - X_3, X_2, X_3\} \). The equation of the PPC is obtained e.g. in terms of \( x \equiv \sigma_p, y \equiv R_P \) and \( z \equiv \sigma_X \) by eliminating \( X_2 \) between the equations for expected return and standard deviation given by eqs. (3.2) & (3.3) and we obtain

\[ x^2 - \frac{3}{4}y^2 - 306z^2 + 12y - 162z + 18yz - 57 = 0 \] (11.1)

It is easily seen that the projection of the above curve on each of the three planes is a hyperbola. However, we need to focus on the \( XY \) plane. We can write eq. (11.1) as
The above characteristics reveal that the projection of the PPC on the \( XY \) plane shall consist of a family of hyperbola \( \{H_z\} \) with each hyperbola corresponding to a value of \( z = X_3 \). The centre of the hyperbola moves up along the \( Y \) axis as more of the security \( C (15, 20) \) is inducted into the portfolio and the point of inflexion also moves away from the abscissa as well as the ordinate axes showing that the minimum variance portfolio increases both in terms of the expected return and variance. Hence, the portfolio optimization problem, in essence, boils down to (i) identifying that hyperbola out of the family \( \{H_z\} \) which is such that the value of \( \tan \theta = \frac{R_F - R_i}{\sigma_F} \), i.e. slope of the tangent drawn from the riskfree asset to the hyperbola is maximum. Let this hyperbola be \( H_{\alpha} \); (ii) once the hyperbola is identified, to obtain the coordinates of the point of contact of the tangent (that has the maximum slope) with the hyperbola \( H_{\alpha} \) (to which it is tangent). The efficient frontier then simply becomes the straight line joining the riskfree asset with the point of contact.

The procedure is purely an extension of the one set out in Section 7(c)(ii). Since a generalization to the \( N \) securities is absolutely straightforward, we set out the procedure for the latter, in view of its practical importance. Setting \( \tan \theta = \frac{R_F - R_i}{\sigma_F} \), making use of eqs. (4.1) & (4.2), we obtain

\[
\tan \theta = \frac{\sum_{i=1}^{N} X_i (R_i - R_F)}{\sqrt{\sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} X_i X_j \sigma_{ij}}}
\]

Taking partial derivatives, with respect to each \( X_i \) and equating them to zero, writing \( \frac{R_F - R_i}{\sigma_F} = \lambda, Z_k = \lambda X_k, \) \( \sum_{i=1}^{N} Z_i = \lambda \), we obtain the following eqs. for the composition vector:

\[
R_i - R_F = Z_i \sigma_i^2 + \sum_{j=1, j \neq i}^{N} Z_j \sigma_{ij}, i = 1, 2, 3, ..., N
\]

Thus, we get a set of \( N \) equations for an equal number of unknowns, being the components of the composition vector \( X = \{X_i, i = 1, 2, 3, ..., N\} \) which would, in the normal course, have a unique solution corresponding to the point of contact of the tangent to the hyperbola \( H_{\alpha} \), identified as above. Knowing the composition vector, it is rudimentary to calculate the corresponding coordinates in risk-return space. The point so obtained would be the point of contact of the tangent of greatest slope with the hyperbola \( H \).

The efficient frontier, in that case is the arc of the hyperbola \( H \), extending from the point of minimum variance on \( H \), upwards to infinity. To trace out the arc, we may follow the following:

(i) Let the point of contact of the maximum slope tangent with the hyperbola \( H \), be designated \( P \) and let the corresponding composition vector be \( X_P \). Taking a different riskfree rate, say, \( R_{F}' \) and solving the set of equations (11.8) corresponding to \( R_{F}' \) we obtain \( Q \) (with composition vector \( X_Q \)) that is the point of contact of the tangent with maximum slope with one of the hyperbola of the family \( H \). Now, it turns out that the hyperbola \( H \) is the optimal hyperbola of the family \( H \) for all the riskfree rates \( R_F \) so that the point \( Q \) also lies on \( H \).
(ii) Let us construct a third portfolio $R$ that is the average of both $P$ & $Q$ so that $X_R = 0.50 (X_P + X_Q)$. Knowing the composition vector $X_R$, we can straight away calculate the standard deviation $\sigma_R$ using eq. (3.3) e.g.

$$
\sigma_R^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} X_i^R X_j^R \sigma_{ij} = 0.25 \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ (X_i^P + X_i^Q) (X_j^P + X_j^Q) \right] \sigma_{ij}
$$

(11.9)

(iii) Treating $P$ & $Q$ as separate securities and knowing their respective variances, we can express the variance of $R$ as

$$
\sigma_R^2 = 0.25 (\sigma_P^2 + \sigma_Q^2 + 2 \rho_{PQ} \sigma_P \sigma_Q)
$$

(11.10)

where $\rho_{PQ}$ is, as yet, unknown. However, by equating the two expressions for $\sigma_R$ given by eqs. (11.9) & (11.10), we can obtain $\rho_{PQ}$ whence the problem of tracing the efficient frontier (i.e. the extended arc of the hyperbola $PQ$ from the point of minimum variance) gets reduced to a two-security problem and can be easily solved as given in the earlier sections of this work. It may be noted that all the relevant parameters of both $P$ & $Q$ viz. their variances, covariance and expected returns are known quantities now.

References

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