Palestine Journal of Mathematics

Vol. 1(2) (2012) , 95–103

FRACTIONAL CALCULUS APPROACH IN THE STUDY OF INSTABILITY PHENOMENON IN FLUID DYNAMICS

J. C. Prajapati, A. D. Patel, K. N. Pathak and A. K. Shukla

Communicated by Jose Luis Lopez-Bonilla

MSC2010 Classifications: 76S05, 35R11, 33E12 .

Keywords: Fluid flow through porous media, Laplace transforms, Fourier sine transform, Mittag - Leffler function, Fox-Wright function, Fractional time derivative.

Authors are indeed extremely grateful to the referees for valuable suggestions which have helped us improve the paper.

Abstract. The work carried out in this paper is an interdisciplinary study of Fractional Calculus and Fluid Mechanics i.e. work based on Mathematical Physics. The aim of this paper is to generalize the instability phenomenon in fluid flow through porous media with mean capillary pressure by transforming the problem into Fractional partial differential equation and solving it by using Fractional Calculus and Special functions.

1 Introduction and Preliminaries

The subject of fractional calculus deals with the investigations of integrals and derivatives of any arbitrary real or complex order, which unify and extend the notions of integer-order derivative and n-fold integral. It has gained importance and popularity during the last four decades or so, mainly due to its vast potential of demonstrated applications in various seemingly diversified fields of science and engineering, such as fluid flow, rheology, diffusion, relaxation, oscillation, anomalous diffusion, reaction-diffusion, turbulence, diffusive transport akin to diffusion, electric networks, polymer physics, chemical physics, electrochemistry of corrosion, relaxation processes in complex systems, propagation of seismic waves, dynamical processes in self-similar and porous structures and others. The importance of this subject further lies in the fact that during the last three decades, three international conferences dedicated exclusively to fractional calculus and its applications were held in the University of New Haven in 1974, University of Glasgow, Scotland in 1984, and the third in Nihon University in Tokyo, Japan in 1989 in which various workers presented their investigations dealing with the theory and applications of fractional calculus Free shear flows are inhomogeneous flows with mean velocity gradients that develop in the absence of boundaries. Turbulence free shear flows are commonly found in natural and engineering environments. The jet of air issuing from one's nostrils or mouth upon exhaling, the turbulent plume from a smoldering cigarette, and the buoyant jet issuing from an erupting volcano - all illustrate both the omnipresence of free turbulent shear flows and the range of scales of such flows in the natural environment. Examples of the multitude of engineering free shear flows are the wakes behind moving bodies and the exhausts from jet engines. Most combustion processes and many mixing processes involve turbulent free shear flows. Free shear flows in the real world are most often turbulent. The tendency of free shear flows to become and remain turbulent can be greatly modified by the presence of density gradients in the flow, especially if gravitational effects are also important. Free share flows deals with incompressible constant-density flows away from walls, which include shear layers, jets and wakes behind bodies. Hydrodynamic stability is of fundamental importance in fluid dynamics and is a well-established subject of scientific investigation that continues to attract great interest of the fluid mechanics community. Hydrodynamic instabilities of prototypical character are, for example, the Rayleigh-Bènard, the Taylor-Couette, the Bènard-Marangoni, the Rayleigh-Taylor, and the Kelvin-Helmholtz instabilities. Modeling of various instability mechanisms in biological and biomedical systems is currently a very active and rapidly developing area of research with important biotechnological and medical applications (biofilm engineering, wound healing, etc.). The understanding of breaking symmetry in hemodynamics could have important consequences for vascular biology and diseases and its implication for vascular interventions (grafting, stenting, etc.). When in a porous medium filled with one fluid and another fluid is injected which is immiscible in nature in ordinary condition, then instability occurs in the flow depending upon viscosity difference in two flowing phases. When a fluid flow through porous medium displaced by another fluid of lesser viscosity then instead of regular displacement of whole front protuberance take place which shoot through the porous medium at a relatively high speed. This phenomenon is called fingering phenomenon (or instability phenomenon). Many researchers have studied this phenomenon with different point of view.

Fractional calculus is now considered as a practical technique in many branches of science including physics (Oldham and Spainier [13]). A growing number of works in science and engineering deal with dynamical system described by fractional order equations that involve derivatives and integrals of non-integer order (Benson *et al.* [2], Metzler and Klafter [9], Zaslavsky [23]). These new models are more adequate than the previously used integer order models, because fractional order derivatives and integrals describe the memory and hereditary properties of different substances (Poddulony [14]). This is the most significant advantage of the fractional order models in comparison with integer order models, in which such effects are neglected. In the context of flow in porous media, fractional space derivatives model large motions through highly conductive layers or fractures, while fractional time derivatives describe particles that remain motionless for extended period of time (Meerscheart *et al.* [8]).

In the recent years, the fluid flow through porous media has become highly emerging area of research for the enhanced recovery of crude oil from the pores of reservoir rock. The phenomenon of instability in polyphasic flow is playing very important role in the study of fluid flow through porous media in two ways *viz.* with capillary pressure and without capillary pressure. The statistical view point was studied by Scheidegger and Johnson [18], Bhathawala and Shama Parveen [3] considering instability phenomenon in porous media without mean Capillary pressure. Verma [21] has also studied the behavior of instability in a displacement process through heterogeneous porous media and existence and uniqueness of solution of the problem was discussed by Atkinson and Peletier [1]. El-Shahed and Salem [11, 12] have used the fractional calculus approach in fluid dynamics, which has been described by fractional partial differential equation and the exact solution of these equations have been obtained by using the discrete Laplace transform, Fourier transform and some well-known Special functions.

Flow in a porous medium is described by Darcy's Law (El-Shahed and Salem [11]) which relates the movement of fluid to the pressure gradients acting on a parcel of fluid. Darcy's Law is based on a series of experiments by Henry Darcy in the mid-19th century showing that the flow through a porous medium is linearly proportional to the applied pressure gradient and inversely proportional to the viscosity of the fluid. In one dimension, q represents "mass flow rate by unit area" and is defined as,

$$q = -\frac{K}{\delta}\frac{dP}{dx},$$

where K is permeability, a parameter intrinsic to the porous network. The unit of permeability K is $\frac{\mathrm{m}}{\mathrm{s}}$. δ is the kinematics viscosity has dimension L^2T^{-1}, e.g., $\mathrm{cm}^2\ \mathrm{sec}^{-1}$ and $\frac{dP}{dx}$ is the non-hydrostatic part of pressure gradient has dimension $ML^{-2}T^{-2}$ e.g., g cm^{-2} ec^{-2}. Thus the mass flow rate by unit area (q) has dimension $\frac{cm^2}{cm^2s^{-1}}\frac{g}{cm^2s^2} = \frac{g}{cm^2s}$. Here, we considered homogeneous dimensions but in fractional calculus dimensions are inhomogeneous.

Physical Interpretation of Fractional calculus:

Heymans Nicole and Podlubny Igor [6], Podlubny Igor [14] and [15] have discussed the physical interpretation of the Riemann-Liouville fractional differentiation and integration and proposed it in terms of inhomogeneous and changing (non-static, dynamic) time scale. The contributions of Barrows and Newton to the development of mathematics and physics in the XVII century which led to the appearance of the "mathematical time", which is postulated to "flow equably" and which is usually depicted as a semi-infinite straight line.

Newton himself postulated:

"Absolute, true and mathematical time of it self, and from its own nature, flows equably without relation to anything external."

Such a postulate was absolutely necessary for developing Newton's differential calculus and applying it to problems of mechanics.

The outstanding mathematical achievement associated with the geometrization of time was, of course, the invention of the calculus of fluxions by Newton. Mathematically, Newton seems to have found support for his belief in absolute time by the need, in principle, for an ideal rate-measurer. The invention of differential and integral calculus and today's use of them is the strongest reason for continuing using homogeneous equably flowing time.

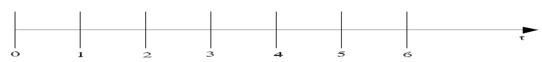

Figure 1. Homogeneous time axis

Time is often depicted using the time axis, and the geometrically equal intervals of the time axis are considered as corresponding to equal time intervals (Figure 1).

G. Clemence wrote:

“The measurement of time is essentially a process of counting. Any recurring phenomenon whatever, the occurrences of which can be counted, is in fact a measure of time.”

Clocks, including atomic clocks, repeat their “ticks”, and we simply count those ticks, calling them hours, minutes, seconds, milliseconds, etc. But we are not able to verify if the *absolute* time which elapsed between, say, the fifth and the sixth tick (the sixth “second”) is exactly the same as the time, which elapsed between the sixth and the seventh tick (the seventh “second”). The possible inhomogeneity of the time scale is illustrated in Figure 2.

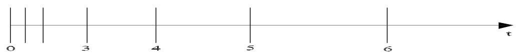

Figure 2. Homogeneous time axis

The following well-known facts are considered for studying the instability phenomenon in fluid flow through Porous Media by employing Fractional Calculus approach.

The *Laplace Transform* (Sneddon [19])is defined as,

$$L\{f(x)\} = \int_0^\infty e^{-st} f(t)\, dt \quad (Re\,(s) > 0)\,. \tag{1.1}$$

The *Fourier sine transform* (Debnath [5]) is defined as,

$$u\,(n,t) = \frac{2}{\sqrt{\pi}} \int_0^\infty u\,(x,t)\ \sin nx\ dx.. \tag{1.2}$$

The Error Function (Rainville [16]) of x is defined as

$$erf(x) = \frac{2}{\pi} \int_0^x \exp(-t^2)\, dt \tag{1.3}$$

and *The complimentary error function* of x is defined as

$$erf_c(x) = \frac{2}{\pi} \int_x^\infty \exp(-t^2)\, dt \tag{1.4}$$

In 1903, the Swedish mathematician Gosta Mittag-Leffler [10] introduced the function $E_\alpha(z)$ defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\,(\alpha\, n + 1)}, \tag{1.5}$$

where z is a complex variable and $\Gamma(s)$ is a gamma function, α. The Mittag–Leffler function is direct generalization of the exponential function to which it reduces for $\alpha = 1$. For $0 < \alpha < 1$, $E_\alpha(z)$ interpolates between the pure exponential and a hypergeometric function $\frac{1}{1-z}$. Its importance is realized during the last two decades due to its involvement in the problems of physics, chemistry, biology, engineering and applied sciences. Mittag–Leffler function naturally occurs as the solution of fractional order differential equation or fractional order integral equations. The generalization of $E_\alpha(z)$ was studied by Wiman [22] in 1905 and defined the function as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\,(\alpha\, n + \beta)}, (\alpha, \beta \in \mathbb{C};\ Re\,(\alpha) > 0,\ Re\,(\beta) > 0) \tag{1.6}$$

which is known as Wiman's function or generalized Mittag–Leffler function as $E_{\alpha,1}(z)$=$E_{\alpha}(z)$.
The Laplace transform of (1.6) takes in the form (Shukla and Prajapati [20])

$$\int_0^\infty e^{-st}\, t^{\alpha j+\beta-1}\, E^{(j)}_{\alpha,\beta}(x\,t^{\alpha})\; dt \;=\; \frac{j\,!\; s^{\alpha-\beta}}{(s^{\alpha}-x)^{j+1}} \tag{1.7}$$

where $E^{(j)}_{\alpha,\beta}(z) \;=\; \dfrac{d^j}{dz^j}\, E_{\alpha,\beta}\,(z)$.
The *Fox-Wright function* (Craven and Csordas [4]) is defined as,

$${}_p\Psi_q(x) = \sum_{k=0}^{\infty} \frac{\prod\limits_{j=1}^{p}\Gamma(a_j k + b_j)}{\prod\limits_{j=1}^{q}\Gamma(c_j k + d_k)}\; \frac{x^k}{k\,!}, \tag{1.8}$$

where $\Gamma(x)$ denotes the Gamma function and p and q are nonnegative integers. If we set $b_j = 1$ $(j = 1, 2, 3, ..., p)$ and $d_j = 1$ $(j = 1, 2, 3, ..., q)$ then (1.8) reduces to the familiar generalized hypergeometric function (Craven and Csordas [4])

$${}_pF_q(a_1, ..., a_p\,; b_1, ..., b_q\,; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k\,...(a_p)_k}{(b_1)_k\,...(b_p)_k}\; \frac{x^k}{k\,!}. \tag{1.9}$$

For the study of generalized Navier - Stokes equations, El - Shahed and Salem [12] used the very special case of (1.1), given as

$$w(\alpha,\, \beta;\, z) = \sum_{j=0}^{\infty} \frac{z^j}{j\,!\;\Gamma\,(\alpha\, j+\beta)}. \tag{1.10}$$

The Laplace Transforms of (1.10) is given by

$$\int_0^\infty e^{-st}\, w\,(\alpha,\, \beta;\, t)\, dt = \frac{1}{s}\; E_{\alpha\,,\beta}\left(\frac{1}{s}\right). \tag{1.11}$$

The relationship between the Wright function and the Complementary Error function is given as,

$$w\left(-\frac{1}{2},\, 1;\, z\right) = erf_c\, \left({}^{z}\!/_{2}\right). \tag{1.12}$$

Riemann-Liouville fractional integrals of order μ (Khan and Abukhammash [7])
Let $f(x) \in L(a,b)$, $\mu \in \mathbb{C}$ $\quad (Re(\mu) > 0)$ then

$${}_aI^{\mu}_x\, f(x) = I^{\mu}_{a+}\, f(x) = \frac{1}{\Gamma(\mu)} \int_a^x \frac{f(t)}{(x-t)^{1-\mu}} dt (x > a) \tag{1.13}$$

is called R-L left-sided fractional integral of order μ.
Let $f(x) \in L(a,b)$, $\mu \in \mathbb{C}$ $\quad (Re(\mu) > 0)$ then

$${}_xI^{\mu}_b\, f(x) = \frac{1}{\Gamma(\mu)} \int_x^b \frac{f(t)}{(t-x)^{1-\mu}} dt (x < b) \tag{1.14}$$

is called R-L right-sided fractional integral of order μ.
The Laplace Transform of the fractional derivative is given by (El - Shahed and Salem [12])

$$\int_0^\infty e^{-st}\, D^{\alpha}_t f\,(t)\, dt \;=\; s^{\alpha}\, f(s) \;-\; \sum_{j=0}^{n-1} s^{\alpha-j-1}\, D^{\alpha}_t f(0) (n-1 < \alpha < n) \tag{1.15}$$

Theorem (Asymptotic expansion of Wiman function $E_{\alpha,\beta}(z)$): Let $0 < \alpha < 1$ and β be an arbitrary complex number then

$$E_{\alpha,\beta}(z) = \frac{1}{2\alpha\pi\, i} \int \frac{\exp\left(\xi^{\frac{1}{\alpha}}\right)\, \xi^{\frac{1-\beta}{\alpha}}}{\xi - z}\, d\xi. \tag{1.16}$$

We also use following integral (El - Shahed and Salem [12]) in terms of Wright function as,

$$\int_0^\infty n\, \sin\, nx\, E_{\alpha,\, \alpha+1}\, (-n^2 C\, t^{\alpha})\, dn = \frac{\pi}{2C\, t^{\alpha}}\;\; W\left(\frac{-\alpha}{2},\; 1\,;\, \frac{-x}{\sqrt{C\; t^{\alpha}}}\right). \tag{1.17}$$

2 Statement of the Problem

When water is injected into oil saturated porous medium, as a result perturbation (instability) occurs and develops the finger flow (Scheidegger and Johnson [18]). In this paper, our aim is to study one dimensional flow, x-indicating the direction of fluid flow with the origin at the surface, due to presence of large quantity of water at $x = 0$. We assume that water saturation at $x = 0$ is almost equal to one i.e.1 and water saturation remain constant during the displacement process. Our particular interest in this paper is to explore the possibilities of transforming the problem in form of fractional partial differential equation with appropriate initial and boundary conditions.

3 Formation of the Problem

The seepage velocity of water (V_w) and oil (δ_w) are given by Darcy's law [17] as

$$V_w = -\frac{K_w}{\delta_w}\, K\, \frac{\partial P_w}{\partial x} \tag{3.1}$$

$$V_o = -\frac{K_o}{\delta_o}\, K\, \frac{\partial P_o}{\partial x} \tag{3.2}$$

and equation of continuity

$$\phi\, \frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \tag{3.3}$$

$$\phi\, \frac{\partial S_o}{\partial t} + \frac{\partial V_o}{\partial x} = 0 \tag{3.4}$$

where K is the permeability of the homogeneous medium, K_w and K_o are the relative permeability of the water and oil, S_w and S_o are saturation of water and oil respectively, P_w and P_o are the pressure in water and oil, phases δ_w and δ_o are the kinematics viscosities of water and oil respectively and ϕ is the porosity of medium.

For inhomogeneous dimensions, considering the Time-fractional partial differential equations of the two phases as under:

$$\phi\, \frac{\partial^\alpha S_w}{\partial t^\alpha} + \frac{\partial V_w}{\partial x} = 0, \qquad (0 < \alpha < 1) \tag{3.5}$$

$$\phi\, \frac{\partial^\alpha S_o}{\partial t^\alpha} + \frac{\partial V_o}{\partial x} = 0, \qquad (0 < \alpha < 1)\,. \tag{3.6}$$

For $\alpha = 1$, equations (3.5) and (3.6) reduce to equations of continuity (3.3) and (3.4) respectively, and from the definition of phase saturation [17], we have

$$S_w + S_o = 1 \tag{3.7}$$

The capillary pressure P_C is defined as pressure discontinuity between the flowing phases across their common interface and assume the function of the phase saturation is a continuous linear functional relation as

$$P_c = \beta\, S_w \tag{3.8}$$

$$P_c = P_o - P_w, \tag{3.9}$$

where β is constant.
Relationship between phase saturation and relative permeability [18] is given by

$$\left.\begin{aligned} K_w &= S_w \\ K_0 &= 1 - S_w \\ &= S_0 \end{aligned}\right\} \tag{3.10}$$

4 Formation of Fractional partial differential equation

By substituting the values of V_w and V_0(from (3.1) and (3.2)) in (3.5) and (3.6) respectively, we get

$$\phi\, \frac{\partial^\alpha S_w}{\partial t^\alpha} = \frac{\partial}{\partial x}\left\{\frac{K_w}{\delta_w}\, K\, \frac{\partial P_w}{\partial x}\right\} \quad (0 < \alpha < 1) \tag{4.1}$$

$$\phi\, \frac{\partial^\alpha S_o}{\partial t^\alpha} = \frac{\partial}{\partial x}\left\{\frac{K_o}{\delta_o}\, K\, \frac{\partial P_o}{\partial x}\right\} \quad (0 < \alpha < 1)\,. \tag{4.2}$$

Eliminating $\frac{\partial P_w}{\partial x}$ from (4.1) and (3.9),

$$\phi\, \frac{\partial^\alpha S_w}{\partial t^\alpha} = \frac{\partial}{\partial x}\left\{K\, \frac{K_w}{\delta_w}\left(\frac{\partial P_o}{\partial x} - \frac{\partial P_c}{\partial x}\right)\right\}. \tag{4.3}$$

From (4.2), (4.3) and (3.7) we obtained,

$$\frac{\partial}{\partial x}\left[K\left\{\frac{K_w}{\delta_w}+\frac{K_o}{\delta_o}\right\}\frac{\partial P_o}{\partial x}-\frac{K_w}{\delta_w}K\frac{\partial P_c}{\partial x}\right]=0. \tag{4.4}$$

Integrating (4.4) with respect to x, we get

$$K\left\{\frac{K_w}{\delta_w}+\frac{K_o}{\delta_o}\right\}\frac{\partial P_o}{\partial x}-\frac{K_w}{\delta_w}K\frac{\partial P_c}{\partial x}=-B, \tag{4.5}$$

where B is the constant of integration, whose value can be determined.
Equation (4.5) can be written as

$$\frac{\partial P_o}{\partial x}=\frac{-B}{K\frac{K_w}{\delta_w}\left\{1+\frac{K_o}{K_w}\frac{\delta_w}{\delta_o}\right\}}+\frac{\frac{\partial P_c}{\partial x}}{1+\frac{K_o}{K_w}\frac{\delta_w}{\delta_o}}. \tag{4.6}$$

Substituting the value of $\frac{\partial P_o}{\partial x}$ from (4.6) in (4.3), we arrived at

$$\phi\frac{\partial^\alpha S_w}{\partial t^\alpha}+\frac{\partial}{\partial x}\left[\frac{K\frac{K_o}{\delta_o}\frac{\partial P_c}{\partial x}}{1+\frac{K_o}{K_w}\frac{\delta_w}{\delta_o}}+\frac{B}{1+\left\{\frac{K_o}{K_w}\frac{\delta_w}{\delta_o}\right\}}\right]=0. \tag{4.7}$$

Pressure of oil (P_o) can be written as,

$$P_o=\frac{1}{2}(P_o+P_w)+\frac{1}{2}(P_o-P_w)=\overline{P}+\frac{1}{2}P_c. \tag{4.8}$$

where $\bar{P}$ is the mean pressure, which is constant.
From (4.5) and (4.8) we get,

$$B=\frac{K}{2}\left\{\frac{K_w}{\delta_w}-\frac{K_o}{\delta_o}\right\}\frac{\partial P_c}{\partial x}. \tag{4.9}$$

Substituting (4.9) in (4.7), we get

$$\phi\frac{\partial^\alpha S_w}{\partial t^\alpha}+\frac{1}{2}\frac{\partial}{\partial x}\left\{K\frac{K_w}{\delta_w}\frac{\partial P_c}{\partial S_w}\frac{\partial S_w}{\partial x}\right\}=0,\quad (0<\alpha<1) \tag{4.10}$$

taking $K\frac{K_w}{\delta_w}\frac{\partial P_c}{\partial S_w}=-\lambda$ then (4.10) reduces in the form,

$$\frac{\partial^2 S_w}{\partial x^2}=\frac{1}{C}\frac{\partial^\alpha S_w}{\partial t^\alpha}, \tag{4.11}$$

where $C=\frac{\lambda}{2P}$.

Equation (4.11) is the desired fractional partial differential equation of motion for water saturation, which governed by the flow of two immiscible phases in a homogenous porous medium and appropriate initial and boundary conditions are associated with the description as to

$$\left.\begin{aligned}&S_w(x,0)=0,\\&S_w(0,T)=S_{w_0}<1\,,\\&\lim_{x\to\infty}S_w(x,T)=0\,;\ 0<x<\infty\end{aligned}\right\} \tag{4.12}$$

5 Solution of Problem

From (4.11), we have

$$\frac{\partial^\alpha S_w}{\partial t^\alpha}=C\frac{\partial^2 S_w}{\partial x^2}. \tag{5.1}$$

Applying Fourier sine transform (1.2) on (5.1), yields

$$\frac{\partial^\alpha S_w(n,t)}{\partial t^\alpha}=C\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{\partial^2 S_w}{\partial x^2}\sin nx\,dx\,,$$

integrating by parts, gives

$$= C\sqrt{\frac{2}{\pi}}\left[\sin nx\,\frac{\partial S_w}{\partial x}\right]_0^{\infty} - n\,C\sqrt{\frac{2}{\pi}}\int\limits_0^{\infty}\cos nx\,\frac{\partial S_w}{\partial x}\,dx$$

$$= C\sqrt{\frac{2}{\pi}}\left[\sin nx\,\frac{\partial S_w}{\partial x}\right]_0^{\infty} - n\,C\sqrt{\frac{2}{\pi}}\left[\cos nx\,S_w\right]_0^{\infty} - n^2C\sqrt{\frac{2}{\pi}}\int\limits_0^{\infty}\sin nx\;S_w\,dx\,,$$

and applying boundary conditions (4.12), we can write

$$\frac{\partial^{\alpha}S_w(n,t)}{\partial t^{\alpha}} = C\sqrt{\frac{2}{\pi}}\,(0) - n\,C\sqrt{\frac{2}{\pi}}\,(-S_{w_0}) - n^2C\,S_w\,(n,\ t)$$

$$\frac{\partial^{\alpha}S_w(n,t)}{\partial t^{\alpha}} = n\,C\,S_{w_0}\sqrt{\frac{2}{\pi}} - n^2C\,S_w\,(n,\ t), \tag{5.2}$$

use of (1.15) and Laplace transform of (5.2), gives

$$L\left\{D^{\alpha}S_w(n,t)\right\} = L\left\{n\,C\,S_{w_0}\sqrt{\frac{2}{\pi}} - n^2C\,S_w\,(n,\ t)\right\}$$

or

$$s^{\alpha}\,S_w(n,s) - \sum_{j=0}^{n-1}s^{\alpha-j-1}\,D_t^{\alpha}\,S_w(n,0) = n\,C\,\sqrt{\frac{2}{\pi}}\,S_{w_0}\,L\{1\} - n^2C\,L\{S_w(n,\,t)\}$$

or

$$s^{\alpha}\,S_w(n,s) - 0 = n\,C\,\sqrt{\frac{2}{\pi}}\,\frac{S_{w_0}}{s} - n^2C\,S_w\,(n,\ s)$$

or

$$s^{\alpha}\,S_w(n,s) + n^2C\,S_w\,(n,\ s) = n\,C\,\sqrt{\frac{2}{\pi}}\,\frac{S_{w_0}}{s}$$

or

$$S_w\,(n,\ s) = \sqrt{\frac{2}{\pi}}\,n\,C\,S_{w_0}\,\frac{s^{-1}}{s^{\alpha}+n^2C}, \tag{5.3}$$

the inverse Laplace transform of (5.3), yields

$$S_w\,(n,\ t) = \sqrt{\frac{2}{\pi}}\,n\,C\,S_{w_0}\,L^{-1}\left\{\frac{1}{s(s^{\alpha}+n^2C)}\right\}$$

$$= \sqrt{\frac{2}{\pi}}\,n\,C\,S_{w_0}\,\frac{1}{2\pi i}\int\limits_L\frac{e^{st}}{s(s^{\alpha}+n^2C)}ds,$$

the inverse Fourier sine transform of this equation, gives

$$S_w(x,t) = \sqrt{\frac{2}{\pi}}\;C\,S_{w_0}\,\frac{1}{2\pi i}\,\sqrt{\frac{2}{\pi}}\int\limits_0^{\infty}n\left\{\int\limits_L\frac{e^{st}}{s(s^{\alpha}+n^2C)}ds\right\}\sin nx\,dn. \tag{5.4}$$

On substituting $\xi = s^{\alpha}t^{\alpha}$, $z = -n^2Ct^{\alpha}$, $\beta = 1+\alpha$ in (1.16), we get

$$E_{\alpha,\alpha+1}(-n^2Ct^{\alpha}) = \frac{1}{2\alpha\pi i}\int\limits_L\frac{e^{st}(s^{\alpha}t^{\alpha})^{-\frac{\alpha}{\alpha}}}{s^{\alpha}t^{\alpha}+n^2Ct^{\alpha}}\;t^{\alpha}\alpha\,s^{\alpha-1}\,ds,$$

or

$$E_{\alpha,\alpha+1}(-n^2Ct^{\alpha}) = \frac{1}{2\pi i}\int\limits_L\frac{e^{st}\,s^{-\alpha}t^{-\alpha}}{s^{\alpha}+n^2C}\;s^{\alpha-1}\,ds,$$

or

$$\frac{1}{2\pi i}\int\limits_L\frac{e^{st}\,s^{-\alpha}}{s^{\alpha}+n^2C}\;s^{\alpha-1}\,ds = t^{\alpha}\,E_{\alpha,\alpha+1}(-n^2Ct^{\alpha})\,, \tag{5.5}$$

equation (5.4) can also be written as

$$S_w\,(x,\,t) = \frac{2\,C\,S_{w_0}}{\pi}\,t^{\alpha}\int_0^{\infty} n\,\sin\,nx\,E_{\alpha,\,\alpha+1}\,(-n^2 C\,t^{\alpha})\,dn, \tag{5.6}$$

this is easy to write in the form of Wright function as

$$S_w\,(x,\,t) = S_{w_0}\,W\left(\frac{-\alpha}{2},\,1\,;\frac{-x}{\sqrt{C\,t^{\alpha}}}\right). \tag{5.7}$$

On setting $\alpha = 1$ and using (1.12), (5.7) reduces to,

$$S_w\,(x,\,t) = S_{w_0}\,erf_c\left(\frac{x}{2\sqrt{C\,t}}\right). \tag{5.8}$$

6 Conclusion

The fractional calculus approach in the constitutive relationship model of generalization of the instability phenomenon in fluid flow through porous media with mean capillary pressure is introduced. We have obtained the exact solution of the fractional partial differential equation in term of well-known Wright function by using Laplace transform, Fourier transform and Special functions with appropriate initial and boundary conditions. If $\alpha = 1$ then equation (5.7) reduces to (5.8), i.e. we obtained the solution in the form of complementary error function. This method certainly useful than conventional method as the conventional method derived only for $\alpha = 1$ (equations (3.3) and (3.4)) whose solution given by equation (5.8) while this fractional calculus together with Fourier and Laplace transforms method presented in this paper also applicable for $0 < \alpha < 1$ whose solution given by equation (5.7).

References

[1] Atkinson, F. V. and Peletier, L. A., Similarity profiles of flow through porous media, Arch. Rational Mech. Annual. Vol. 42, 376 – 379 (1971).

[2] Benson D. A., Wheatcraft S. W. and Meerschaert M. M., Application of a fractional advection-dispersion equation. *Water Resources Research* **36**(**6**), 1403-1412(2000).

[3] Bhathawala, P.H. and Shama Parveen, Analytic study of three Phase flow through Porous Media, J. Indian Acad. Math. Vol. 23, 17-24 (2001).

[4] Craven Thomas and Csordas George, The Fox-Wright functions and Laguerre multiplier sequences, J. Math. Anal. and Appl., Vol. 314, 109-125 (2006).

[5] Debnath, L., Integral Transforms and their Applications, CRC Press, New York-London- Tokyo, 1995.

[6] Heymans Nicole and Podlubny Igor, Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives, Rheologica Acta, Vol. 45(5), 765-771 (2006).

[7] Khan M. A. and Abukhammash G. S., A study on two variable analogues of certain fractional operators, Pro Mathematica, Vol. 17(33), 32-48 (2003).

[8] Meerscheart M. M., Benson D. A., Scheffler H. P. and Baeumer B., Stochastic Solution of space-time fractional diffusion equations. Physical Review E 65(4), 1103-1106 (2002).

[9] Metzler R. and Klafter J., The random walk's guide to anomalous diffusion: A fractional dynamic approach. Physics Reports, Vol. 339, 1-77 (2000).

[10] Mittag-Leffler G.M., Sur la nouvelle fonction $E_{\alpha}\,(x)$, C. R. Acad. Sci. Paris, Vol. 137, 554–558 (1903).

[11] El - Shahed Moustafa and Salem Ahmed, On the generalized Navier - stokes equations, J. Applied Mathematics and Computation, Vol. 156(1), 287-293 (2004).

[12] El - Shahed Moustafa and Salem Ahmed, Decay of vortex velocity and diffusion of the temperature for fractional viscoelastic fluid through porous medium. International J. of Comm. in Heat and Mass Transfer, Vol. 33, 240-248 (2006).

[13] Oldham K. B. and Spainier J., The Fractional Calculus. Academic Press, New York, (1974).

[14] Poddulony, Igor J., Fractional differential equation, Academic Press, New York, (1999).

[15] Poddulony, Igor J., Geometric and Physical Interpretation of Fractional Integration and Fractional Differentiation, Fractional Calculus and Applied analysis, Vol. 5(4), 367-386 (2002).

[16] Rainville, E.D., Special Functions, The Macmillan Company, New York, (1960).

[17] Scheidegger, A. E., The physics of flow through porous media, Univ. of Toronto press, Toronto, (1960).

[18] Scheidegger, A.E. & Johnson, E. F., The statical behavior of instabilities in displacement process in porous media, Can. J. Physics. Vol. 39 (1961).

[19] Sneddon, I. N., Fourier Transform, McGraw - Hill, New York, (1951).

[20] Shukla, A.K. and Prajapati J.C., On a generalization of Mittag–Leffler function and its properties, J. Math. Anal. and Appl., Vol. 336, 797-811 (2007).

[21] Verma, A. P., Statical behavior of fingering in displacement processes in heterogeneous porous medium with capillary pressure, Can. J. Physics, Vol. 47(3), 319-324 (1969).

[22] Wiman A., Über den fundamental Satz in der Theorie der Funktionen $E\alpha(x)$, Acta Math., Vol. 29, 191–201 (1905).

[23] Zaslavsky G. M., Choas Fractional kinetics and anomalous transport, Physics Reports 371(6), 461-580 (2002).

Author information

J. C. Prajapati, Department of Mathematical Sciences, Faculty of Applied Sciences, Charotar University of Science and Technology, Changa, Anand- 388 421, India.
E-mail: jyotindra18@rediffmail.com

A. D. Patel, Department of Mathematics and Humanities, S. V. National Institute of Technology, Surat - 395 007, India.
E-mail: ptlamit83@yahoo.com

K. N. Pathak, Department of Physics, S. V. National Institute of Technology, Surat - 395 007, India.
E-mail: knp@ashd.svnit.ac.in

A. K. Shukla, Department of Mathematics and Humanities, S. V. National Institute of Technology, Surat - 395 007, India.
E-mail: ajayshukla2@rediffmail.com

Received: November 15, 2011

Accepted: February 12, 2012