Related fixed point on two metric spaces

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Abstract. In this paper, a new fixed point theorem for two pairs of mappings on two metric spaces is proved. This result generalizes the main theorem from [2].

1 Introduction

The following related fixed point theorem was proved in [2], See also [1].

Theorem 1.1. Let \((X, d)\) and \((Y, \rho)\) be complete metric spaces, let \(T\) be a mappings of \(X\) into \(Y\), and let \(S\) be a mappings of \(Y\) into \(X\) satisfying the inequalities

\[
d(Sy, Sy')d(STx, STx') \leq c \max \{d(Sy, Sy')\rho(Tx, Tx'), d(x', Sy)\rho(y', Tx), d(x, x')d(Sy, SY)d(Sy', STx')\},
\]

\[
\rho(Tx, Tx')\rho(TSy, TSy') \leq c \max \{d(Sy, Sy')\rho(Tx, Tx'), d(x', Sy)\rho(y', Tx),
\]

\[
\rho(y, y')\rho(Tx, Tx')\rho(TSy, TSy')\}
\]

for all \(x, x'\) in \(X\) and \(y, y'\) in \(Y\), where \(0 \leq c \leq 1\). If either the mappings \(T\) or \(S\) is continuous then \(ST\) has a unique fixed point \(z\) in \(X\) and \(TS\) has a unique fixed point \(w\) in \(Y\). Further, \(Tz = w\) and \(Sw = z\).

2 Main results

We now prove the following related fixed point theorem.

Theorem 2.1. Let \((X, d)\) and \((Y, \rho)\) be complete metric spaces, let \(A, B\) be mappings of \(X\) into \(Y\), and let \(S, T\) be mappings of \(Y\) into \(X\) satisfying the inequalities

\[
d(Sy, Ty')d(SAx, TBx') \leq c \max \{d(Sy, Ty')\rho(Ax, Bx'), d(x', Sy)\rho(y', Ax),
\]

\[
d(x, x')d(Sy, SY)d(Ty', TBx')\},
\]

\[
\rho(Ax, Bx')\rho(BSy, ATy') \leq c \max \{d(Sy, Ty')\rho(Ax, Bx'), d(x', Sy)\rho(y', Ax),
\]

\[
\rho(y, y')\rho(Ax, Bx')\rho(BSy, ATy')\}
\]

for all \(x, x'\) in \(X\) and \(y, y'\) in \(Y\), where \(0 \leq c \leq 1\). If one of the mappings \(A, B, S\) and \(T\) is continuous then \(SA\) and \(TB\) have a common fixed point \(z\) in \(X\) and \(BS\) and \(AT\) have a common fixed point \(w\) in \(Y\). Further, \(Az = Bz = w\) and \(Sw = Tw = z\).

Proof. Let \(x\) be an arbitrary point in \(X\), we define the sequences \(\{x_n\}\) in \(X\) and \(\{y_n\}\) in \(Y\) by

\(Sy_{2n-1} = x_{2n-1}, Bx_{2n-1} = y_{2n}, Ty_{2n} = x_{2n}, Ax_{2n} = y_{2n+1}\)

Applying inequality (2.1), we get

\[
d(Sy_{2n-1}, Ty_{2n})d(SAx_{2n}, TBx_{2n-1}) \leq c \max \{d(Sy_{2n-1}, Ty_{2n})\rho(Ax_{2n}, Bx_{2n-1}),
\]

\[
d(x_{2n-1}, y_{2n})\rho(y_{2n}, Ax_{2n}), d(x_{2n}, x_{2n-1})d(Sy_{2n-1}, Ty_{2n}),
\]

\[
d(Sy_{2n-1}, SAx_{2n})d(Ty_{2n}, TBx_{2n-1})\}
\]

or

\[
d(Sy_{2n-1}, Ty_{2n})d(SAx_{2n}, TBx_{2n-1}) \leq c \max \{d(Sy_{2n-1}, Ty_{2n})\rho(Ax_{2n}, Bx_{2n-1}),
\]

\[
d(x_{2n-1}, y_{2n})\rho(y_{2n}, Ax_{2n}), d(x_{2n}, x_{2n-1})d(Sy_{2n-1}, Ty_{2n}),
\]

\[
d(Sy_{2n-1}, SAx_{2n})d(Ty_{2n}, TBx_{2n-1})\}
\]
from which it follows that
\[ d(x_{n+1}, x_n) \leq c \max \{d(x_{n-1}, x_n), d(x_n, x_{n-1})\}. \] (2.4)

Applying inequality (2.2), we get
\[ \rho(y_{2n}, y_{2n+1}) \rho(y_{2n}, y_{2n+1}) \leq c \max \{d(x_{n-1}, x_n), d(x_n, x_{n-1})\}, \]
\[ d(x_{n+1}, x_n) \leq c \max \{d(x_n, x_{n-1}), d(y_{n+1}, y_n)\}, \]
\[ \rho(y_{2n}, y_{2n+1}) \rho(y_{2n}, y_{2n+1}) \leq c \max \{d(x_{n-1}, x_n), d(x_n, x_{n-1})\}, \]
(2.5)

from which it follows that
\[ \rho(y_{2n}, y_{2n+1}) \leq c \max \{d(x_{n-1}, x_n), \rho(y_{n-1}, y_n)\}. \] (2.6)

It now follows from inequalities (2.3), (2.4), (2.5) and (2.6) that, for some \( n \)
\[ d(x_{n+1}, x_n) \leq c \max \{\rho(y_n, y_{n+1}), d(x_n, x_{n-1})\}, \]
\[ \rho(y_{n+1}, y_n) \leq c \max \{d(x_n, x_{n-1}), \rho(y_{n-1}, y_n)\}, \]
and easily by induction that
\[ d(x_{n+1}, x_n) \leq c^n \max \{\rho(y_1, y_2), d(x_1, x_2)\}, \]

similarly,
\[ \rho(y_{n+1}, y_n) \leq c^n \max \{d(x_1, x_2), \rho(y_1, y_2)\}, \]

for \( n=1,2,3,\ldots \). Since \( 0 \leq c < 1 \), it follows that \( \{x_n\} \) and \( \{y_n\} \) are the cauchy sequences with the limits \( z \) in \( X \) and \( w \) in \( Y \).

Now suppose that \( A \) is continuous. Then
\[ \lim A x_{2n} = Az = \lim y_{2n+1} = w \]
and so \( Az = w \).

Using inequality (2.1), we are successively obtained
\[ d(Sy_{2n-1}, Ty_{2n}) \leq c \max \{d(Sy_{2n-1}, Ty_{2n}), d(Ax_{2n-1}, Bx_{2n-1})\}, \]
\[ d(x_{n-1}, Sy_{2n}) \rho(y_{2n}, Ax_{2n}), \]
\[ d(x_{n-1}, x_{2n}) \rho(Sy_{2n-1}, Ty_{2n}), \]
\[ d(Sy_{2n-1}, Ax_{2n}) \rho(Ty_{2n}, Bx_{2n-1})\],

which implies
\[ d(SA z, TB x_{2n-1}) \leq c \max \{\rho(A z, B x_{2n-1}), \rho(y_{2n}, z), d(x_{2n-1}, x_{2n})\}, \]

Letting \( n \) approaches to infinity, we have
\[ d(Sw, z) \leq c \max \{\rho(A z, w), \rho(w, A z), 0\}. \]

Then \( Sw = z = SAz \).

Further, Applying inequality (2.2) we obtain
\[ \rho(A x_{2n}, B x_{2n-1}) \rho(B y_{2n-1}, A T y_{2n}) \leq c \max \{d(Sy_{2n-1}, Ty_{2n}) \rho(A x_{2n}, B x_{2n-1}), \]
\[ d(x_{n-1}, Sy_{2n-1}) \rho(y_{2n}, Ax_{2n}), \]
\[ \rho(y_{2n}, y_{2n-1}) \rho(A x_{2n}, B x_{2n-1}), \]
\[ \rho(A x_{2n}, B y_{2n-1}) \rho(B x_{2n-1}, AT y_{2n})\}, \]

thus,
\[
\rho(BSy_{2n}, ATy_{2n}) \leq c \max\{d(Sy_{2n}, Ty_{2n}), d(x_{2n}, Sy_{2n-1}), \\
\rho(y_{2n}, y_{2n-1}), \rho(Az, BSy_{2n-1})\},
\]

Letting \( n \) approaches to infinity, we have

\[
\rho(w, Az) \leq c \max\{d(z, Tw), d(z, Sw), 0, \rho(Az, w)\},
\]

Then \( Tw = z = TBz \).

By the symmetry, the same results again hold if one of the mappings \( B, S, T \) is continuous instead of \( A \).

To prove the uniqueness, suppose that \( TB \) and \( SA \) have a second distinct common fixed point \( z' \). Then, using inequality (2.1), we get

\[
d(Sy, Ty')d(SAz, TBz') \leq c \max\{d(Sy, Ty')\rho(Az, Bz'), d(z', Sy)\rho(y', Az), \\
d(z, z')d(Sy, Ty'), d(Sy, SAz)d(Ty', TBz')\},
\]

that is,

\[
d(z, z')d(SAz, TBz') = \left[ d(z, z') \right]^2 \leq c \max\{d(z, z')\rho(Az, Bz'), d(z', z)\rho(Bz', Az), \\
d(z, z')d(z, z')d(z', z')\},
\]

and hence

\[
d(z, z') \leq c \max\{\rho(Az, Bz'), \rho(Bz', Az), \\
d(z, z')d(z, z')d(z', z')\}.
\]

Therefore,

\[
d(z, z') \leq cd(z, z'). \quad (2.7)
\]

Further, applying inequality (2.2), we obtain

\[
\rho(Az, Bz')\rho(BSy, ATy') \leq c \max\{d(Sy, Ty')\rho(Az, Bz'), d(z', Sy)\rho(y', Az), \\
\rho(y, y')\rho(Az, Bz'), \rho(Az, BSy)\rho(Bz', ATy')\},
\]

that is,

\[
\rho(Az, Bz')\rho(BSy, ATy') = \left[ \rho(Az, Bz') \right]^2 \leq c \max\{d(Sy, Ty')\rho(Az, Bz'), d(z', Sy)\rho(Bz', Az), \\
\rho(Az, Bz')\rho(Az, Bz'), \rho(Az, BSy)\rho(Bz', ATy')\}.
\]

Therefore,

\[
\rho(Az, Bz') \leq cd(z, z'). \quad (2.8)
\]

It now follows from inequalities (2.7) and (2.8) that

\[
d(z, z') \leq cp(Az, Bz') \leq c^2d(z, z').
\]

So \( z = z' \). The uniqueness of \( w \) is proved similarly. This complete the proof of the theorem.

\( \Box \)

References


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