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Min-max Game Theory for Coupled PDE Systems - Abstract Theory with Applications to Fluid Structure Interactions

Jing Zhang

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Abstract. We present the features of a min-max game theory problem for a linear fluid-structure interaction model in both elastic and visco-elastic case, with (possibly) control and disturbance exercised at the interface between the two media. This problem will be embedded in a more general abstract framework. The sought-after saddle solutions are expressed in a pointwise feedback form, which involves a Riccati operator; that is, an operator satisfying a suitable non-standard Riccati differential equation. We shall show that the inclusion of viscoelastic effects allows for applicability of larger class of boundary controls. Motivation, application as well as a brief historical account are also provided.

1 A Brief Historical Account of Game Theory

Game Theory is “the study of mathematical models of conflict and cooperation between intelligent rational decision-makers” [M.1]. Nowadays, game theory has been extensively studied as a fast developing branch in the mathematical sciences and is applied to a wide range of disciplines, such as economics, political science, military science, biology, ecology, etc. The mathematical foundation for game theory was established by John von Neumann [N.1]. In Late 1950s, Rufus Isaacs initiated the study of differential games [I.1], a theme later taken up in [B.1] and [F.1]. It considers problems involving modeling and analysis of conflicts in the context of dynamic systems. The systems, modeled by differential equations, usually consist of two players, the good one and the bad one, with opposite goals. From the technical point of view, the differential games are generalized version to the optimal control problem. The latter has a single control and a single criterion to be optimized in the system, whereas the former has two players and two criteria, one for each player. A min-sup problem, in the context of reachability of perturbed systems, was studied by Delfour and Mitter in 1969 [D-M.1]. Two-player differential game problems on linear dynamics with quadratic utility function over finite or infinite time horizons, in the context of finite ordinary differential equations was later studied by Bernhard in 1979 [B.2]. This was followed by the well-known work of Basar and Bernhard [B-B.1] in 1991. This particular differential min-max game problem involves the following two-step “optimization problem”: an “inf” over the set of “good players” (also called control) for a fixed “bad player” (also called disturbance), followed by a “sup” over all disturbances. This algorithm then singles out “the worst of all possible best outcomes” as a design strategy. The Basar-Bernard’s work was influential in stimulating a corresponding study of min-max game theory for partial differential equations (PDEs); initially, single PDEs, either parabolic or hyperbolic type, with both control and disturbance acting on the boundary of the spatial domain (or as Dirac masses). An extensive, yet not complete list, is given in [M-T.1]-[M-T.4]. More recently, min-max game theory problems have also been studied for systems of coupled PDEs of different types: a parabolic equation coupled with a hyperbolic equation with control acting at the interface between the two respective media. The fluid-structure interaction is an illustration of this kind. On another front, some recent studies on linear quadratic differential games in the finite dimensional settings are in Delfour et al [D.1], [D-DS.1].

2 Game Theory for a Fluid-Structure Interaction Model: Literature Overview and Engineering and Biological Motivations

2.1 Literature Overview

The fluid-structure interaction model is well established in both mathematical and engineering literatures [Li.1], [D-G-H-L.1] and the applications range from naval and aerospace engineering to cell biology and biomedical engineering. numerous works on this model emerge over the last few decades. Here, we list recent results closely related to our work of min-max game problem on this model. In [A-T.1, A-T.2, A-T.3, A-T.4, A-T.5], the wellposedness and boundary stabilization of a linear fluid-structure interaction model was studied, where the fluid part was modeled by Stokes equation. In these works, the pressure in the equation was eliminated by the introduction of a suitable Green's map. In [B-G-L-T.1, B-G-L-T.2], the existence and regularity of a nonlinear fluid-structure interaction model was studied, in which a nonlinear Navier-Stokes equation was coupled with an elastic equation in three dimensional case. These work made the ground work of establishing the optimal control problem of [L-Tu.2] for a linearized fluid-structure model, where the control was placed on the interface between fluid and the structure. [T-Z.1] established the general abstract theory of min-max game problem for coupled hyperbolic/parabolic PDE systems that possess the singular estimate property (introduced in Section 3). [L-T-Z.1] adapted this abstract theory to the linearized model introduced in [L-Tu.2], where an additional disturbance was placed either in the interior of two media (the fluid and the structure) or on the interface between the media. In a more recent work of [L-T-Z.2], the visco-elastic effects to the fluid-structure interaction model was considered. It turns out that when the visco-elastic term is included in the model, the system becomes a parabolic-analytic coupling and the overall dynamics generates a strongly continuous analytic semigroup. Thus with suitable Dirichlet type of control on the interface, the overall model satisfies the singular estimates assumption in [T-Z.1].

2.2 An Example of Application in Biology and Engineering

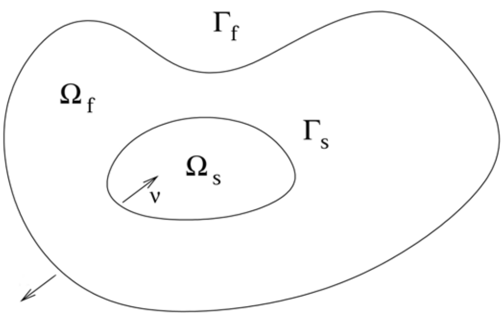
One concrete application of this model in the min-max game scenario is the measurement of the dosage of antibiotics prescribed to a patient. The research carried out in this area has raised wide attention as prescription drug dosage has become a fast growing problem recent years. It is documented that in 2008, a total of 36,450 deaths were attributed to drug overdose, and this number is still increasing. Our PDE model provides a mathematical approach for determining the optimal dosage for the antibiotics. Here the "optimal dosage" is defined as the dosage that gives the desired effect with minimum side effects. We work under the assumption that doctors always want to minimize the antibiotics dosage prescribed to patients (so as to minimize the side effects). While on the other hand, if the dosage is too low, the antibiotics can not kill the bacteria; worse still, bacteria can even get antibiotic-resistant. In this scenario, the antibiotics and the bacteria become two players in a noncooperative game. Our research targets the goal of finding the minimum antibiotics dosage that kills the worst bacteria in the patient, thus cures the bacteria infection completely. This is a typical minimax algorithm in game theory. In the PDE model, the effects of the antibiotics and the bacteria to the patient is described by the control and the disturbance respectively.

Our objective is to develop the feedback optimal dosage for the antibiotics based on the solution (the feedback operator) of a certain Riccati equation. It is known that the Riccati theory is a very powerful tool in designing and computing feedback controls in the linear quadratic regulator problems. In our study, we extend the classic Riccati equations to so called non-standard Riccati equations. The difference between a non-standard Riccati equation and the usual Riccati equation is an extra term corresponding to the present disturbance in the PDE system, which, however, occurs with a "bad" (negative) sign in front. The construction of the non-standard Riccati equation associated to the PDE model provides us with a feedback algorithm for computing the optimal antibiotic dosage for a particular patient due to his physical conditions, which can be monitored by his blood test, such as the white blood cells counts, etc; whereas our PDE model describes the blood status by the interaction between the blood cells (modeled by the equation of the system of elasticity) and the blood plasma (modeled by the linearized Navier-Stokes equation).

3 The Fluid-Structure Interaction Model

We consider the following fluid-structure interaction model which accounts for both*elastic and viscoelastic effects* :

Let $\Omega \in \mathbb{R}^n$, $n = 2, 3$ be an open bounded domain consisting of an interior region Ω_s and an exterior region Ω_f, see picture below. We denote by Γ_f the outer boundary of the domain while Γ_s is the boundary of region Ω_s which also borders the exterior region Ω_f and where the interaction of the two systems take place. Thus Γ_s is the interface between the two media. Let u be an n-dimensional vector function defined on Ω_f representing the velocity of the fluid, while the scalar function p represents the pressure. Additionally, let v and v_t be the n-dimensional displacement and velocity functions of the solid Ω_s. We also denote by ν the unit normal vector outward with respect to the domain Ω_f. The boundary-interface control is represented by $g \in L_2(0, T; (L_2(\Gamma_s))^n)$ and is active on the boundary Γ_s. For simplicity (but it also seems a physically reasonable situation), we first assume that the deterministic disturbance $w = \{w_1, w_2\}$ acts with one component w_1 in Ω_f and one component w_2 in Ω_s. We work under the assumption of small but rapid oscillations of the solid body, so that the interface Γ_s may be assumed to be static, see [D-G-H-L.1] for more modeling details. Additional reference to the non-linear model of the Navier-Stokes equations include [Li.1, p126], which, in turn, makes reference to a biological model [C-F]. The correct model of a moving structure immersed in a Navier-Stokes fluid (say a boat that drifts under the action of the fluid; not a self-propelled structure) appears to be still unresolved. A discussion leading to an arbitrary Lagrange-Euler formulation is given in [M-Z.1, equation (8.6), p219]. A more recent effort is in [B-Z.1].



Boundary Control/Disturbance Model

Given the boundary control $g \in L_2(0, T; (L_2(\Gamma_s))^n)$ at the interface, and the interior deterministic disturbance $w = \{w_1, w_2, w_3\} \in L_2(0, T; (L_2(\Omega_f))^n \times (L_2(\Omega_s))^n \times (L_2(\Gamma_s))^n)$, we consider the following coupled system of partial differential equations in the unknowns $\{u, v, v_t, p\}$,

$$\begin{cases} u_t - \Delta u + Lu + \nabla p = w_1 \quad \text{in} \quad Q_f \equiv \Omega_f \times (0,T] & (3.1a) \\ \operatorname{div} u = 0 \quad \text{in} \quad Q_f \equiv \Omega_f \times (0,T] & (3.1b) \\ v_{tt} - \operatorname{div}\sigma(v) - \rho \operatorname{div}\sigma(v_t) = w_2 \quad \text{in} \quad Q_s \equiv \Omega_s \times (0,T] & (3.1c) \\ v_t = u \quad \text{in} \quad \Sigma_s \equiv \Gamma_s \times (0,T] & (3.1d) \\ u = 0 \quad \text{in} \quad \Sigma_f \equiv \Gamma_f \times (0,T] & (3.1e) \\ \sigma(v + \rho v_t) \cdot \nu = \epsilon(u) \cdot \nu - p\nu - g + w_3 \quad \text{in} \quad \Sigma_s \equiv \Gamma_s \times (0,T] & (3.1f) \\ u(0, \cdot) = u_0 \quad \text{in} \quad \Omega_f & (3.1g) \\ v(0, \cdot) = v_0, \;\; v_t(0, \cdot) = v_1 \quad \text{in} \quad \Omega_s & (3.1h) \end{cases}$$

The elastic strain tensor ϵ and the stress tensor σ, are $n \times n$ symmetric matrices respectively given by

$$\epsilon_{ij}(u) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) = \epsilon_{ji}(u) \tag{3.2}$$

and

$$\sigma(u) = \lambda \operatorname{tr}\epsilon(u) I + 2\mu\epsilon(u) = \big(\sigma_{ij}(u)\big)_{i,j=1}^{n} \tag{3.3a}$$

$$\sigma_{ij}(u) = \lambda\Big(\sum_{k=1}^{n} \epsilon_{kk}(u)\Big)\delta_{ij} + 2\mu\epsilon_{ij}(u) = \sigma_{ji}(u) \tag{3.3b}$$

where $\lambda > 0$ and $\mu > 0$ are the Lamé constants. Obviously we have

$$|\epsilon(u)| \leq |\nabla u|; \;\; |\sigma(u)| \leq 2\max\{\lambda, 2\mu\}|\epsilon(u)| \leq 2\max\{\lambda, 2\mu\}|\nabla u| \tag{3.4}$$

The constant ρ plays an important role. When $\rho = 0$, model (3.1) is purely parabolic-hyperbolic coupling. When $\rho > 0$, the term $\rho \operatorname{div}\sigma(v_t)$ adds the "visco-elastic" effects to the structure, which is typical in modeling blood passing through the arteries [C-M-T, C-L-M-T]. And because of this "visco" term $\rho \operatorname{div}\sigma(v_t)$, the overall fluid-structure dynamics become parabolic-analytic coupling.

The term Lu is a linearization of the convective term of the Navier-Stokes term $(u \cdot \nabla)u$ and is defined as

$$Lu = (u \cdot \nabla)y_e + (y_e \cdot \nabla)u, \quad \operatorname{div} y_e = 0, \;\; y_e|_{\Gamma_f} = 0 \tag{3.5}$$

where y_e is a steady-state or equilibrium solution; that is, a time independent smooth vector function in $[C^2(\Omega_f)]^n$ with the properties: $\operatorname{div} y_e = 0, \;\; y_e|_{\Gamma_f} = 0$.

4 Semigroup Framework for Min-Max Game Theory Problems

To study the min-max game theory for parabolic-hyperbolic type coupled PDEs, our main difficulty would be the construction of the associated non-standard Riccati equation in the context of boundary control/boundary disturbance, which yields a mathematically challenging analysis (in the setting of "highly unbounded" operators), where the standard approach (within the setting of "bounded" operator) fails. This is due to the fact that the feedback operator along with the associated coefficients in the non-standard Riccati equation may not be well defined. The intrinsic reason for this is the lack of analyticity in the PDE system. The theoretical framework of constructing the Riccati equation associated with systems generating analytic semigroups has indeed been established [M-T.3],[M-T.4],[L-T.2, Chapter 6]; whereas for each of these parabolic-hyperbolic coupled PDE models, the overall dynamics is not parabolic due to the presence of the hyperbolic component. Therefore, the semigroup generated from the whole PDE system lacks the analyticity property, which is typical for parabolic dynamics. Nevertheless, the parabolic component does impose a partial smoothing on the overall dynamical evolution, which is captured in the "singular estimate" property. Paper [T-Z.1] consider the following abstract dynamics:

$$\text{Abstract Model:} \quad \begin{cases} \dot{y}(t) = Ay(t) + Bu(t) + Gw(t) \ \text{ on } [\mathcal{D}(A^*)]' \\ y(0) = y_0 \in Y \end{cases} \tag{4.1}$$

where $\{y(t), u(t), w(t)\}$ represent the state, control and disturbance respectively, A is the generator of a strongly continuous semigroup on the Hilbert state space Y, B is a linear, highly unbounded operator over the control space U, such that $B : U \to [\mathcal{D}(A^*)]$ or $A^{-1}B \in \mathcal{L}(U; Y)$, similarly for G over the disturbance space V. Moreover, both satisfy the following "singular estimate" at the origin:

$$\|e^{At}Bu\|_Y \leq \frac{C_T}{t^\alpha}\|u\|_U, \ \ \|e^{At}Gw\|_Y \leq \frac{C_T}{t^\alpha}\|w\|_V, \ \ 0 < \alpha < 1, \ \ 0 < t \leq T \tag{4.2}$$

The min-max game theory problem for the dynamics (4.1) satisfying (4.2) is as follows:

Find a solution $\{y^*(t, y_0), u^*(t, y_0), w^*(t, y_0)\}$ on $(0, T)$, such that it solves the following problem:

$$\sup_{w \in V} \inf_{u \in U} \int_0^T \{\|Ry(t)\|_Z^2 + \|u(t)\|_U^2 - \gamma^2\|w(t)\|_V^2\}\, dt \tag{4.3}$$

where R is a linear bounded observation operator from Y to another Hilbert space Z, and γ is a positive constant.

Thus, qualitatively, the min-max game problem consists in finding the minimal cost of the system under the worst disturbance. For this problem, we obtain a critical value γ_c for γ, and prove the existence and uniqueness of the solution under the condition $\gamma > \gamma_c$ for all initial conditions through a variational approach, while for $\gamma < \gamma_c$ there is no solution. The key result is that the solution pair (control u^*, disturbance w^*) can be expressed in pointwise feedback form of the dynamic y^*:

$$u^*(t, y_0) = -B^*P(t)y^*(t, y_0); \quad w^*(t, y_0) = \gamma^{-2}G^*P(t)y^*(t, y_0) \ \ 0 \leq t \leq T \tag{4.4}$$

where $P(t)$ satisfies an operator Riccati differential equation. Because the abstract system (4.1) is infinite dimensional with unbounded generator A, and highly unbounded control and disturbance operator B and G, as noted before, the wellposedness of all quantities as well as of the associated Riccati differential equation is a fundamental technical issue. The presence of the "singular estimate" - which substitutes for the lack of analyticity of semigroup - turns out to guarantee that the feedback operators $B^*P(t)$ and $G^*P(t)$ are both bounded.

Remark 4.1. The corresponding case, under singular estimates (4.2), for $T = \infty$ is studied in [T.1]. Instead, [T.2] studies the min-max problem in the abstract hyperbolic case for $T < \infty$.

The theory we established is based on the abstract model (4.1), and the result is optimal and complete in showing the validity for the construction of the associated non-standard Riccati equation. This allows one to include several physically relevant illustrations such as structural acoustic chambers, thermoelastic plates, composite beams, etc, for this abstract theory. However, the elastic fluid-structure interaction model (without the visco-elastic term, i.e. $\rho = 0$ in model(3.1)) we are interested in happens to be more pathological. Its study requires solving additional technical and conceptual obstacles because such model fails the critical assumption of our original theory, the"singular estimates" (4.2) from the control/disturbance space to the state space Y. In fact, in the present case, a weaker "singular estimate" holds true, not in the original state space, but in a space slightly larger. This is due, intrinsically, to the fact that there is a mismatch between the regularity of the hyperbolic component and the regularity of the parabolic component at the interface Γ_s. As a consequence, the terms "$e^{At}Bu$" and "$e^{At}Gw$" no longer belong to the state space Y, and thus (4.2) fails in the present model. Our treatment in this part is to introduce a weaker singular estimate condition [L-T-Z.1], the Output Singular Estimate (the original singular estimate is then called the State Singular Estimate correspondingly). Then we extend our original theory to this new Output Singular Estimate PDE system.

5 Abstract Min-Max Game Theorem for the Output Singular Estimate PDE System

In this section, we develop the general min-max game theory in the context of abstract dynamics, particularly for elastic fluid-structure interaction model, with unbounded control and disturbance actions.

Abstract Dynamics Let U (control), Y (the state), Z (observation) and V (disturbance) be given Hilbert spaces. We consider the dynamics governed by the state equation

$$y_t = Ay + Bg + Gw; \;\; on\ [\mathcal{D}(A^*)]^{'}; \;\; y(0) = y_0 \in Y, \tag{5.1}$$

and subject to the following assumptions, to be maintained throughout the paper.

Hypotheses Let U (control), Y (state), Z (observation) and V (disturbance) be given Hilbert space.

(H.1) $A : Y \supset \mathcal{D}(A) \to Y$ is the infinitesimal generator of a strongly continuous (s.c.) semigroup e^{At} on Y, $t \geq 0$.

(H.2) B is a linear operator on $U = \mathcal{D}(B) \to [\mathcal{D}(A^*)]'$, satisfying the condition $R(\lambda, A)B \in \mathcal{L}(U;Y)$, for some $\lambda \in \rho(A)$, where $R(\lambda, A)$ is the resolvent of A and $\rho(A)$ is the resolvent set of the A. (This assumption is automatically satisfied if $B \in \mathcal{L}(U,Y)$ i.e. B is bounded as in the standard case)

(H.3) G is a linear operator on $V = \mathcal{D}(G) \to [\mathcal{D}(A^*)]'$, satisfying the condition $R(\lambda, A)G \in \mathcal{L}(V;Y)$, for some $\lambda \in \rho(A)$. (This assumption is automatically satisfied if $G \in \mathcal{L}(V,Y)$ i.e. G is bounded as in the standard case)

(H.4) Let $R : \mathcal{D}(R) \to Z$ be a linear operator, such that

$$\mathcal{D}(R) \supset \{e^{At}BU,\ 0 < t \leq T\} \cup \{e^{At}GV,\ 0 < t \leq T\} \cup Y \tag{5.2}$$

(H.5) The triple $\{A, B, R\}$ satisfies the following Output Singular Estimate Condition: There exists $0 < \alpha < 1$ and a constant $C_T > 0$ such that

$$\|Re^{At}B\|_{\mathcal{L}(U;Z)} = \|B^*e^{A^*t}R^*\|_{\mathcal{L}(Z;U)} \leq \tfrac{C_T}{t^\alpha},\;\; 0 < t \leq T \tag{5.3a}$$

$$\text{or}\;\; Re^{At}Bg \in {}_\alpha C([0,T];Z),\;\; \forall g \in U \tag{5.3b}$$

where $(Bg, y)_Y = (g, B^*y)_U$, $g \in U$, $y \in \mathcal{D}(B^*) \supset \mathcal{D}(A^*)$. The function space ${}_\alpha C([0,T]; \cdot)$ is defined in (5.6) below.

(H.6) With R as in (H.4), the triple $\{A, G, R\}$ satisfies the following Output Singular Estimate Condition:

$$\|Re^{At}G\|_{\mathcal{L}(V;Z)} = \|G^* e^{A^* t} R^*\|_{\mathcal{L}(Z;V)} \leq \tfrac{C_T}{t^\alpha}, \;\; 0 < t \leq T \tag{5.4a}$$

$$\text{or } \; Re^{At}Gw \in {}_\alpha C([0,T]; Z), \;\; \forall w \in V \tag{5.4b}$$

where $(Gw, y)_Y = (w, G^* y)_V$, $w \in V$. For notational simplification, we take the same constant α in (5.3) and (5.4).

(H.7) With R as in (H.4),

$$R \in \mathcal{L}(Y; Z), \;\; \text{this then implied that } Re^{At} : \text{ continuous } Y \to C([0,T]; Z) \tag{5.5}$$

Remark 5.1. If B, G and R are all bounded operators, then assumptions (H.2)-(H.7) are automatically satisfied. Moreover, all of the above abstract assumptions (H.1)-(H.7) are *a-fortiori* satisfied in the case of fluid-structure interaction problem (3.1a-h) in the following specific setting $Y \equiv Z = \mathcal{H}$, $U = L_2(\Gamma_s)$, $R \in \mathcal{L}(\mathcal{H}_{-\delta}; \mathcal{H})$, where δ is arbitrarily positive, $\alpha = 1/4 + \epsilon$. Actually, assumption (H.6) accommodates also the case where the deterministic disturbance acts at the interface Γ_s. Thus, model (3.1) where however now both control g and disturbance w act on, say, two distinct portions of the interface Γ_s is included in the present abstract setting. This is explicitly given in problem (5.18a-i) at the end of Section 5.

Remark 5.2. When $\rho > 0$ in the model (3.1), i.e., the system has "visco-elastic effects". As shown in [L-T-Z.2] the overall dynamics generates a s.c. analytic semigroup. It is also shown there that Dirichlet control action applied on the interface satisfies the singular estimate property in (4.2) . Thus assumptions (H.1)-(H.7) are satisfied automatically.

Notation For further reference, we next define the Banach space ${}_rC([s,T]; X)$, where $0 < r < 1$, which was already introduced in (5.3b). This space measures the singularity on the left point s. The topology on ${}_rC([s,T], X)$ is defined as ([L-T.2, p 3]):

$${}_rC([s,T]; X) = \{f \in C((s,T]; X) : \; \|f\|_{{}_rC([s,T];X)} \equiv \sup_{s<t\leq T} (t-s)^r \|f(t)\|_X\} \tag{5.6}$$

Min-max game theory problem over finite time interval $[0,T]$ For a fixed $0 < T < \infty$ and fixed $\gamma > 0$, we associate with (5.1) the cost functional

$$J(g, w; y_0) \equiv J(g, w; y(g,w); y_0) = \int_0^T [\|Ry(t)\|_Z^2 + \|g(t)\|_U^2 - \gamma^2 \|w(t)\|_V^2]\, dt \tag{5.7a}$$

where $y(t) = y(t; y_0)$ is the solution of (5.1) due to $g(t)$ and $w(t)$.

What we are going to study is the following game theory problem

$$\sup_w \inf_u J(g, w; y_0) \tag{5.7b}$$

where the infimum is taken over all $g \in L_2(0,T;U)$ for $w \in L_2(0,T;V)$ fixed, and the supremum is then taken over all $w \in L_2(0,T;V)$.

Abstract Theorem for Output Singular Estimate Model Equipped with the above assumptions and properties, we have following result for the abstract dynamical model (5.1).

Theorem 5.3. *Assume (H.1)-(H.7). Then there exists a critical value $\gamma_c \geq 0$, such that:*

(i) If $\gamma_c > 0$ and $0 < \gamma < \gamma_c$, then taking the supremum in w as dictated by (5.7b) leads to $+\infty$ for all initial conditions $y_0 \in Y$; that is, there is no finite solution of the game theory problem (5.7)

(ii) If $\gamma > \gamma_c$, then:

(a) There exists a unique solution $\{g^*(\,\cdot\,; y_0), w^*(\,\cdot\,; y_0), y^*(\,\cdot\,; y_0)\}$ of the game theory problem (5.7)

(b) There exists a bounded non-negative self-adjoint operator $P(t) = P^*(t) \in \mathcal{L}(Y)$, $0 \le t \le T$, defined explicitly in terms of the problem data, such that:

$$P(t) \text{ continuous} : Y \to C([0,T]; Y) \tag{5.8}$$

(c)

$$\|B^*P(t)x\|_U + \|G^*P(t)x\|_V \le C_T(T-t)^{1-\alpha}\|x\|_Y, \;\; 0 \le t \le T;\; 0 \le \alpha < 1 \tag{5.9}$$

(d) $P(t)$ defines the cost functional in (5.7b) of the solution of the min-max game problem initiating at the point $x \in Y$ and at the time t, over the interval $[t,T]$, for all $t \in [0,T]$:

$$(P(t)x, x)_Y \equiv \sup_{w\in L_2(t,T;V)} \inf_{g\in L_2(t,T;U)} J(u, w; x) \tag{5.10}$$

(e) $P(t)$ satisfies a non-standard differential Riccati equation: for all $x, y \in \mathcal{D}(A)$, we have

$$\begin{cases} (\dot{P}(t)x, y)_Y &= -(Rx, Ry)_Y - (P(t)x, Ay)_Y - (P(t)Ax, y)_Y \\ & \qquad +(B^*P(t)x, B^*P(t)y)_U - \gamma^{-2}(G^*P(t)x, G^*P(t)y)_V \\ P(T) = 0 & \end{cases} \tag{5.11a}$$

$$\tag{5.11b}$$

(f) The following pointwise feedback relations hold true for the min-max game theory solution:

$$g^*(t;x) \;\; = \;\; -B^*P(t)y^*(t;x) \in C([0,T];U), \;\; x \in Y \tag{5.12}$$

$$w^*(t;x) \;\; = \;\; \gamma^{-2}G^*P(t)y^*(t;x) \in C([0,T];V), \;\; x \in Y \tag{5.13}$$

(g) The operator

$$R\Phi(t,s)x \equiv Ry^*(t,s;x) \in C([s,T];Z), \;\; x \in Y \tag{5.14}$$

is an evolution type operator, which is strongly continuous in each variable t and s separately, holding the other fixed. Moreover, for all $x \in \mathcal{D}(A)$, $\Phi(t,s)x$ is differentiable with respect to t in the dual sense on $[\mathcal{D}(A^*)]'$; that is

$$\frac{d}{dt}\Phi(t,s)x = [A - BB^*P(t) + \gamma^{-2}GG^*P(t)]\Phi(t,s)x \in C([s,T];Y) \tag{5.15}$$

(h) $R\Phi(t,s)B$ and $R\Phi(t,s)G$ satisfy the same singular estimates as $Re^{A(t-s)}B$ and $Re^{A(t-s)}G$ in (H.5), (H.6); that is:

$$R\Phi(t,s)Bg, \;\; R\Phi(t,s)Gw \;\in\; {}_{\alpha}C([s,T];\,\cdot\,), \;\; \text{"}\cdot\text{" denotes } U \text{ or } V; \tag{5.16}$$

continuously in $u \in U$ and $v \in V$.

(i) *Finally, for all $x \in \mathcal{D}(A)$, $R\Phi(t,s)x$ is differentiable with respect to s in the space ${}_{\alpha}C([s,t];Z)$, and*

$$\frac{d}{ds}[R\Phi(t,s)x] = -R\Phi(t,s)[A - BB^*P(s) + \gamma^{-2}GG^*P(s)]x \in {}_{\alpha}C([s,t];Z) \quad (5.17)$$

Remark 5.4. If B, G, R are bounded, Then Theorem 5.1 holds true under only one assumption (H.1).

The above abstract theorem applies, *a-fortiori*, also to the following:

Boundary control/disturbance model: In the present model, both control g and disturbance w act at the interface between the two media.

$$\begin{cases} u_t - \Delta u + Lu + \nabla p = w_1 & \text{in} \quad Q_f \equiv \Omega_f \times (0,T] & (5.18a) \\ \operatorname{div} u = 0 & \text{in} \quad Q_f \equiv \Omega_f \times (0,T] & (5.18b) \\ v_{tt} - \operatorname{div}\sigma(v) - \rho\operatorname{div}\sigma(v_t) = w_2 & \text{in} \quad Q_s \equiv \Omega_s \times (0,T] & (5.18c) \\ v_t = u & \text{in} \quad \Sigma_s \equiv \Gamma_s \times (0,T] & (5.18d) \\ u = 0 & \text{in} \quad \Sigma_f \equiv \Gamma_f \times (0,T] & (5.18e) \\ \sigma(v + \rho v_t)\cdot\nu = \epsilon(u)\cdot\nu - p\nu - g & \text{in} \quad \Sigma_{s1} \equiv \Gamma_{s1} \times (0,T] & (5.18f) \\ \sigma(v + \rho v_t)\cdot\nu = \epsilon(u)\cdot\nu - p\nu - w_3 & \text{in} \quad \Sigma_{s2} \equiv \Gamma_{s2} \times (0,T] & (5.18g) \\ u(0,\cdot) = u_0 & \text{in} \quad \Omega_f & (5.18h) \\ v(0,\cdot) = v_0, \ \ v_t(0,\cdot) = v_1 & \text{in} \quad \Omega_s & (5.18i) \end{cases}$$

with $\Gamma_{s1} \cup \Gamma_{s2} = \Gamma_s$, $\bar{\Gamma}_{s1} \cap \bar{\Gamma}_{s2} = \emptyset$. That is, control g and disturbance w act on two distinct portion of the interface Γ_s.

6 Specialization of Abstract Model to the Fluid-Structure Interaction Model

In this section,We adapt the abstract theory we have developed to the fluid structure interaction model. We shall see below that the abstract spaces of Section 5, corresponding to the coupled PDE model (3.1) are

$$\mathcal{H} \equiv \mathbb{H} \times (H^1(\Omega_s))^n \times (L_2(\Omega_s))^n \text{ for } \{u, v, v_t\}; \ \mathcal{U} \equiv (L_2(\Gamma_s))^n; \ \mathcal{V} \equiv (L_2(\Omega_f))^n \times (L_2(\Omega_s))^n \times (L_2(\Gamma_s))^n \quad (6.1)$$

where

$$\mathbb{H} \equiv \{u \in (L_2(\Omega_f))^n : \operatorname{div} u = 0, \ \ u\cdot\nu|_{\Gamma_f} = 0\} \quad (6.2)$$

We moreover define the space

$$E \equiv \{u \in (H^1(\Omega_f))^n : \operatorname{div} u = 0, \ \ u|_{\Gamma_f} = 0\} \quad (6.3)$$

Notation: Henceforth, we shall drop the symbol $(\,\cdot\,)^n$, $n = 2, 3$ for all Sobolev spaces H^s and L_2 spaces pertaining to u and v, for the sake of simplicity of notation. Accordingly, say on the domain Ω_s and corresponding boundary Γ_s, the L_2-inner products are denoted by

$$(u_1, u_2) = \int_{\Omega_s} u_1 \cdot u_2 \, d\Omega_s; \ \ \langle u_1, u_2 \rangle = \int_{\Gamma_s} u_1 \cdot u_2 \, d\Gamma_s \quad (6.4a)$$

On the domain Ω_f, the space E is topologized with respect to the inner product given by

$$(u_1, u_2)_E = (u_1, u_2)_{1,f} \equiv \int_{\Omega_f} \epsilon(u_1) \cdot \epsilon(u_2)\, d\Omega_f \tag{6.4b}$$

We also denote the induced norm by $\|\cdot\|_{1,\Omega_f}$ which is equivalent to the usual $H^1(\Omega_f)$ norm via Korn's inequality and Poincaré's inequality [K.1]:

$$\|u\|_{1,\Omega_f} = \Big(\int_{\Omega_f} |\epsilon(u)|^2\, d\Omega_f\Big)^{1/2} \tag{6.5}$$

The space $H^1(\Omega_s)$ is topologized with respect to the inner product given by

$$(v, z)_{1,s} \equiv \int_{\Omega_s} v \cdot z\, d\Omega_s + \int_{\Omega_s} \sigma(v) \cdot \epsilon(z)\, d\Omega_s \tag{6.6a}$$

so that the $\|\cdot\|_{1,\Omega_s}$ norm induced by the inner product above is

$$\|v\|^2_{1,\Omega_s} = \int_{\Omega_s} \sigma(v) \cdot \epsilon(v)\, d\Omega_s + |v|^2_{0,\Omega_s} \tag{6.6b}$$

This again equivalent to the usual $H^1(\Omega_s)$ norm by Korn's inequality [K.1].

With $T > 0$ preassigned, the quadratic cost functional corresponding to (3.1) is then

$$\begin{aligned} J(u, v, g, w) = &\int_0^T \big(|g(t)|^2_{L_2(\Gamma_s)} + |u(t)|^2_{L_2(\Omega_f)} + |v(t)|^2_{H^1(\Omega_s)} + |v_t(t)|^2_{L_2(\Omega_s)}\big)\, dt \\ &- \gamma^2 \int_0^T |w_1(t)|^2_{L_2(\Omega_f)} + |w_2(t)|^2_{L_2(\Omega_s)} + |w_3(t)|^2_{L_2(\Gamma_s)}\, dt \end{aligned} \tag{6.7}$$

The min-max game problem corresponds to (6.7) is

$$\sup_{w \in L_2(\Omega_f) \times L_2(\Omega_s)} \inf_{g \in L_2(\Gamma_s)} J(u, v, g, w) \tag{6.8}$$

We next transform the PDE System (3.1) into the abstract state equation and then use the abstract theory in Section 5 to maxi-minimize the cost functional (6.7). The literature already contains two treatments leading to the abstract model corresponding to this PDE system. One approach - introduced in [A-T.1] and pursued in [A-T.2]-[A-T.5] - eliminates the pressure by introducing two suitable Green's maps of appropriate elliptic problems in Ω_f; one with Dirichlet datum on Γ_s and homogeneous Neumann datum on Γ_f; the second, the other way around: homogeneous Dirichlet datum in Γ_s and Neumann datum on Γ_f. This approach was actually studied for the case of interest in these references, where $g \equiv 0$ in Γ_s. The second approach, instead, is variational: it was presented in [B-G-L-T.1]-[B-G-L-T.2],[L-Tu.1]-[L-Tu.2]. We shall use this approach here. We first define the following operators:

$A : E \to E'$, such that

$$(Au, \phi)_{\Omega_f} = -(\epsilon(u), \epsilon(\phi))_{\Omega_f}, \;\; \phi \in E \tag{6.9}$$

The Neumann map $N : L_2(\Gamma_s) \to \mathbb{H}$, such that

$$Ng = h \Leftrightarrow \{h \in \mathbb{H} : \; (\epsilon(h), \epsilon(\phi))_{\Omega_f} = \langle g, \phi\rangle_{\Gamma_s}, \, \phi \in E\} \tag{6.10}$$

By Lax-Milgram Theorem, A is linear, bounded from E to E'. N is linear bounded from $H^{1/2}(\Gamma_s)$ to $H^1(\Omega_f)$. This allows we to consider operator A as acting on $\mathbb{H}$ with domain

$\mathcal{D}(A) = \{u \in E, |(\epsilon(u), \epsilon(\phi))| \leq C|\phi|_E\}$. Thus A is negative, self-adjoint and generate an analytic semigroup e^{At} on $\mathbb{H}$. Let

$$\mathcal{A} = \begin{pmatrix} A - L & AN\sigma() \cdot \nu & \rho AN\sigma() \cdot \nu \\ 0 & 0 & I \\ 0 & \operatorname{div} \sigma() & \rho \operatorname{div} \sigma() \end{pmatrix}, \ \mathcal{B} = \begin{pmatrix} AN \\ 0 \\ 0 \end{pmatrix}, \ \mathcal{G} = \begin{pmatrix} P & 0 & -AN \\ 0 & 0 & 0 \\ 0 & I & 0 \end{pmatrix} \tag{6.11}$$

By [L-Tu.2], because L is compact from $\mathcal{D}(A)$ to $\mathbb{H}$, the perturbation $A - L$ still generates an analytic semigroup. It was shown in [B-G-L-T.1] $\mathcal{A}$ with domain $\mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ defined as follows

$$\mathcal{D}(\mathcal{A}) = \{(u, v, z) \in \mathcal{H} : u \in E,\ A(u + N\sigma(v + \rho z) \cdot \nu) - Lu \in \mathbb{H},\ z \in H^1(\Omega_s), \\ \operatorname{div} \sigma(v + \rho z) \in L_2(\Omega_s),\ z|_{\Gamma_s} = u|_{\Gamma_s} \text{ in } H^{1/2}(\Gamma_s)\} \tag{6.12}$$

will generate a strongly continuous semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$. Thus let $y = (u, v, v_t)$ and $y_0 = (u_0, v_0, v_1)$, we have following results:

Theorem 6.1. *[B-G-L-T.1],[L-T-Z.2]. For all $\rho \geq 0$, the PDE system (3.1) can be modeled abstractly as follows:*

$$y_t = \mathcal{A}y + \mathcal{B}g + \mathcal{G}w \in [\mathcal{D}(\mathcal{A}^*)]',\ \ y_0 \in \mathcal{H} \tag{6.13}$$

Where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{G}$ are defined as in (6.11). Moreover, $\mathcal{A}$ is the infinitesimal generators of s.c. semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$ with the domain $\mathcal{D}(\mathcal{A})$ defined in (6.12). When $\rho > 0$, $\mathcal{A}$ generates a s.c. analytic semigroup.

Control Operator $\mathcal{B}$ The operator $\mathcal{B}$ defined in (6.11) is unbounded from $L_2(\Gamma_s)$ to $\mathcal{H}$ as the operator AN is unbounded from $L_2(\Gamma_s)$ to H. To apply Theorem 5.1 to (6.13), we need to verify the validity of Assumption (H.2) on the operator $\mathcal{B}$. The following proposition follows:

Proposition 6.2. *[L-Tu.2], [L-T-Z.2]. Let $\rho \geq 0$, there exists some $\omega > 0$, such that the resolvent $R(\lambda, \mathcal{A})$ satisfies $R(\lambda, \mathcal{A})\mathcal{B} \in \mathcal{L}(L_2(\Gamma_s) \to \mathcal{H})$ for all $\lambda > \omega$.*

Disturbance Operator $\mathcal{G}$ The operator $\mathcal{G}$ defined in (6.11) is also bounded from $L_2(\Omega_f) \times L_2(\Omega_s)$ to $\mathcal{H}$ because of the operator AN. More specifically, the presence of w_1 and w_2 does not bring any problem in verifying the singular estimates property of $\mathcal{G}$ as they are interior disturbance, however, the boundary disturbance w_3 raise the unboundedness issue just as the boundary control g. And it is easy to verify that $\mathcal{G}$ has the similar property with $\mathcal{B}$. The following proposition holds:

Proposition 6.3. *Let $\rho \geq 0$, there exists some $\omega > 0$, such that the resolvent $R(\lambda, \mathcal{A})$ satisfies $R(\lambda, \mathcal{A})\mathcal{G} \in \mathcal{L}(L_2(\Gamma_s) \to \mathcal{H})$ for all $\lambda > \omega$.*

Singular Estimate Property We now investigate the dynamical property of the pair $\{\mathcal{A}, \mathcal{B}\}$ since $\mathcal{B}$ is unbounded, which is critical for Applying Theorem 5.1 to the fluid-structure interaction model. To this end, we define the following scale of Hilbert spaces parameterized by the parameter $\delta \geq 0$:

$$\mathcal{H}_{-\delta} \equiv \mathbb{H} \times H^{1-\delta}(\Omega_s) \times H^{-\delta}(\Omega_s), \quad \delta \geq 0 \tag{6.14}$$

We then remark that with the above notation we have: $\mathcal{H} = \mathcal{H}_0$, where $\mathcal{H}$ is the energy space defined in (6.1). The following result from [L-Tu.2] is critical.

Theorem 6.4. *[L-Tu.2, Theorem 5.1] The semigroup $e^{\mathcal{A}t}$ generated in $\mathcal{H}$ by the operator $\mathcal{A}$ in (6.11) and the control operator $\mathcal{B}$ in (6.11) satisfy the following Singular Estimate (SE)*

$$\|e^{\mathcal{A}t}\mathcal{B}g\|_{\mathcal{H}_{-\delta}} \leq \frac{C_T}{t^{1/4+\epsilon}}\|g\|_{L_2(\Gamma_s)}, \quad \|e^{\mathcal{A}t}\mathcal{G}(0,0,w_3)\|_{\mathcal{H}_{-\delta}} \leq \frac{C_T}{t^{1/4+\epsilon}}\|w_3\|_{L_2(\Gamma_s)}, \quad 0 < t \leq T \tag{6.15}$$

for every $g \in L_2(\Gamma_s)$, and any $\delta > 0$.

Note that in (6.4), we use, without loss of generality, $(0, 0, w_3)$ to substitute (w_1, w_2, w_3) because w_3 is the only component in (w_1, w_2, w_3) that gives rise to the singular estimate property for the operator $\mathcal{G}$.

From Theorem (6.4), we are ready to obtain the sought-after Output Singular Estimate property for the pair $\{\mathcal{A}, \mathcal{B}, \mathcal{G}\}$.

Corollary 6.5. *Consider the PDE system (3.1) and corresponding abstract model (6.13) satisfying, in particular, Theorem 6.3, with $\delta > 0$ arbitrary and henceforth fixed. Assume that the observation operator $\mathcal{R}$ satisfies*

$$\mathcal{R} \in \mathcal{L}(\mathcal{H}_{-\delta}; \mathcal{Z}) \tag{6.16}$$

where $\mathcal{Z}$ is the observation or output space (possibly $\mathcal{H} = \mathcal{Z}$). Then (6.15) yields the desired Output Singular Estimate

$$\|\mathcal{R}e^{\mathcal{A}t}\mathcal{B}g\|_{\mathcal{Z}} \leq \frac{C_T}{t^{1/4+\epsilon}}\|g\|_{L_2(\Gamma_s)}, \ \|\mathcal{R}e^{\mathcal{A}t}\mathcal{G}w\|_{\mathcal{Z}} \leq \frac{C_T}{t^{1/4+\epsilon}}\|w_3\|_{L_2(\Gamma_s)} \quad 0 < t \leq T. \tag{6.17}$$

Next, we illustrate examples of the observation operator $\mathcal{R}$ satisfying the smoothing property (6.16), (6.17) in Corollary 6.4.

Example 4.1: Let $v_1 \in H^{1-\delta}(\Omega_s)$, $v_2 \in H^{-\delta}(\Omega_s)$, $\delta > 0$ small. Define the operators $\mathbb{R}_1$ and $\mathbb{R}_2$ on v_1 and v_2 respectively by

$$\mathbb{R}_1 v_1 = (v_1, \phi_1)\psi_1; \quad \mathbb{R}_2 v_2 = (v_2, \phi_2)\psi_2 \tag{6.18}$$

where ϕ_i, ψ_i are fixed vectors:

$$\phi_1 \in [H^{1-\delta}(\Omega_s)]'; \ \ \psi_1 \in H^1(\Omega_s); \ \ \phi_2 \in H^\delta(\Omega_s); \ \ \psi_2 \in L_2(\Omega_s) \tag{6.19}$$

Thus, $\mathbb{R}_1$ is bounded (thus smoothing): $H^{1-\delta}(\Omega_s) \to H^1(\Omega_s)$, $\mathbb{R}_1 \in \mathcal{L}(H^{1-\delta}(\Omega_s); H^1(\Omega_s))$, but not smoothing above $H^1(\Omega_s)$. Similarly, $\mathbb{R}_2$ is bounded (thus smoothing): $H^{-\delta}(\Omega_s) \to L_2(\Omega_s)$, $\mathbb{R}_2 \in \mathcal{L}(H^{-\delta}(\Omega_s); L_2(\Omega_s))$, but not smoothing above $L_2(\Omega_s)$. Finally, for $\{u_1, v_1, v_2\} \in \mathbb{H} \times H^{1-\delta}(\Omega_s) \times H^{-\delta}(\Omega_s)$, define $\mathcal{R} \in \mathcal{L}(\mathcal{H}_{-\delta}; \mathcal{H})$ (but not smoothing above $\mathcal{H}$) by:

$$\mathcal{R}\begin{bmatrix} u_1 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_f & 0 & 0 \\ 0 & \mathbb{R}_1 & 0 \\ 0 & 0 & \mathbb{R}_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ \mathbb{R}_1 v_1 \\ \mathbb{R}_2 v_2 \end{bmatrix} \in \mathcal{H} \equiv \mathbb{H} \times H^1(\Omega_s) \times L_2(\Omega_s) \tag{6.20}$$

Example 4.2: On Ω_s, let

$$\bar{\Delta}_N f = (-\Delta)f, \quad \mathcal{D}(\bar{\Delta}_N) = \{f \in H^2(\Omega_s) : \frac{\partial f}{\partial \nu}|_{\partial\Gamma_s} = 0\} \tag{6.21}$$

So that $\mathcal{D}(\bar{\Delta}_N^{1/2}) = H^1(\Omega_s)$, $\mathcal{D}(\bar{\Delta}_N^{\delta/2}) = H^\delta(\Omega_s)$, $\delta \geq 0$. Let $\{u_1, v_1, v_2\} \in \mathcal{H}_{-\delta}$, so that, by (1.3b), $v_1 \in H^{1-\delta}(\Omega_s)$ and $v_2 \in H^{-\delta}(\Omega_s)$. Thus,

$$\bar{\Delta}_N^{-\delta/2} v_1 \in H^1(\Omega_s), \quad \bar{\Delta}_N^{-\delta/2} v_2 \in L_2(\Omega_s) \tag{6.22}$$

Thus, the operator $\mathcal{R} \in \mathcal{L}(\mathcal{H}_{-\delta}; \mathcal{H})$ defined by

$$\mathcal{R}\begin{bmatrix} u_1 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_f & 0 & 0 \\ 0 & \bar{\Delta}_N^{-\delta/2} & 0 \\ 0 & 0 & \bar{\Delta}_N^{-\delta/2} \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ \bar{\Delta}_N^{-\delta/2} v_1 \\ \bar{\Delta}_N^{-\delta/2} v_2 \end{bmatrix} \in \mathcal{H} \tag{6.23}$$

where I_f is the identity on $\mathbb{H}$ and satisfies property (4.3) with respect to the output space $\mathcal{Z} = \mathcal{H}$. In this example, $\mathcal{R}$ is also smoothing above $\mathcal{H}$, unlike the case of Example 4.1.

From the above analysis, we have verified that the operator $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{G}$ defined in (6.11) satisfy the Assumptions (H.1)-(H.3). The existence of the smoothing operator $\mathcal{R}$ is presented in Example 4.1 and Example 4.2, for which the pair $(\mathcal{A}, \mathcal{B}, \mathcal{R})$ as well as $(\mathcal{A}, \mathcal{G}, \mathcal{R})$ satisfy the Assumptions (H.4)-(H.7). Thus we conclude that the fluid-structure interaction model (3.1) satisfies the Assumptions of Theorem 5.1 with $\alpha = 1/4 + \epsilon$. Followed by Theorem 5.1, we have following important results for the min-max game problem for fluid-structure interaction model.

Min-Max Game for Elastic Fluid-Structure Interaction Model: Main Results

Theorem 6.6. *In reference of model (3.1) with $\rho = 0$ and the min-max game problem (6.8), there exists a critical $\gamma_c > 0$, for each initial condition in $\mathcal{H}$, that is, $y_0 = (v_0, v_1, u_0) \in H^1(\Omega_s) \times L_2(\Omega_s) \times \mathbb{H}$,*

(i) if $0 < \gamma < \gamma_c$, $J(u, v, g, w)$ will grow to infinity as we take supremum over $w = (w_1, w_2)$. Thus there is not finite solution for (6.8) for any initial condition $y_0 \in \mathcal{H}$.

(ii) if $\gamma > \gamma_c$, then for each initial condition $y_0 \in \mathcal{H}$ there exists a unique control $g^* \in C([0, T], L_2(\Gamma_s))$, a unique disturbance $(w_0^*, w_1^*) \in C([0, T], L_2(\Omega_f) \times L_2(\Omega_s))$ and corresponding optimal state

$$y^*(t) = (v^*(t), v_t^*(t), u^*(t)) \in C([0, T], H^1(\Omega_s) \times L_2(\Omega_s) \times H)$$

such that

$$J(u^*, v^*, g^*, w^*) = \sup_{w \in L_2(\Omega_f) \times L_2(\Omega_s)} \inf_{g \in L_2(\Gamma_s)} J(u, v, g, w)$$

Furthermore,

(a) $\|g^*(t)\|_{L_2(\Gamma_s)} \le C(T - t)^{3/4 - \epsilon}$,
$\|w_1^*(t)\|_{L_2(\Omega_f)} + \|w_2^*(t)\|_{L_2(\Omega_s)} \le C$.

(b) $\|v^*(t)\|^2_{H^1(\Omega_s)} + \|v_t^*(t)\|^2_{L_2(\Omega_s)} + \|u^*(t)\|^2_H \le C$.

(c) There exists a positive self-adjoint $n \times n$ operator matrix $Q(t)$ on $\mathcal{H}$, let

$$P(t)(v^*(t), v_t^*(t), u^*(t)) = (p_1(t), p_2(t), p_3(t))$$

with $p_1(t)$, $p_2(t)$ and $p_3(t)$ being n-dimensional vector function, such that the control is given by

$$g^*(t) = -\mathcal{B}^* P(t) y^*(t) = -(p_1(t))|_{\Gamma_s}$$

the desired cost functional is given by

$$J(u^*, v^*, g^*, w^*) = \int_\Omega \{\epsilon(p_1(0)) \cdot \epsilon(v_0) + p_2(0) \cdot v_1 + p_3(0) \cdot u_0\} \, d\Omega$$

(d) The feedback operator $\mathcal{B}^* P(t) \in \mathcal{L}(\mathcal{H}, L_2(\Gamma_s))$ for all $0 \le t < T$ with singular estimate

$$\|\mathcal{B}^* P(t) y\|_{L_2(\Gamma_s)} \le \frac{C \|y\|_{\mathcal{H}}}{(T - t)^{1/4 + \epsilon}}$$

(e) $P(t)$ *is the unique solution to the following non-standard Riccati equation:*

$$\begin{cases} (\dot{P}(t)x, y)_{\mathcal{H}} & = -(\mathcal{R}x, \mathcal{R}y)_{\mathcal{H}} - (P(t)x, \mathcal{A}y)_{\mathcal{H}} - (P(t)\mathcal{A}x, y)_{\mathcal{H}} \\ & \quad +(\mathcal{B}^*P(t)x, \mathcal{B}^*P(t)y)_{L_2(\Gamma_s)} - \gamma^{-2}(\mathcal{G}^*P(t)x, \mathcal{G}^*P(t)y)_{L_2(\Omega_f)\times L_2(\Omega_s)} \\ P(T) = 0 \end{cases} \tag{6.24}$$

where $\mathcal{R}$ *is given by Example 4.2.*

7 Visco-Elastic Fluid-Structure Interaction Model with Dirichlet Control on the Interface

The inclusion of visco-elastic effects ($\rho > 0$) has two advantages: (1) it allows for applicability of a much richer class of controls and (2) it does not require an incremental smoothing hypothesis imposed on the observation R. One can $\sigma = 0$ in the definition of the observation operator. Indeed, as shown in [L-T-Z.2] one may use Dirichlet type of control action (physically attractive but mathematically notoriously difficult -see [L-T.4, B-D-D-M]). This is to say that model (3.1) can also account for a Dirichlet type of control added to the interface. the resulting model is the following.

$$\begin{cases} u_t - \Delta u + Lu + \nabla p = w_1 \quad \text{in} \quad Q_f \equiv \Omega_f \times (0,T] & (7.1a) \\ \operatorname{div} u = 0 \quad \text{in} \quad Q_f \equiv \Omega_f \times (0,T] & (7.1b) \\ v_{tt} - \operatorname{div}\sigma(v) - \rho\operatorname{div}\sigma(v_t) = w_2 \quad \text{in} \quad Q_s \equiv \Omega_s \times (0,T] & (7.1c) \\ v_t = u + g_0 \quad \text{in} \quad \Sigma_s \equiv \Gamma_s \times (0,T] & (7.1d) \\ u = 0 \quad \text{in} \quad \Sigma_f \equiv \Gamma_f \times (0,T] & (7.1e) \\ \sigma(v + \rho v_t)\cdot\nu = \epsilon(u)\cdot\nu - p\nu - g_1 \quad \text{in} \quad \Sigma_s \equiv \Gamma_s \times (0,T] & (7.1f) \\ u(0,\cdot) = u_0 \quad \text{in} \quad \Omega_f & (7.1g) \\ v(0,\cdot) = v_0, \ \ v_t(0,\cdot) = v_1 \quad \text{in} \quad \Omega_s & (7.1h) \end{cases}$$

The significant difference between model (7.1) and (3.1) is that in (7.1d), one more control g_0 is added and $\rho > 0$ must be positive. This difference changes the entire setup. The overall fluid-structure dynamics will satisfy the *singular estimate* (4.2) in section 4, which is much stronger than the *output singular estimate* in section 5. Here we also assume that the disturbances only lie in the interior of two media. We again want to transform (7.1) into abstract dynamical system.

Let A and N defined as in (6.9) and (6.10) respectively. We define the Dirichlet map D: $(L_2(\Gamma_s))^n \to (H^{1/2}(\Omega_s))^n$ as follows:

$$Dg = h \iff \{h \in H^{1/2}(\Omega_s) : \operatorname{div}\sigma(h) = 0, \ h|_{\Gamma_s} = g\} \tag{7.2}$$

Denote by A_D the corresponding elastic operator $\operatorname{div}\sigma(\cdot)$ with zero Dirichlet condition.
In this new model (7.1), the operator $\mathcal{A}$ has a corresponding representation $\mathcal{A}_D$ as follows:

$$\mathcal{A}_D = \begin{pmatrix} A - L_f & AN\sigma()\cdot\nu & \rho AN\sigma()\cdot\nu \\ 0 & 0 & I \\ -\rho A_D D & A_D - A_D D & \rho A_D \end{pmatrix} \tag{7.3}$$

with domain defined in (6.12).
The control operator $\mathcal{B}$ and disturbance operator $\mathcal{G}$ are changed to

$$\mathcal{B}_D = \begin{pmatrix} AN \\ 0 \\ \rho A_D D \end{pmatrix} \quad \mathcal{G}_D = \begin{pmatrix} P & 0 \\ 0 & 0 \\ 0 & I \end{pmatrix} \tag{7.4}$$

With the above preparation, and assume y and y_0 remains the same as in section 6, $g = (g_0, g_1)$, $w = (w_1, w_2)$ we have the following theorem

Theorem 7.1. *[L-T-Z.2] For all $\rho > 0$, the PDE system (7.1) can be modeled abstractly as follows:*

$$y_t = \mathcal{A}_D y + \mathcal{B}_D g + \mathcal{G}_D w \in [\mathcal{D}(\mathcal{A}^*)]', \quad y_0 \in \mathcal{H} \tag{7.5}$$

where $\mathcal{A}_D$, $\mathcal{B}_D$, $\mathcal{G}_D$ are defined as in (7.3) and (7.4). Moreover, $\mathcal{A}_D$ is the infinitesimal generators of s.c. analytic semigroup $e^{\mathcal{A}_D t}$ on $\mathcal{H}$ with the domain $\mathcal{D}(\mathcal{A}_D)$ defined in (6.12).

Control Operator $\mathcal{B}_D$ The operator $\mathcal{B}$ defined in (7.4) is unbounded from $L_2(\Gamma_s)$ to $\mathcal{H}$ because the operator AN is unbounded from $L_2(\Gamma_s)$ to $\mathbb{H}$ and the operator $A_D D$ is unbounded from $L_2(\Gamma_s)$ to $H^1(\Omega_s)$ as well. [L-T-Z.2] proved the following important result of $\mathcal{B}_D$ regarding its singular estimate property in assumption (H.2).

Theorem 7.2. *The semigroup $e^{\mathcal{A}_D t}$ generated in $\mathcal{H}$ by the operator $\mathcal{A}_D$ in (7.3) and the control operator $\mathcal{B}_D$ in (7.4) satisfy the following Singular Estimate (SE)*

$$\|e^{\mathcal{A}t}\mathcal{B}_D(g_0, g_1)\|_{\mathcal{H}} \leq \frac{C_T}{t^{3/4+\epsilon}}(\|g_1\|_{H^{-1/2}(\Gamma_s)} + \|g_0\|_{H^{1/2}(\Gamma_s)}), \quad 0 < t \leq T \tag{7.6}$$

for any $\epsilon > 0$.

Disturbance Operator $\mathcal{G}_D$ The operator $\mathcal{G}_D$ defined in (7.4) is bounded from $L_2(\Omega_f) \times L_2(\Omega_s)$ to $\mathcal{H}$, thus it satisfies the singular estimate (4.2) automatically.

Theorems (7.1), (7.2) and the boundedness of the operator $\mathcal{G}_D$ suggest that the dynamics of model (7.1) satisfies the singular estimate (4.2) with parameter $\alpha = 3/4 + \epsilon$, which is stronger than the output singular assumption in (H.5) and (H.6). Thus Theorem 6.6 applies to this model with α changed to $3/4 + \epsilon$ and with no requirement on the smoothness of the operator $\mathcal{R}$.

Remark 7.3. Recent developments in the area of Riccati Equations with unbounded controls include a new class of system where the notion of "singular estimate" is substantially relaxed. In fact, this new theory encompasses classes of problems where the dynamics can be split into singular estimate part and another one that is purely hyperbolic subject to a suitable "trace condition". Complete theory of solvability of Riccati equations for this new class has been developed in [A-B-L.1, A-B-L.2, B-L]. However, the referenced work deals only with minimization problems. Potential extensions of this new framework in the direction of game theory is envisioned as a future project.

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Author information

Jing Zhang, Department of Mathematics and Computer Science, Virginia State University, Petersburg, VA 23806, USA.
E-mail: jizhang@vsu.edu

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