Palestine Journal of Mathematics

Vol. 3(1) (2014) , 70–76

Characterization and Subordination Properties for λ-Spirallike Generalized Sakaguchi Type Functions

Trilok Mathur, Ruchi Mathur and Deepa Sinha

Communicated by S.P. Goyal

MSC 2010 Classifications: 30C45 , 30C50 , 30C80. 11R32.

Keywords and phrases: Analytic functions; Starlike functions; Convex functions; Sakaguchi type function; λ-spirallike functions.

The authors are thankful to Prof. S.P. Goyal, Emeritus Scientist(CSIR), University of Rajasthan, Jaipur, India, for his kind help and valuable suggestions during the preparation of this paper.

Abstract. In this paper we shall introduce and study subclasses $R^{\lambda}(\alpha, s, t)$ and $P^{\lambda}(\alpha, s, t)$ of the class of λ- spirallike generalized Sakaguchi type function. Here we shall prove characterization and subordination properties for these subclasses and point out several interesting consequences of our results.

1. INTRODUCTION

Let A be the class of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are analytic in the unit disc $\Delta = \{z \in C : |z| < 1\}$. For two functions $f, g \in A$, we say that the function $f(z)$ is subordinate to $g(z)$ in Δ and write $f \prec g$, or $f(z) \prec g(z)(z \in \Delta)$ if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1(z \in \Delta)$, such that
$f(z) = g(w(z)), (z \in \Delta)$. In particular, if the function g is univalent in Δ, the above subordination is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.
An analytic function $f(z) \in A$ is said to be in the generalized Sakaguchi class $S(\alpha, s, t)$ defined by Frasin [[1], see also [2], [10], [11]] if it satisfies

$$Re\left\{\frac{(s-t)zf'(z)}{f(sz) - f(tz)}\right\} > \alpha, \quad z \in \Delta$$

for some $\alpha(0 \le \alpha < 1), s, t \in C, |t| \le 1, s \neq t$ and for all $z \in \Delta$.
For $s = 1$ the generalized Sakaguchi class $S(\alpha, s, t)$ reduces to the subclass $\sum(\alpha, t)$ studied by Owa et al. [6] and Goyal and Goswami [8].
A function $f(z)$ is said to be in the class $S_p(\lambda)$ if it satisfies the condition

$$Re\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\} > 0 \quad (|\lambda| < \frac{\pi}{2}) \tag{1.2}$$

Špaěck [5] proved that members of $S_p(\lambda)$ known as λ- spirallike functions are univalent in the unit disc Δ. Silverman [3], Singh [9] and several others have discussed various properties for spirallike functions.
Recently Goyal and Goswami [8] have introduced and studied subclasses $P^{\lambda}(\alpha, t)$ and $M^{\lambda}(\alpha, t)$ of the classes of λ- spirallike functions. A function $f(z) \in A$ is said to be in the class $P^{\lambda}(\alpha, t)$

if it satisfies

$$Re\left\{\frac{e^{i\lambda}(1-t)zf'(z)}{f(z)-f(tz)}\right\} > \alpha\cos\lambda$$

If $zf^{'}(z) \in P^{\lambda}(\alpha, t)$ then $f(z) \in M^{\lambda}(\alpha, t)$.
Now we introduce a subclass $R^{\lambda}(\alpha, s, t)$ of the class of λ- spirallike generalied Sakaguchi functions as follows.

Definition 1.1 A function $f(z) \in A$ is said to be in the class $R^{\lambda}(\alpha, s, t)$ if it satisfies

$$Re\left\{\frac{e^{i\lambda}(s-t)zf'(z)}{f(sz)-f(tz)}\right\} > \alpha\cos\lambda \quad (|t| \leq 1, s \neq t, |\lambda| < \frac{\pi}{2}) \tag{1.3}$$

for some $\alpha (0 \leq \alpha < 1)$ and for all $z \in \Delta$.
obviously $R^0(\alpha, s, t) = S(\alpha, s, t)$, $R^{\lambda}(\alpha, 1, t) = P^{\lambda}(\alpha, t)$ and $R^{\lambda}(0, 1, 0) = S_p(\lambda)$
We also denote by $P^{\lambda}(\alpha, s, t)$, the subclass of A consisting of all functions $f(z)$ such that $zf'(z) \in R^{\lambda}(\alpha, s, t)$. To prove our main results, we need the following definition and lemma:

Definition 1.2 [4] A sequence $\{b_n\}_1^{\infty}$ of complex numbers is said to be a subordinating factor sequence, whenever $f(z)$ given by (1.1) is regular, univalent and convex in Δ, and

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z) \quad \text{in} \quad \Delta \tag{1.4}$$

Lemma 1.3 [4] The sequence $\{b_n\}_1^{\infty}$ is a subordinating factor sequence if and only if

$$Re\left[1 + 2\sum_{n=1}^{\infty} b_n z^n\right] > 0, \quad z \in \Delta \tag{1.5}$$

The purpose of the present paper is to investigate the characterization and subordination properties for the class of functions $R^{\lambda}(\alpha, s, t)$ and $P^{\lambda}(\alpha, s, t)$. Some interesting consequences of the main results are also discussed.

2. MAIN RESULTS

We first prove the following theorems dealing with characterization properties for the classes $R^{\lambda}(\alpha, s, t)$ and $P^{\lambda}(\alpha, s, t)$.

Theorem 2.1 Let $f(z) \in A$ such that

$$\left|\frac{(s-t)zf'(z)}{f(sz)-f(tz)} - 1\right| < 1-\gamma, \quad (s, t \in C, s \neq t, |t| \leq 1; 0 \leq \gamma \leq 1, z \in \Delta) \tag{2.1}$$

then $f(z) \in R^{\lambda}(\alpha, s, t)$, provided that

$$|\lambda| \leq \cos^{-1}\left(\frac{1-\gamma}{1-\alpha}\right) \tag{2.2}$$

for some $\alpha (0 \leq \alpha < 1)$ and $z \in \Delta$.
Proof: Suppose that

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} - 1 = (1-\gamma)\omega(z), \ \text{where} |\omega(z)| < 1, \ \text{for all} \ \ z \in \Delta$$

Now

$$Re\left\{e^{i\lambda}\frac{(s-t)zf'(z)}{f(sz)-f(tz)}\right\} = \cos\lambda + (1-\gamma)Re\left\{e^{i\lambda}\omega(z)\right\}$$
$$\geq \cos\lambda - (1-\gamma)\left|e^{i\lambda}\omega(z)\right|$$
$$\geq \cos\lambda - (1-\gamma) \geq \alpha\cos\lambda$$

provided that $|\lambda| \leq \cos^{-1}\left(\frac{1-\gamma}{1-\alpha}\right)$. This completes the proof of Theorem 2.1. ■

If we set $\gamma = 1-(1-\alpha)\cos\lambda$, where $|\lambda| < \pi/2$, in Theorem 2.1, we obtain the following

Corollary 2.2 Let $f(z) \in A$ such that

$$\left|\frac{(s-t)zf'(z)}{f(sz)-f(tz)} - 1\right| < (1-\alpha)\cos\lambda \tag{2.3}$$

then $f(z) \in R^{\lambda}(\alpha, s, t)$ for $|\lambda| < \pi/2$ and $\alpha(0 \leq \alpha < 1)$.

Remark: On putting $s = 1$ in Theorem 2.1 we get the known result due to Goyal et al. [8], and by putting $s = 1, t = 0$, and $\alpha = 0$ in Theorem 2.1 we get the result due to Silverman [3].

Theorem 2.3 If $f(z) \in A$ satisfies the following inequality

$$\sum_{n=2}^{\infty}\left[|n-u_n|\sec\lambda\ + (1-\alpha)\,|u_n|\right]|a_n| \leq 1-\alpha \tag{2.4}$$

for some $\alpha(0 \leq \alpha < 1)$, then $f(z) \in R^{\lambda}(\alpha, s, t)$, where

$$|\lambda| < \pi/2,\ \ u_n = \sum_{j=1}^{n} s^{n-j}t^{j-1}$$

such that $s, t \in C, |t| \leq 1, s \neq t$.

Proof: To prove the Theorem 2.3, we show that if $f(z)$ satisfies the inequality (2.4) then

$$\left|\frac{(s-t)zf'(z)}{f(sz)-f(tz)} - 1\right| < (1-\alpha)\cos\lambda$$

Since

$$\left|\frac{(s-t)zf'(z)}{f(sz)-f(tz)} - 1\right| = \frac{\sum\limits_{n=2}^{\infty}|(n-u_n)|\,a_n}{1-\sum\limits_{n=2}^{\infty}|u_n|\,a_n}$$

Thus if $f(z)$ satisfies (2.4), then we have

$$\left|\frac{(s-t)zf'(z)}{f(sz)-f(tz)} - 1\right| < (1-\alpha)\cos\lambda$$

This completes the proof of the Theorem 2.3 . ■

Theorem 2.4 If $f(z) \in A$ satisfies the following inequality

$$\sum_{n=2}^{\infty} n\left[|n-u_n|\sec\lambda\ + (1-\alpha)\,|u_n|\right]|a_n| \leq 1-\alpha \tag{2.5}$$

for some $\alpha(0 \le \alpha < 1)$, then $f(z) \in P^{\lambda}(\alpha, s, t)$, where

$$|\lambda| < \pi/2, \;\; u_n = \sum_{j=1}^{n} s^{n-j} t^{j-1}$$

such that $s, t \in C, |t| \le 1, s \ne t$.

Remark: For $\lambda = 0$, Theorems 2.3 and 2.4 reduce to the known results due to Owa et al. [6] By setting $s = 1$ in Theorem 2.1, 2.3 and 2.4 we obtain results of Goyal and Goswami [8]

By setting $t = -1$ in Theorem 2.4, we obtain

Corollary 2.5 If $f(z) \in A$ satisfies the following inequality

$$\sum_{n=2}^{\infty} \left[|n - u_n| \sec\lambda \; + (1-\alpha) |u_n| \right] |a_n| \le 1 - \alpha \tag{2.6}$$

for some $\alpha(0 \le \alpha < 1)$, where $|\lambda| < \pi/2$,

$$u_n = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

then $f(z) \in S(\alpha, 1, -1)$.

3. SUBORDINATION PROPERTY

Theorem 3.1 Let $f(z) \in A$ satisfies the inequality (2.4), and K denote the familiar class of the convex univalent functions in Δ. Then for every $g \in K$, we have

$$\frac{|2-s-t| \sec\lambda + (1-\alpha)|s+t|}{2((1-\alpha) + |2-s-t| \sec\lambda + (1-\alpha)|s+t|)} (f * g)(z) \prec g(z) \tag{3.1}$$

where

$$z \in \Delta, |t| \le 1, s \ne t, s+t \ne 2, 0 \le \alpha < 1 \text{ and } |\lambda| < \pi/2$$

In particular

$$Re\{f(z)\} > -\frac{((1-\alpha) + |2-s-t|\sec\lambda + (1-\alpha)|s+t|)}{|2-s-t|\sec\lambda + (1-\alpha)|s+t|} \;\; (z \in \Delta) \tag{3.2}$$

The following constant factor

$$\frac{|2-s-t|\sec\lambda + (1-\alpha)|s+t|}{2\left((1-\alpha) + |2-s-t|\sec\lambda + (1-\alpha)|s+t|\right)} \tag{3.3}$$

is the best dominant.

Proof: Let $f(z) \in A$ satisfies the inequality (2.4) and suppose that $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in K$. Then

$$\frac{|2-s-t| \sec\lambda + (1-\alpha)|s+t|}{2((1-\alpha) + |2-s-t| \sec\lambda + (1-\alpha)|s+t|)} (f * g)(z)$$

$$= \frac{|2-s-t| \sec\lambda + (1-\alpha)|s+t|}{2((1-\alpha) + |2-s-t| \sec\lambda + (1-\alpha)|s+t|)} (z + \sum_{n=0}^{\infty} a_n c_n z^n) \tag{3.4}$$

Thus by definition (1.2), the assertion of our theorem will hold if the sequence

$$\left\{\frac{|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|}{2((1-\alpha)+|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|)}a_n\right\}_{n=1}^{\infty}$$

is subordinating factor sequence, with $a_1 = 1$. By virtue of Lemma (1.3), this will be the case if and only if

$$Re\left\{1+2\sum_{n=1}^{\infty}\frac{|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|}{2((1-\alpha)+|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|)}a_n z^n\right\}>0 \quad z\in\Delta \tag{3.5}$$

Now

$$\begin{aligned}
&Re\left\{1+\sum_{n=1}^{\infty}\frac{|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|}{(1-\alpha)+|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|}a_n z^n\right\}\\
&=Re\left\{1+\frac{|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|}{(1-\alpha)+|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|}z\right.\\
&\left.+\sum_{n=2}^{\infty}\frac{|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|}{(1-\alpha)+|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|}a_n z^n\right\}\\
&\geq 1-\sum_{n=1}^{\infty}\frac{|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|}{(1-\alpha)+|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|}r\\
&-\sum_{n=2}^{\infty}\frac{|n-u_n|\sec\lambda+(1-\alpha)|u_n|}{((1-\alpha)+|2-s-t|\sec\lambda+(1-\alpha)|s+t|}|a_n|r^n\\
&>1-\frac{|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|}{(1-\alpha)+|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|}r\\
&-\frac{(1-\alpha)}{(1-\alpha)+|2-s-t|\sec\lambda+(1-\alpha)|s+t|}r>0 \quad (|z|\leq r<1)
\end{aligned} \tag{3.6}$$

Thus (3.6) holds true in Δ. This proves the subordination result (3.1). The inequality (3.2) follows from (3.1) upon setting

$$g(z)=\frac{z}{1-z}=\sum_{n=1}^{\infty}z^n\in K \tag{3.7}$$

To prove sharpness of the constant given by (3.3), we consider the function f_0 defined by

$$f_0(z)=z-\frac{(1-\alpha)\sec\lambda}{(|2-s-t|\sec\lambda+(1-\alpha)|s+t|)}z^2$$

where

$$z\in\Delta, |t|\leq 1, s\neq t, s+t\neq 2, 0\leq\alpha<1 \text{ and } |\lambda|<\pi/2 \tag{3.8}$$

Then by using (3.1), we have

$$\frac{|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|}{2((1-\alpha)+|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|)}f_0(z)\prec\frac{z}{1-z} \tag{3.9}$$

It can be easily verified for the function $f_0(z)$ defined by (3.8) that

$$\min_{|z|\leq 1}Re\left\{\frac{|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|}{2((1-\alpha)+|2-s-t|\sec\lambda+(1-\alpha)\,|s+t|)}f_0(z)\right\}=-\frac{1}{2} \tag{3.10}$$

This shows that constant given by (3.3) is the best dominant. ■

We also consider the following useful consequence of the subordination Theorem (3.1). Upon setting $s = 1, t = -1$, we get

Corollary 3.2 Let $f(z) \in A$ is in $S(\alpha, 1, -1)$ and satisfies the inequality (2.4). Then for every $g \in K$, we have

$$\frac{\sec\lambda}{((1-\alpha)+2\sec\lambda)}(f*g)(z) \prec g(z) \tag{3.11}$$

In particular

$$Re\{f(z)\} > -\frac{((1-\alpha)+2\sec\lambda)}{2\sec\lambda} \quad (z \in \Delta) \tag{3.12}$$

The following constant factor

$$\frac{\sec\lambda}{((1-\alpha)+2\sec\lambda)} \tag{3.13}$$

is the best dominant.

Remark: Putting $s = 1$ in Theorem 3.1 we get known results of Goyal and Goswami [8] and by putting $t = 0, s = 1$ and $\alpha = 0$ in Theorem (3.1) we get a known result obtained by Singh [9].

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Author information

Trilok Mathur, Birla Institute of Technology and Science, Pilani, India.
E-mail: tmathur@pilani.bits-pilani.ac.in

Ruchi Mathur, Jaipur Engineering College and Research Centre, Jaipur, India.
E-mail: ruchs_21@yahoo.co.in

Deepa Sinha, South Asian University, New Delhi, India.
E-mail: deepasinha2001@gmail.com

Received: January 26, 2013.

Accepted: February 13, 2013.