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Weak Reflections and Remainders in Compactifications

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Abstract. In this paper, we introduce Fan-Gotttesman remainder.We characterize the spaces having a weak reflection in compact spaces as the spaces with the finite Fan-Gottesman remainder.

1 Introduction and Some Preliminaries

A compactification of a space X is a compact space containing X as a dense subspace. Recall the Fan-Gottesman compactification is defined by Ky Fan and Noel Gottesman. In 1952, they constructed the compactification for a space X with a base β for the open sets,containing $\emptyset$ and satisfying the following conditions:

(i) $A, B \in \beta$ implies $A \cap B \in \beta$

(ii) $A \in \beta$ implies $X - clA \in \beta$

(iii) For every open set U in X and every $A \in \beta$ such that $clA \subset U$, there exists a set $B \in \beta$ such that $clA \subset B \subset clB \subset U$

The compactification associated with β is obtained as follows. A centered system x^* is defined as a family of elements in β such that

$$\cap_{i=1}^{m} clA_i \neq \varnothing$$

for every finite family $A_1, A_2, ..., A_m$ in x^*. By Teichmuller-Tukey lemma each centered system of β is contained in a maximal one. The set of all maximal centered systems of β is denoted by X^*. For each $A \in \beta, A^*$ is defined as the set of all maximal centered systems x^* such that $B \in x^*$ for some $B \in \beta$ where $clA \subset B$. Namely;

$$A^* = \{x^* \in X^* : \text{there exists a } B \in x^* \text{ with } clA \subset B\}$$

A topology on X^* is defined by taking

$$\beta^* = \{A^* : A \in \beta\}$$

as a base for its open sets. X^* is compact and Hausdorff space with this topology. X^* is called Fan-Gottesman compactification [2].

Now,we will define this compactification via ultra-open filter

$$X^* = X \cup \{cl\,(G) : G \in y,\ y \text{ is a nonconvergent ultra-open filter in } X\ \}$$

where "ultra-open" means maximal among all filters, having a base consisting of open sets.The sets

$$S\,(G) = G \cup \{cl\,(G) : y \in X^* - X, cl\,(G) \in y\}$$

where G is open in X, constitute an open base of X^*.

In this paper,we will say that $X^* - X$ is Fan-Gottesman remainder.

The notion of weak reflection is a natural generalization of the concept of *reflection.* It is well known that any continuous mapping $f : X \to Y$ to compact Hausdorff space Y may be uniquely factorized through the Stone-Ćech compactification βcX of the completely regular T_1-modification cX of X. Then the space BcX is the reflection of X in compact Hausdorff spaces.

Any compactification γX of X is said to be *a weak reflection* of X in the class of compact space if for every compact Y and every continuos mapping $f : X \to Y$ there exists a mapping

$g : \gamma X \to Y$ continuously extending f. Note that for a compact Hausdorff space Y, the Fan-Gottesman compactification X^* also has the extension property described above.

It is natural to ask whether every topological space has, at least, a weak reflection in general compact space. This question was asked by J. Adámek and J.Rosický [1] and was answered in the negative by M.Hus̈ek [3]. In fact, he described some spaces having a weak reflection in compact spaces and some spaces having no weak reflection in compact spaces. He also fully characterized all normal spaces which have a weak reflection in compact spaces; they are exactly the spaces with the finite Wallman (or, equivalently, Stone-Ćech) remainder. In this paper we show that every space with finite Fan-Gottesman remainder has a weak reflection in compact regular space.

2 Main Results

Theorem 2.1. *If the Fan-Gottesman remainder of X is finite, then the Fan-Gottesman compactification of X is the weak reflection of X in compact space.*

Proof. Let $X^* - X$ is finite, $f : X \to Y$ be continuous, Y be compact. For $x \in X^* - X$ put $f^{\sim}(x)$ to be an accumulation point of $\{A : clA \text{ is in } Y,\ f^{-1}(A) \in x\}$ for $x \in X$ define $f^{\sim}(x) = f(x)$. We shall prove that $f^{\sim} : X^* \to Y$ is continuous. Clearly, $f^{\sim}$is continuous on X since the restriction of $f^{\sim}$ to X coincides with f and X is open in X^*. Take $x \in X^* - X$ and an open set G in Y containing $f^{\sim}(x)$. Then $f^{-1}(G)$ is open in X. Since there is some $F \in x$ such that $F \subset f^{-1}(G)$,we can choose an open subset U of $f^{-1}(G)$ such that $X - U$ belongs to all elements of $X^* - X$ but not to x. Then $U \cup (x)$ is neighborhood of in X^* and $f^{\sim}$ maps this neigborhood into G. □

Lemma 2.2. *Let X be an infinite topological T_1 space.Then X contains an infinite subspace with discrete space or an infinite subspace with the topology of finite complements [4].*

Lemma 2.3. *If X contains an infinite family $\{F_i\}$ of closed noncompact subsets such that $F_p \cap F_q$ is compact for $p \neq q$, then X has no weak reflection in compact space [3].*

Lemma 2.4. *Let X be a topological space. Suppose that $X^* - X$ contains an infinite subspace with discrete topology. Then there exists a sequence $F_1, F_2, \ldots$of closed noncompact subsets of X which are pairwise disjoint.*

Proof. Assume that $\mathbb{N} \subseteq X^* - X$ and the topology of $\mathbb{N}$, induced from X^*, is discrete. Let ß be collection of all open sets in X. There exist open sets $G_n \in$ß , $n \in \mathbb{N}$ such that $n \in S(G_n)$ and $m \notin S(G_n)$ for $n \neq m$. Since $G_n \in n$, there is some open set U_n such that $U_n \in n$ with $U_n \subseteq G_n$. The sets $F_n = U_n - \cup_{i=1}^{n-1} G_i$, where $n \in \mathbb{N}$, constitutes the desired family. □

Indeed, every F_n is closed and disjoint from F_m for $n \neq m$. The noncompactness of F_n follows from the fact that every $n \in \mathbb{N} \subseteq X^* - X$ constitutes a nonconvergent ultra-closed filter in X.

Lemma 2.5. *Let X be a topological space. Suppose that $X^* - X$ contains an infinite subspace having the topology of finite complements. Then there exists a sequence $H_1, H_2, \ldots$of closed noncompact subsets of X which are pairwise disjoint.*

Proof. We may assume that $\mathbb{N} \subseteq X^* - X$ and the topology of $\mathbb{N}$, induced from X^*, is the topology of finite complements**.** Denote by ß the collection of all open sets in X. By induction we define the desired sequence:

(i) Let $N_1 = \mathbb{N}, x_1 = 1, y_1 = 2$. There exists an open set $G_1 \in$ ß such that $x_1 \notin S(G_1)$ and $y_1 \in S(G_1)$. Since $N_1 \cap S(G_1) \neq \varnothing$, the set $N_1 - S(G_1)$ is finite and then the set $N_2 = N_1 \cap S(G_1)$ is infinite. Since $x_1 \notin S(G_1)$ it follows that $G_1 \notin x_1$ which implies that $X - G_1 \in x_1$. We put $F_1 = X - G_1$. Evidently, F_1 is closed in X and nonempty since $F_1 \in x_1$. Morever, it is noncompact because x_1 is a nonconvergent ultraclosed filter in X.

(ii) Suppose that for some $k \geq 1$ there exist open set $G_1, G_2, \ldots G_k \in$ß sets $N_1, N_2, \ldots N_k \subseteq \mathbb{N}$ and noncompact closed sets $F_1, F_2, \ldots, F_k \subseteq X$ such that

i $G_1 \supseteq G_2 \supseteq \ldots \supseteq G_k$,

ii $G_{i+1} = G_i \cap S(G_i)$ for $i = 1, 2, \ldots, k$,

iii G_{k+1} is finite ,

iv $F_i \subseteq (X - G_i) \cap G_{i-1}$ for $i = 2, 3, \ldots k$.

We shall prove that (2) is fullfilled for $k + 1$.

By (iii) there are two distinct points $x_{k+1}, y_{k+1} \in N_{k+1}$ and an open set $G_{k+1} \in$ß such that $x_{k+1} \notin S(G_{k+1})$ and $y_{k+1} \in S(G_{k+1})$. Because $N_{k+1} \subseteq S(G_k)$ by (ii), one can easily check that we may assume $G_k \supseteq G_{k+1}$. Hence (i) is fulfilled. We put $N_{k+2} = N_{k+1} \cap S(G_{k+1})$. The set $\mathbb{N} \cap S(G_{k+1})$ is open in $\mathbb{N}$ and nonempty because it contains y_{k+1}. Hence its complement $\mathbb{N} - S(G_{k+1})$ is finite; therefore $N_{k+1} - S(G_{k+1})$ is also finite.It follows that N_{k+2} is infinite. Notice that (ii) and (iii) are satisfied for $k + 1$. Since $x_{k+1} \notin S(G_{k+1})$,it follows $G_{k+1} \notin x_{k+1}$ and then $X - G_{k+1} \in x_{k+1}$. On the other hand, since $x_{k+1} \in N_{k+1} \subseteq S(G_k)$,we have $G_k \in x_{k+1}$. Then there exists a set $U_{k+1} \in x_{k+1}$, closed in X, such that $U_{k+1} \subset G_k$. We put

$$F_{k+1} = U_{k+1} \cap (X - G_{k+1})$$

Evidently, F_{k+1} is closed in X and since $F_{k+1} \in x_{k+1}$ it is nonempty. Moreover, it is noncompact because x_{k+1} is a nonconvergent ultraclosed filter in X. Since $U_{k+1} \subset G_k$ it follows that $F_{k+1} \subseteq (X - G_{k+1}) \cap G_k$. Hence (iv) is fulfilled for $k + 1$, which completes the induction.

Now let $p, r \in \mathbb{N}, p < r$. Then, by (iv) and (i), it follows that $F_r \subseteq G_{r-1} \subseteq \cdots \subseteq G_p$. On the other hand, by (iv) we have $F_p \subseteq X - G_p$,which implies that $F_p \cap F_s = \emptyset$. It follows that $\{F_i\}_{i \in \mathbb{N}}$ is the desired sequence. □

Theorem 2.6. *Let X be a topological space. Then the following statements are equivalent:*

(i) The Fan-Gotteman compactification of X is its weak reflection in compact spaces.

(ii) The space X has a weak reflection in compact spaces.

(iii) There exists such that any pairwise disjoint family of closed sets in X contains at most k noncompact elements.

(iv) Every infinite sequences $F_1, F_2, \ldots$ of closed sets such that $F_p \cap F_q$ is compact for $p \neq q$ has a compact member

(v) The Fan-Gottesman remainder of X is finite.

Proof. We will show that $(i) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i)$ and $(vi) \Rightarrow (iii) \Rightarrow (v)$. But $(i) \Rightarrow (ii), (iii) \Rightarrow (v)$ and $(iv) \Rightarrow (v)$ are clear; Lemma 2.2 implies that $(ii) \Rightarrow (iv)$. The implication $(vi) \Rightarrow (i)$ follows from Theorem 2.1

$(v) \Rightarrow (vi)$ Suppose that Fan-Gottesman remainder is infinite. Since the Fan-Gottesman remainder is a T_1-space, it follows from Lemma 2.1 that Fan-Gottesman remainder contains an infinite discrete subspace or an infinite subspace with the topology of finite complements. Then by Lemma 2.3 or Lemma 2.4, we obtain that there is a sequence $F_1, F_2, \ldots$ of subsets of X which are closed, noncompact and pairwise disjoint.

$(vi) \Rightarrow (iii)$ Let $k \in \mathbb{N}$ be the cardinality of Fan-Gottesman remainder. Assume that for $m \in \mathbb{N}$ there are pairwise disjoint, closed and noncompact sets $F_1, F_2, \ldots, F_m \subseteq X$. Then since every F_i is noncompact and closed, there are nonconvergent ultra-closed filters $y_1, y_2, \ldots, y_m \in X^* - X$ such that $F_i \in y_i$ for every $i = 1, 2, \ldots, m$. Since $F_p \cap F_q = \emptyset$ for $p \neq q$,it follows that $y_p \neq y_q$. Therefore $m \leq k$, which completes the proof. □

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