DERIVED FUNCTORS USING \mathcal{X} -INJECTIVE CORESOLUTIONS AND DIMENSIONS

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Abstract In this paper, we take \mathcal{X} is the class of all pure projective modules and introduce a right derived functor $Fixt_R^n(-,-)$ using \mathcal{X} -injective coresolutions. It is shown that $Fixt_R^1(M,N) \to Ext_R^1(M,N)$ is an isomorphism for all R modules N if and only if M is \mathcal{X}^{\perp} -projective. If R is a Noetherian ring, then we show that $glLcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M}) \leq n-2$ if and only if $Fcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M}) \leq n$. Finally, we show that every left R-module has an \mathcal{X} injective cover with the unique mapping property if and only if every pure injective R-module is \mathcal{X} -injective.

1 Introduction

The notions of covers and envelopes of modules were introduced by Enochs in [1]. Let \mathscr{C} be a class of left *R*-modules. Following [1], we say that a map $f \in Hom_R(C, M)$ with $C \in \mathscr{C}$ is a \mathscr{C} -precover of *M*, if the group homomorphism $Hom_R(C', f) : Hom_R(C', C) \to Hom_R(C', M)$ is surjective for each $C' \in \mathcal{C}$. A \mathscr{C} -precover $f \in Hom_R(C, M)$ of *M* is called a \mathscr{C} -cover of *M* if *f* is right minimal. That is, if fg = f implies that *g* is an automorphism for each $g \in End_R(C)$. $\mathscr{C} \subseteq R$ -Mod is a precovering class (covering class) provided that each module has a \mathscr{C} -precover (\mathcal{C} -cover). Dually, we have the definition of \mathscr{C} preenvelope (\mathscr{C} envelope).

Given a class \mathscr{C} of left *R*-modules, we write

$$\begin{split} \mathscr{C}^{\perp} &= & \left\{ N \in R\text{-}Mod \,|\, Ext^{1}_{R}(M,N) = 0, \,\,\forall \, M \in \mathscr{C} \right\} \\ ^{\perp}\mathscr{C} &= \,\, \left\{ N \in R\text{-}Mod \,|\, Ext^{1}_{R}(N,M) = 0, \,\,\forall \, M \in \mathscr{C} \right\} \end{split}$$

A \mathscr{C} -precover f of M is said to be *special* if f is an epimorphism and ker $f \in \mathscr{C}^{\perp}$.

A \mathscr{C} -preenvelope f of M is said to be *special* if f is a monomorphism and *coker* $f \in {}^{\perp}\mathscr{C}$.

A module is said to be *pure projective* [5] if it is projective with respect to pure exact sequence.

The notions of FP-injective modules and FP-injective dimensions of modules and rings were first introduced by Stenström in [9]. The FP-injective dimension of an R-module M, denoted by FP-id(M), is defined to be the smallest nonnegative integer n such that M has an FP-injective resolution of length n. L. Mao and N. Ding in [4] introduced by \mathcal{X} -injective modules. An R-module M is called \mathcal{X} -injective if $Ext_R^1(A, M) = 0$ for all R-modules $A \in \mathcal{X}$.

In this paper, we take \mathcal{X} is the class of all pure projective modules. \mathcal{X}^{\perp} is the class of all \mathcal{X} -injective modules. An R-module M is \mathcal{X}^{\perp} -projective if $Ext_R^1(M, U) = 0$ for all R-modules $U \in \mathcal{X}^{\perp}$. $^{\perp}(\mathcal{X}^{\perp})$ is the class of all \mathcal{X}^{\perp} -projective modules. Clearly, $(^{\perp}(\mathcal{X}^{\perp}), \mathcal{X}^{\perp})$ is a cotorsion theory.

Throughout this paper, R denotes a non-trivial associative ring with identity and \mathcal{M} denotes a category of left R-modules. Clearly, \mathcal{M} is an abelian category with enough injectives. The class \mathcal{X}^{\perp} of \mathcal{X} -injective modules is a full subcategory which is closed under isomorphisms. Similarly, a subcategory of a subcategory \mathcal{X}^{\perp} of \mathcal{M} always means a full subcategory of \mathcal{X}^{\perp} which is closed under isomorphisms. By $\mathcal{I}(\mathcal{M})$ we denote the classes of all injective objects of a category \mathcal{M} .

If \mathcal{X} is the class of all pure projective modules, then every left *R*-module has an \mathcal{X}^{\perp} coresolution over an arbitrary ring by [8, Theorem 2.2]. Then, the functor Hom(-, -) is right
balanced on *R*-*Mod* × *R*-*Mod* by $\mathcal{X}^{\perp} \times \mathcal{X}^{\perp}$. In [7], if \mathcal{X} is the class of all pure projective
modules, then every *R*-module has an \mathcal{X} -injective precover over a Noetherian ring *R*. It follows

that every *R*-module has an \mathcal{X}^{\perp} -resolution. Hence, if *R* is a Noetherian ring, then Hom(-, -) is left balanced.

Let $Fixt^n(-,-)$ denote the *n*th right derived functor of Hom(-,-) with respect to the pair $\mathcal{X}^{\perp} \times \mathcal{X}^{\perp}$. Then, for two left *R*-modules *M* and *N*, $Fixt^n(M,N)$ can be computed using a \mathcal{X}^{\perp} -resolution of *M* or a \mathcal{X}^{\perp} -coresolution of *N*. Also we denote $Fixt_n(-,-)$ is the *n*th left derived functor of Hom(-,-) with respect to the pair $\mathcal{X}^{\perp} \times \mathcal{X}^{\perp}$. Then, for two left *R*-modules *M* and *N*, $Fixt_n(M,N)$ can be computed using a \mathcal{X}^{\perp} -coresolution of *M* or a \mathcal{X}^{\perp} -resolution of *N*.

The left \mathcal{X}^{\perp} -dimension of a left R-module M, denoted by $Lcores.dim_{\mathcal{X}^{\perp}}(M)$, is defined as inf{n: there is a \mathcal{X}^{\perp} -resolution of the form $0 \to G_n \to \cdots \to G_0 \to M \to 0$ of M}. If there is no such n, set left $Lcores.dim_{\mathcal{X}^{\perp}}(M) = \infty$. The global left \mathcal{X}^{\perp} -dimension of \mathcal{M} , denoted by $glLcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M})$, is defined to be $sup\{Lcores.dim_{\mathcal{X}^{\perp}}(M): M \in \mathcal{M}\}$ and is infinite otherwise. The right versions can be defined similarly. We denote by $Rcores.dim_{\mathcal{X}^{\perp}}(M)$ the right \mathcal{X}^{\perp} -dimension of a left R-module M and we denote by $glRcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M})$ the global right \mathcal{X}^{\perp} -dimension of \mathcal{M} .

This paper is organized as follows: In Section 2, we take \mathcal{X} is the class of all pure projective modules and we introduce a right derived functor $Fixt_R^n(-,-)$ using \mathcal{X}^{\perp} -injective coresolutions. It is shown that $Fixt_R^1(M, N) \to Ext_R^1(M, N)$ is an isomorphism for all R modules N if and only if M is \mathcal{X}^{\perp} -projective.

In Section 3, we investigate the \mathcal{X} -injective dimension of modules and rings and the left derived functors $Fixt_n(-,-)$. Let R be a Noetherian ring. We prove that $Fcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M}) \leq 1$ if and only if the canonical map μ : $Fixt_0(_RR, N) \rightarrow Hom(_RR, N)$ is a monomorphism for any left R-module N. Then, it is shown that $Fcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M}) \leq n(n \geq 2)$ if and only if $Fixt_{n+k}(\mathcal{M}, N) = 0$ for all left R-modules N and all $k \geq -1$. Moreover, $glLcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M}) \leq n - 2(n \geq 2)$ if and only if $Fixt_{n+k}(\mathcal{M}, N) = 0$ for all pure injective R-modules N and all $k \geq -1$. Finally, every left R-module has an \mathcal{X}^{\perp} -injective cover with the unique mapping property if and only if $Fcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M}) \leq 2$.

2 The Right Derived Functors using \mathcal{X} -injective coresolutions:

In this section, we investigate right derived functors using \mathcal{X}^{\perp} -coresolution. By [8, Theorem 2.2], every *R*-module has an \mathcal{X}^{\perp} -coresolution. Let $0 \to N \to G^0 \to G^1 \to \cdots$ be an \mathcal{X}^{\perp} -coresolution of *N*. This complex is unique up to homotopy. This leads us to new derived functors $Fixt_R^n(-,-)$, which are well defined. Applying the functor $Hom_R(M,-)$, we obtain the deleted complex

$$\mathbb{G}^{\bullet}: 0 \to Hom_R(M, G^0) \to Hom_R(M, G^1) \to \cdots$$

Then, we can define $Fixt_{R}^{n}(M, N) = H^{n}(Hom(M, \mathbb{G}^{\bullet})).$

Theorem 2.1. For $M, N \in Obj(\mathcal{M})$, then the $Fixt_R^n(M, N)$ are well defined.

Proof. For any *R*-module *N*, there is an \mathcal{X}^{\perp} -coresolution of *M*

$$0 \to N \stackrel{\epsilon}{\to} G^0 \stackrel{\alpha_1}{\to} G^1 \stackrel{\alpha_2}{\to} \cdots$$

and an *R*-module \overline{N} , there is an \mathcal{X}^{\perp} -coresolution of \overline{N}

$$0 \to \overline{N} \xrightarrow{\epsilon'} \overline{G^0} \xrightarrow{\alpha'_1} \overline{G^1} \xrightarrow{\alpha'_2} \cdots$$

Let $\gamma \in Hom_R(N, \overline{N})$. We only need to show that there is a commutative diagram

and the associated map of \mathcal{X}^{\perp} -coresolutions is unique up to homotopy.

Every *R*-module has a special \mathcal{X}^{\perp} -preenvelope. Then, *N* has an \mathcal{X}^{\perp} -preenvelope $\overline{G^0}$ and \overline{N} has an \mathcal{X}^{\perp} -preenvelope $\overline{G^0}$ such that $\gamma_0 \in Hom_R(\overline{G^0}, \overline{G^0}), \epsilon' \in Hom_R(\overline{N}, \overline{G^0})$ and $\gamma_0 \epsilon = \epsilon' \gamma$. Then the following diagram is commutative

where L^1 and $\overline{L^1}$ are cokernel of ϵ and ϵ' . Let G^1 and $\overline{G^1}$ are \mathcal{X}^{\perp} -preenvelop of L^1 and $\overline{L^1}$, respectively. Then, the diagram

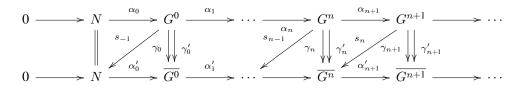
$$\begin{array}{cccc} 0 & \longrightarrow & L^{1} & \longrightarrow & G^{1} \\ & & & & & \\ & & & & & \\ l_{1} & & & & & \\ \gamma_{1} & & & & \\ \gamma_{1} & & & & \\ \psi & & & & \\ 0 & \longrightarrow & \overline{L^{1}} & \longrightarrow & \overline{G^{1}} \end{array}$$

is commutative. Continuing this process, there is $\gamma_{n-1} \in Hom_R(G^{n-1}, \overline{G^{n-1}})$ such that the following diagram

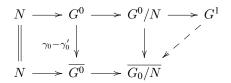
is commutative, where G^i and $\overline{G^i}$ are \mathcal{X} -injective for each $i \in \{0, 1, \dots, n-1\}$, L^n and $\overline{L^n}$ are cokernals. Let G^n and $\overline{G^n}$ are \mathcal{X}^{\perp} -preenvelop of L^n and $\overline{L^n}$, respectively. Then, there exists $\gamma_n \in Hom_R(G^n, \overline{G^n})$ such that the diagram

is commutative. It follows that we can complete the diagram.

We are now to prove the uniqueness up to homotopy, that is, to prove that from the following diagram

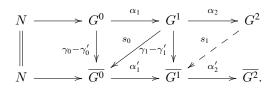


there exist s_0, \dots, s_n, \dots , with $s_n: G^{n+1} \to \overline{G^n}$ such that $\gamma_n - \gamma'_n = \alpha'_n \circ s_{n-1} + s_n \circ \alpha_{n+1}$, where $s_{-1} = 0$. We know that $\gamma_0 \alpha_0 = \alpha'_0 = \gamma'_0 \alpha_0$. It follows that $(\gamma_0 - \gamma'_0)\alpha_0 = 0$. Then, we have the diagram

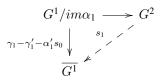


which can be completed since G^1 is an \mathcal{X}^{\perp} -preenvelope, that is, there exists $s_0 \in Hom_R(G^1, \overline{G^0})$ such that $\gamma_0 - \gamma'_0 = s_0 \circ \alpha_1$.

We will show that the following diagram is commutative, that is, a map on G^0 which is zero.



Let s_1 be the map which completes the following diagram



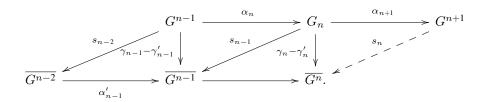
Hence,

$$\begin{aligned} (\gamma_1 - \gamma_1' - \alpha_1' s_0) \alpha_1 &= (\gamma_1 - \gamma_1') \alpha_1 - (\alpha_1' s_0) \alpha_1 \\ &= (\gamma_1 - \gamma_1') \alpha_1 - \alpha_1' (\gamma_0 - \gamma_0') \\ &= 0, \end{aligned}$$

as desired. Continuing this process, we can find s_2, \dots, s_{n-1} . Define s_n as the completion of the following diagram

$$\begin{array}{c} G^{1}/im\alpha_{1} \longrightarrow G^{2} \\ \gamma_{1} - \gamma_{1}' - \alpha_{1}'s_{0} \\ \downarrow \\ \hline G^{1} \end{array} \xrightarrow{s_{1}} \overbrace{C^{1}}^{s_{1}} \overbrace{}^{s_{1}} \end{array}$$

It follows that we can get a following commutative diagram



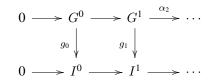
Since the previous diagram has exact rows, then

$$\begin{aligned} (\gamma_n - \gamma'_n - \alpha'_n s_{n-1})\alpha_n &= (\gamma_n - \gamma'_n)\alpha_n - \alpha'_n (s_{n-1}\alpha_n) \\ &= (\gamma_n - \gamma'_n)\alpha_n - \alpha'_n (\gamma_{n-1} - \gamma'_{n-1} - \alpha'_{n-1}s_{n-1}) \\ &= (\gamma_n - \gamma'_n)\alpha_n - \alpha'_n (\gamma_{n-1} - \gamma'_{n-1}) + (\alpha'_{n-1}s_{n-1}) \\ &= 0. \end{aligned}$$

Hence, the \mathcal{X}^{\perp} -coresolution is unique up to homotopy.

From the homology groups of this \mathcal{X}^{\perp} -coresolution gives a well defined derived functor which we will call $Fixt_R^n(M, N)$. Let $0 \to M \to G^0 \to G^1 \to \cdots$ be an \mathcal{X}^{\perp} -coresolution of M and $0 \to M \to I^0 \to I^1 \to \cdots$ an $\mathcal{I}(\mathcal{M})$ -coresolution of M. Since I_0 is injective, there exists $g_0 \in Hom_R(G^0, I^0)$ such that $\alpha' = g \circ \alpha$. By the injectivity of I^1 , there exists $g_1 \in Hom_R(G^1, I^1)$ such that $g_1\alpha_0 = \alpha'_0g_0$. Continuing this process and using analog proof of Theorem 2.1, we can complete the diagram

to a commutative diagram uniquely, up to homotopy. Now applying $Hom_R(M, -)$ to the diagram



gives natural maps $Fixt_{B}^{n}(M,G) \to Ext_{B}^{n}(M,G)$ for all $n \ge 0$.

Proposition 2.2. For $M, G \in Obj(\mathcal{M})$, $Fixt^0_B(M, G) \cong Hom_R(M, G)$.

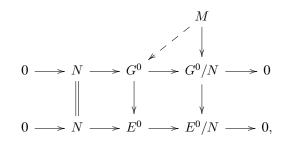
Proof. Let $0 \to M \to G^0 \to G^1 \to \cdots$ be a \mathcal{X}^{\perp} -coresolution of M. Then, the homotopy groups of the complex

$$0 \to Hom_R(M, G^0) \to Hom_R(M, G^1) \to \cdots$$

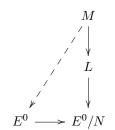
gives us the groups $Fixt_R^n(M,G)$. Hence, $Fixt_R^0(M,G)$ is the kernel of $Hom_R(M,G^0) \rightarrow Hom_R(M,G^1)$. But the functor $Hom_R(M,-)$ is left exact. So if the sequence $0 \rightarrow G \rightarrow G^0 \rightarrow G^1$ is exact, then the sequence $0 \rightarrow Hom_R(M,G^0) \rightarrow Hom_R(M,G^1) \rightarrow \cdots$ is exact. Now $Hom_R(M,G)$ is isomorphic to the kernel of $Hom_R(M,G^0) \rightarrow Hom_R(M,G^1)$, that is, $Hom_R(M,G) \cong Fixt_R^0(M,G)$, as desired.

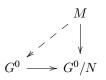
Proposition 2.3. For $M, G \in Obj(\mathcal{M})$, $Fixt^1_{\mathcal{B}}(M, N) \to Ext^1_{\mathcal{B}}(M, N)$ is injective.

Proof. Let $0 \to N \to G^0 \to G^1 \to \cdots$ be an \mathcal{X}^{\perp} -coresolution of N and $0 \to N \to E^0 \to E^1 \to \cdots$ be an $\mathcal{I}(\mathcal{M})$ -coresolution of N. Consider the exact sequences $0 \to N \to G^0 \to G^0/N \to 0$ and $0 \to N \to E^0 \to E^0/N \to 0$. From the following diagram with exact rows



we get the following diagram





can also be completed, as desired.

Theorem 2.4. For $M \in Obj(\mathcal{M})$, M is \mathcal{X}^{\perp} -projective if and only if $Fixt^{1}_{R}(M, N) \rightarrow Ext^{1}_{R}(M, N)$ is an isomorphism for all R modules N.

Proof. Suppose M is \mathcal{X}^{\perp} -projective. By Theorem [8, Theorem 2.2], N has an \mathcal{X} -injective coresolution, $0 \to N \xrightarrow{f} G^0 \to G^1 \to \cdots$. Consider the short exact sequence $0 \to N \xrightarrow{f} G^0 \to G^0/imf \to 0$. Then, we get the following commutative diagram with exact rows:

Hence, $Fixt_R^1(M, N) \cong Ext_R^1(M, N)$ for any *R*-module *N*. Conversely, we may assume that $Fixt_R^1(M, N) \to Ext_R^1(M, G)$ is an isomorphism for all *R*-modules *N*. Then, $Ext_R^1(M, G) = 0$ for all \mathcal{X} -injective *R*-modules *G* since $Fixt_R^1(M, G) = 0$ for all \mathcal{X} -injective *R*-modules *G* and *R*-modules *M*. Thus, *M* is \mathcal{X}^{\perp} -projective.

3 X-injective dimensions and the left derived functors of Hom

In this section, we deals with the \mathcal{X} -injective dimensions of modules and the left derived functors $Fixt_n(-, -)$.

For $M \in Obj\mathcal{M}$, there exists an \mathcal{X} -injective coresolution of M such that $0 \to M \xrightarrow{f} G^0 \xrightarrow{g_0} G^1 \to \cdots$. Then, we obtain the deleted complex

$$\mathbb{G}_{\bullet}: \cdots \to Hom(G', N) \xrightarrow{g_0} Hom(G^0, N) \to \mathbb{C}$$

when we apply the functor Hom(-, N). It follows that $Fixt_n(M, N)$ is exactly *n*th homology of the complex, that is, $Fixt_n(M, N) = H_n(Hom(\mathbb{G}_{\bullet}, N))$. Then, there is a canonical map

$$\mu$$
: $Fixt_0(M, N) = Hom(G^0, N)/im(g_0^{\star}) \rightarrow Hom(M, N)$

defined by $\mu(h + im(g_0^*)) = hg_0$ for $h \in Hom(G^0, N)$.

Proposition 3.1. Let R be a Noetherian ring. For $M \in Obj\mathcal{M}$, M is \mathcal{X} -injective if and only if the canonical map μ : $Fixt_0(M, N) \rightarrow Hom(M, N)$ is an epimorphism for any left R-module N.

Proof. The direct implication is clear if $G^0 = M$. Conversely, if N = M then there exists $h \in Hom(G^0, M)$ such that $\mu(h+im(g_0^{\star})) = hg_0 = 1_M$. Thus, M is isomorphic to a direct summand of G^0 . Since the direct summand of \mathcal{X} -injective module is \mathcal{X} -injective, M is \mathcal{X} -injective. \Box

Corollary 3.2. Let R be a Noetherian ring. Then, the following conditions are equivalent:

- (i) $_{R}R$ is X-injective;
- (ii) The canonical map μ : $Fixt_0(_RR, N) \rightarrow Hom(_RR, N)$ is an epimorphism for any left *R*-module *N*;
- (iii) The canonical map μ : $Fixt_0(RR, RR) \to Hom(RR, RR)$ is an epimorphism;
- *(iv)* Every *R*-module has a special X-injective cover.

Proof. (1) \Leftrightarrow (2) \Leftrightarrow (3) follows from Proposition 3.1.

(1) \Rightarrow (4). For any left *R*-module *M*, *M* has an \mathcal{X} -injective precover. Consider the exact sequence $F \rightarrow M \rightarrow 0$ with *F* a free module. By hypothesis, *F* is \mathcal{X} -injective.

 $(4) \Rightarrow (1)$. Let $\phi: M \rightarrow {}_{R}R$ be a special \mathcal{X} -injective cover. Then, ${}_{R}R$ is isomorphic to a direct summand of M. Thus, ${}_{R}R$ is \mathcal{X} -injective. \Box

Proposition 3.3. Let R be a Noetherian ring. Then, the following are equivalent:

(i) $Rcores.dim_{\mathcal{X}^{\perp}}(M) \leq 1;$

0

(ii) The canonical map μ : $Fixt_0(M, N) \to Hom(M, N)$ is a monomorphism for any left *R*-module *N*.

Proof. (1) \Rightarrow (2). By hypothesis, M has a \mathcal{X}^{\perp} -coresolution $0 \rightarrow G^0 \rightarrow G^1 \rightarrow 0$. It follows that we get an exact sequence $0 \rightarrow Hom(G^1, N) \rightarrow Hom(G^0, N) \rightarrow Hom(M, N)$ for any R-module N. Hence, μ is a monomorphism.

 $(2) \Rightarrow (1)$. Consider the exact sequence $0 \rightarrow M \rightarrow G^0 \rightarrow L^1 \rightarrow 0$, where $M \rightarrow G^0$ is an \mathcal{X} -injective preenvelope. We only need to show that L^1 is \mathcal{X} -injective. By [2, Theorem 8.2.3], we have the commutative diagram with exact rows:

$$Fixt_0(L^1, L^1) \longrightarrow Fixt_0(G^0, L^1) \longrightarrow Fixt_0(M, L^1) \longrightarrow 0$$
$$\mu_1 \downarrow \qquad \mu_2 \downarrow \qquad \mu_3 \downarrow$$
$$\longrightarrow Hom(L^1, L^1) \longrightarrow Hom(G^0, L^1) \longrightarrow Hom(M, L^1).$$

Note that α_2 is an epimorphism by Proposition 3.1 and μ_3 is a monomorphism by hypothesis. Hence, μ_1 is an epimorphism by the Snake Lemma [6, Theorem 6.5]. By Proposition 3.1, L^1 is \mathcal{X} -injective.

Lemma 3.4. Let R be a Noetherian ring. Then, $Rcores.dim_{\chi^{\perp}}(M) = cores.dim_{\chi^{\perp}}(M)$ for any left R-module M.

Proof. Clearly, $cores.dim_{\mathcal{X}^{\perp}}(M) \leq Rcores.dim_{\mathcal{X}^{\perp}}(M)$. Conversely, we assume that $cores.dim_{\mathcal{X}^{\perp}}(M) = n < \infty$. Consider the partial \mathcal{X}^{\perp} -coresolution of M, $0 \to G^0 \to G^1 \to \cdots \to G^{n-1}$. Then, we get an exact sequence $0 \to G^0 \to G^1 \to \cdots \to G^{n-1} \to L \to 0$. Hence, L is \mathcal{X} -injective, and so $Rcores.dim_{\mathcal{X}^{\perp}}(M) \leq n$, as desired. \Box

The finitistic \mathcal{X}^{\perp} -coresolution of dimension, denoted by $Fcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M})$, is defined as $\sup\{cores.dim_{\mathcal{X}^{\perp}}(M): M \in Obj(\mathcal{M})\}.$

Proposition 3.5. Let R be a Noetherian ring. Then, the following are equivalent:

- (i) $Fcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M}) \leq 1;$
- (ii) The canonical map μ : $Fixt_0(RR, N) \rightarrow Hom(RR, N)$ is a monomorphism for any left *R*-module *N*.

Proof. (1) \Leftrightarrow (2). It follows by Proposition 3.3 and Lemma 3.4.

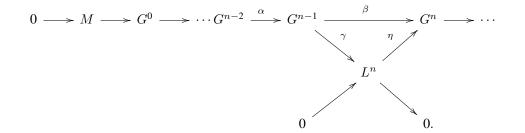
Proposition 3.6. Let R be a Noetherian ring and an integer $n \ge 2$. Then, the following conditions are equivalent:

- (i) $Rcores.dim_{\mathcal{X}^{\perp}}(M) \leq n;$
- (ii) $Fixt_{n+k}(M, N) = 0$ for all R-modules N and all $k \ge -1$;
- (iii) $Fixt_{n-1}(M, N) = 0$ for all R-modules N.

Proof. (1) \Rightarrow (2). Note that $Fixt_{n+k}(M, N) = 0$ for all $k \ge 1$. We only need to show that $Fixt_{n+k}(M, N) = 0$ for all $k \in \{-1, 0\}$. Consider \mathcal{X}^{\perp} -coresolution of $M, 0 \to G^0 \to G^1 \to \cdots \to G^n \to 0$. Which induces an exact sequence $0 \to Hom(G^n, N) \to Hom(G^{n-1}, N) \to Hom(G^{n-2}, N)$ for any R-module N. Hence, $Fixt_n(M, N) = Fixt_{n-1}(M, N) = 0$.

 $(2) \Rightarrow (3)$ is trivial.

(3) \Rightarrow (1). Consider the \mathcal{X}^{\perp} -coresolution of $M, 0 \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots \rightarrow G^n \rightarrow 0$. with $L^n = coker(G^{n-2} \rightarrow G^{n-1})$. We have the following commutative diagram



Since $Fixt_{n-1}(M, L^n) = 0$, the sequence

$$Hom(G^n, L^n) \xrightarrow{\beta^n} Hom(G^{n-1}, L^n) \xrightarrow{\alpha^{\hat{}}} Hom(G^{n-2}, L^n)$$

is exact. Now $\alpha^*(\gamma) = \gamma \alpha = 0$. It follows that $\gamma \in ker(\alpha^*) = im(\beta^*)$. Then, there exists $\nu \in Hom(G^n, L^n)$ such that $\gamma = \beta^*(\nu) = \nu\beta = \nu\eta\gamma$, and hence $\nu\eta = 1$ since γ is epic. Therefore L^n is \mathcal{X} -injective, as desired.

Corollary 3.7. Let R be a Noetherian ring and an integer $n \ge 2$. Then, the following are equivalent:

- (i) cores.dim_{χ^{\perp}}(\mathcal{M}) $\leq n$;
- (ii) $Fixt_{n+k}(M, N) = 0$ for all R-modules M, N and all $k \ge -1$;
- (iii) $Fixt_{n-1}(M, N) = 0$ for all R-modules M and N.

Proposition 3.8. Let R be a Noetherian ring. If M is a pure injective R-module, then M has a minimal \mathcal{X}^{\perp} -resolution $\cdots \rightarrow G_{n-2} \rightarrow G_{n-3} \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ with each G_i an injective module.

Proof. By [7, Theorem 5.5], M has an \mathcal{X} -injective precover $\gamma: G_0 \to M$. Consider the exact sequence $0 \to G_0 \xrightarrow{i} E \to L \to 0$ with E an injective envelope of G_0 . Since the exact sequence is pure and M is pure injective, there exists $h \in Hom(E, M)$ such that $\gamma = hi$. By \mathcal{X} -injective precover γ of M, there exists $\phi \in Hom(E, G)$ such that $\gamma \phi = h$. Hence, $\gamma \phi i = hi = \gamma$. This implies that ϕi is an isomorphism. Then, G is isomorphic to a direct summand of E and hence G is injective. Note that ker $\gamma \in (\mathcal{X}^{\perp})^{\perp}$. Hence, ker γ has an \mathcal{X} -injective precover $\gamma_1: G_1 \to \ker \gamma$ with G_1 an injective module, where ker $\gamma_1 \in (\mathcal{X}^{\perp})^{\perp}$. By continuing the above process, we get the minimal \mathcal{X}^{\perp} -resolution of M.

Theorem 3.9. Let *R* be a Noetherian ring. Consider the following conditions for a pure injective *R*-module *N* and an integer $n \ge 2$:

(i) $Lcores.dim_{\mathcal{X}^{\perp}}(N) \leq n-2;$

(ii) $Fixt_{n+k}(M, N) = 0$ for all *R*-modules *M* and all $k \ge -1$;

(iii) $Fixt_{n+k}(M, N) = 0$ for all R-modules M.

Proof. (1) \Rightarrow (2). By hypothesis, N has a left \mathcal{X}^{\perp} -resolution

$$0 \to G_{n-2} \to \cdots \to G_1 \to G_0 \to N \to 0.$$

Then, we have the following sequence

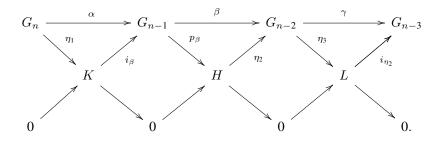
$$0 \rightarrow Hom(M, G_{n-2}) \rightarrow Hom(M, G_{n-3}) \rightarrow \cdots Hom(M, G_0) \rightarrow 0$$

for any *R*-module *M*. Hence, $Fixt_{n+k}(M, N) = 0$ for all $k \ge -1$.

- $(2) \Rightarrow (3)$ is trivial.
- (3) \Rightarrow (1). By pure injectivity of N and Proposition 3.8, N has a minimal \mathcal{X}^{\perp} -resolution:

$$\cdot \to G_n \xrightarrow{\alpha} G_{n-1} \xrightarrow{\beta} G_{n-2} \xrightarrow{\gamma} G_{n-3} \xrightarrow{\mu} \cdots \to G_1 \to G_0 \to M \to 0$$

with each G_i an injective module. Let $i_{\beta} \in Hom(K, G_{n-1})$ be the inclusion and $p_{\beta} \in Hom(G_{n-1}, H)$ the canonical projection, where $K = \ker \beta$ and $H = G_{n-1}/K$. Then, there exists $\eta_1 \in Hom(G_n, K)$ such that $\alpha = i_{\beta}\eta_1$ and there exists a monomorphism $\eta_2 \in Hom(H, G_{n-2})$ such that $\beta = \eta_2 p_{\beta}$. Let $\eta_3 \in Hom(G_{n-2}, L)$ be the canonical projection, where $L = G_{n-2}/im(\eta_2)$. Then, there exists a homomorphism $i_{\eta_2} \in Hom(L, G_{n-3})$ such that $\gamma = i_{\eta_2}\eta_3$. Then, we get the following commutative diagram:



By (3), $Fixt_{n-1}(K, N) = 0$. Then, the sequence

$$Hom(K, G_n) \xrightarrow{\alpha_*} Hom(K, G_{n-1}) \xrightarrow{\beta_*} Hom(K, G_{n-2})$$

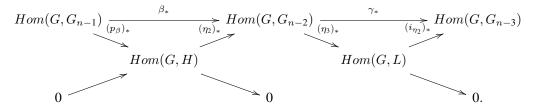
is exact. Since $\beta_*(i_\beta) = \beta i_\beta = 0, i_\beta \in \ker \beta_* = im\alpha_*$. Therefore $i_\beta = \alpha_*(t) = \alpha(t)$ for some $t \in Hom(K, G_n)$. By $\alpha = i_\beta \eta_1, i_\beta = i_\beta \eta_1 t$. Hence, $\eta_1 t = 1_{G_n}$ since i_β is monic. So K is injective. It follows that H and L are injective. We claim that the complex

$$0 \to L \stackrel{\iota_{\eta_2}}{\to} G_{n-3} \to \dots \to G_1 \to G_0 \to N \to 0$$

is an \mathcal{X}^{\perp} -resolution of N. We only need to show that the complex

 $0 \to Hom(G,L) \xrightarrow{(i_{\eta_2})_*} Hom(G,G_{n-3}) \xrightarrow{\mu_*} Hom(G,G_{n-4})$

is exact for any \mathcal{X} -injective R-module G. Note that we have the following commutative diagram:



Now $\operatorname{ker}((i_{\eta_2})_*(\eta_3)_*) = \operatorname{ker}(\gamma_*) = im(\beta_*) = im((\eta_2)_*(p_\beta)_*) = im((\eta_2)_*) = \operatorname{ker}((\eta_3)_*).$ Let $g \in \operatorname{ker}((i_{\eta_2})_*).$ Since $(\eta_3)_*$ is epic, $g = (\eta_3)_*(g_{n-2})$ for some $g_{n-2} \in \operatorname{Hom}(G, G_{n-2}).$ Therefore $(i_{\eta_2})_*(\eta_3)_*(g_{n-2}) = 0.$ Hence, $(\eta_3)_*(g_{n-2}) = 0.$ That is, g = 0. It follows that $(i_{\eta_2})_*$ is monic. On the other hand $\operatorname{ker}(\mu_*) = im(\gamma_*) = im((i_{\eta_2})_*).$ Hence, we obtain the desired exact sequence. This completes the proof.

Corollary 3.10. Let R be a Noetherian ring and an integer $n \ge 2$. Consider the following conditions

- (i) $glLcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M}) \leq n-2;$
- (ii) $Fcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M}) \leq n;$
- (iii) $Lcores.dim_{\chi^{\perp}}(N) \leq n-2$ for all pure injective *R*-modules *N*;

(iv) $Fixt_{n+k}(M, N) = 0$ for all *R*-modules *M*, all pure injective *R*-modules *N* and all $k \ge -1$;

(v) $Fixt_{n-1}(M, N) = 0$ for all *R*-modules *M* and all pure injective *R*-modules *N*.

Then, $(1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5)$.

Proof. It follows from Corollary 3.7 and Theorem 3.9.

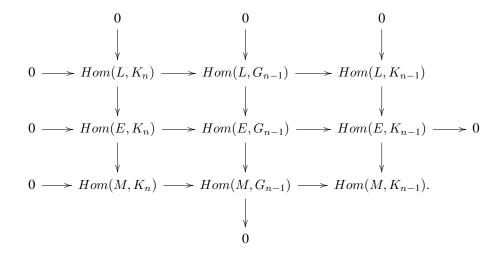
Lemma 3.11. Let R be a Noetherian ring and an integer $n \ge 1$. If M is pure injective R-module, then $id(M) \le n$ if and only if for the minimal left \mathcal{X}^{\perp} -resolution $\cdots \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to N \to 0$ of any pure injective R-module N, $Hom(M, G_n) \to Hom(M, K_n)$ is an epimorphism.

Proof. We prove the result by induction on n. Let n = 0. If M is injective, $Hom(M, G_0) \rightarrow Hom(M, K_0)$ is an epimorphism. Conversely, put N = M. Then, $Hom(M, G_0) \rightarrow Hom(M, M)$ is an epimorphism, and hence M is injective.

Let $n \ge 1$. Consider an exact sequence $0 \to M \to E \to L \to 0$ with E an injective module. Then, we have the following diagrams

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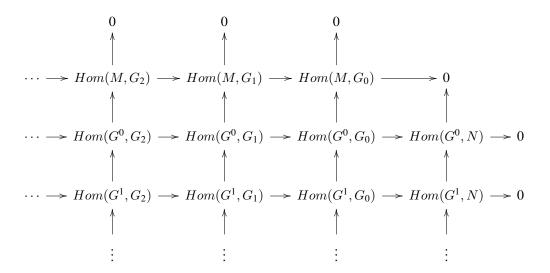
and



are exact and commutative. By [3, Lemma 3.2.10], L is pure injective. Therefore $id(M) \leq n$ if and only if $id(L) \leq n-1$ if and only if $Hom(L, G_{n-1}) \rightarrow Hom(L, K_{n-1})$ is an epimorphism by induction on n if and only if $Hom(E, K_n) \rightarrow Hom(M, K_n)$ is an epimorphism by the second diagram if and only if $Hom(M, G_n) \rightarrow Hom(M, K_n)$ is an epimorphism by the first diagram.

Theorem 3.12. Let R be a Noetherian ring. Then, $Fixt_{R}^{n}(M, N) \cong Fixt_{n}^{R}(M, N)$ for $n \ge 0$.

Proof. By Theorem [8, Theorem 2.2], M has a \mathcal{X}^{\perp} -coresolution. Let $0 \to M \to G^0 \to G^1 \to \cdots$ be an \mathcal{X}^{\perp} -coresolution of M. Since R is hereditary Noetherian, then there exists an \mathcal{X}^{\perp} -resolution $\cdots \to G_2 \to G_1 \to G_0 \to N \to 0$



Since G_i are \mathcal{X} -injective and \mathbb{G}^{\bullet} is an \mathcal{X}^{\perp} -coresolution, all rows but the first are exact. Similarly all columns but the last are exact. Then, by chasing diagrams or by a spectral sequence argument,

$$H^n(Hom(M, \mathbb{G}^{\bullet}(N))) \cong H_n(Hom(\mathbb{G}_{\bullet}(M), N))$$

for each $n \ge 0$. This completes the proof.

Theorem 3.13. Let R be a Noetherian ring. Then, the following conditions are equivalent:

- (i) Every left R-module has an \mathcal{X} -injective cover with the unique mapping property;
- (ii) $Fcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M}) \leq 2;$
- (iii) $Fixt_1(M, N) = 0$ for all left R-modules M and N;
- (iv) $Fixt_k(M, N) = 0$ for all left *R*-modules *M* and *N* and $k \ge 1$.

Proof. (1) \Rightarrow (2). Let M be an R-module. Then, M has an \mathcal{X} -injective cover $\phi: U \to M$ with the unique mapping property. Hence, $0 \to U \to M \to 0$ is a left \mathcal{X} -resolution. Thus, $glLcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M}) = 0$. Hence, by Corollary 3.10, $Fcores.dim_{\mathcal{X}^{\perp}}(\mathcal{M}) \leq 2$.

 $(2) \Rightarrow (1)$. Let M be an R-module. By [7, Theorem 5.6], M has an \mathcal{X} -injective cover $h: U \to M$. It is enough to show that, if for any \mathcal{X} -injective left R-module U' and any homomorphism $j: U' \to U$ such that hj = 0 then j = 0. Consider the natural map $\pi: U \to U/imj$. Then, there exists $\overline{h}: U/imj \to M$ such that $\overline{h\pi} = h$ since $imj \subseteq \ker h$. By hypothesis, U/imj is \mathcal{X} -injective. Then there exists $\mu: U/imj \to U$ such that $\overline{h} = h\mu$. Hence, we get the following commutative diagram with exact row:

$$0 \longrightarrow \ker j \xrightarrow{i} U' \xrightarrow{j} U \xrightarrow{\pi} U/imj \longrightarrow 0.$$

Thus, $h\mu\pi = h$, and hence $\mu\pi$ is an isomorphism. It follow that π is monic. Thus, j = 0. (2) \Leftrightarrow (3) \Leftrightarrow (4). It follows from Corollary 3.7.

Remark 3.14. Let R be a Noetherian ring. Then, every left R-module has an \mathcal{X} -injective cover with the unique mapping property if and only if every pure injective R-module is \mathcal{X} -injective.

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