ON $\phi$-2-ABSORBING ELEMENTS IN MULTIPLICATIVE LATTICES

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Abstract In this paper, we introduce the concept of $\phi$-2-absorbing elements in multiplicative lattices. Let $\phi : L \to L \cup \{\emptyset\}$ be a function. We will say a proper element $q$ of $L$ to be a $\phi$-2-absorbing element of $L$ if whenever $a, b, c \in L$ with $abc \leq q$ and $abc \not\leq \phi(q)$ implies either $ab \leq q$ or $ac \leq q$ or $bc \leq q$. We give some basic properties and establish some characterizations of $\phi$-2-absorbing elements in some special lattices.

1 Introduction

Several authors have studied various extensions of prime and primary ideals. A. Badawi [6] introduced the concept of 2-absorbing ideals in a commutative ring with identity, which is a generalization of prime ideals. A. Badawi and A.Y. Darani [5] studied weakly 2-absorbing ideals which are generalizations of weakly prime ideals [3]. Weakly prime elements in multiplicative lattices are studied in [10]. The concepts of 2-absorbing primary and weakly 2-absorbing primary ideals of commutative rings are studied in [7] and [8]. The concepts of 2-absorbing, weakly 2-absorbing, 2-absorbing primary and weakly 2-absorbing primary elements in multiplicative lattices are studied in [16] and [11] as generalizations of prime and weakly prime elements. Later, the concepts of $\phi$-prime, $\phi$-primary ideals are recently introduced in [12], [9], and generalizations of these are studied in [17]. In this work, our aim is to extend the concepts of 2-absorbing elements to $\phi$-2-absorbing elements and investigate some characterizations in some special lattices.

Throughout this paper $R$ denotes a commutative ring with identity and $L(R)$ denotes the lattice of all ideals of $R$. An element $a$ of $L$ is said to be compact if whenever $a \leq \bigvee a_{\alpha}$ for some finite subset $I_0$ of $I$. A multiplicative lattice, we mean a complete lattice $L$ with the least element $0_L$ and compact greatest element $1_L$, on which there is defined a commutative, associative, completely join distributive product for which $1_L$ is a multiplicative identity. Throughout this paper $L$ denotes a multiplicative lattice and $L_{\ast}$ denotes the set of all compact elements of $L$. By a $C$-lattice we mean a (not necessarily modular) multiplicative lattice which is generated under joins by a multiplicatively closed subset $C$ of compact elements. We note that in a $C$-lattice, a finite product of compact elements is again compact. An element $a \in L$ is said to be idempotent if $a = a^2$. For any $a \in L$, $L/a = \{b \in L : a \leq b\}$ is a multiplicative lattice with the multiplication $c \circ d = cd \lor a$. An element $a \in L$ is said to be proper if $a < 1_L$. A proper element $p$ of $L$ is said to be prime if $ab \leq p$ implies either $a \leq p$ or $b \leq p$. $C$-lattices can be localized. For any prime element $p$ of $L$, $L_p$ denotes the localization at $F = \{x \in C : x \not\in p\}$. If $0_L$ is prime, then $L$ is said to be a domain. A proper element $p$ is called as $\phi$-prime if $ab \leq p$ and $ab \not\leq \phi(p)$ implies either $a \leq p$ or $b \leq p$ for $a, b \in L$. In a $C$-lattice, an element $p$ is $\phi$-prime if and only if $ab \leq p$ and $ab \not\leq \phi(p)$ implies either $a \leq p$ or $b \leq p$ for all $a, b \in L_{\ast}$ by [17]. An element $a \in L_{\ast}$ is said to be maximal in $L$ if $m < x \leq 1_L$ implies $x = 1_L$. It can be easily shown that maximal elements are prime. For $a, b \in L$, we denote $(a : b) = \lor\{x \in L : xb \leq a\}$.

For $a \in L$, we define $\sqrt{a} = \land\{p \in L : p$ is prime and $a \leq p\}$. Note that in a $C$-lattice $L$, $\sqrt{a} = \land\{p \in L : a \leq p$ is a minimal prime over $a\} = \lor\{x \in L_{\ast} : x^n \leq a$ for some $n \in \mathbb{Z}^{+}\}$. A proper element $q$ is said to be primary if $ab \leq q$ implies either $a \leq q$ or $b \leq \sqrt{q}$ for every
pair of elements \(a, b \in L\). A proper element \(q\) is said to be \(\phi\)-primary if for every \(a, b \in L\) with \(ab \leq q\) and \(ab \not\leq \phi(q)\) implies either \(a \leq q\) or \(b \not\leq q\). A proper element \(q\) of \(L\) is said to be a 2-absorbing element if whenever \(a, b, c \in L\) with \(abc \leq q\) implies either \(ab \leq q\) or \(bc \leq q\) or \(ac \leq q\).

A multiplicative lattice is called a Noether lattice if it is modular, principally generated (every element is a join of some principal elements) which satisfies the ascending chain condition. A Noether lattice \(L\) is local if it contains precisely one maximal prime. If \(L\) is a Noether lattice and \(0_L\) is prime, then \(L\) is said to be a Noether domain. In [18], J. F. Wells studied the restricted cancellation law of a Noether lattice. An element \(a\) in a Noether lattice \(L\) satisfies the restricted cancellation law if \(ab = ac \neq 0_L\) implies \(b = c\) for any \(a, b, c \in L\).

2 \(\phi\)-2-absorbing elements

**Definition 2.1.** Let \(\phi : L \to L \cup \{\emptyset\}\) be a function and \(q \in L\) be a proper element. Then \(q\) is said to be a \(\phi\)\(-2\)-absorbing element of \(L\) whenever if \(a, b, c \in L\) with \(abc \leq q\) and \(abc \not\leq \phi(q)\) implies either \(ab \leq q\) or \(ac \leq q\) or \(bc \leq q\).

We can define the following special functions \(\phi_n\) as follows: Let \(q\) be a \(\phi_n\)-2-absorbing element of \(L\). Then we say

\[
\begin{align*}
\phi_0(q) &= 0 & \Rightarrow & & q & \text{is a 2-absorbing element,} \\
\phi_0(q) &= 0 & \Rightarrow & & q & \text{is a weakly 2-absorbing element,} \\
\phi_2(q) &= q^2 & \Rightarrow & & q & \text{is an almost 2-absorbing element,} \\
\phi_n(q) &= q^n & \Rightarrow & & q & \text{is an \(n\)-almost 2-absorbing element for \(n > 2\),} \\
\phi_\infty(q) &= \bigwedge_{n=1}^{\infty} q^n & \Rightarrow & & q & \text{is a \(\omega\)-2-absorbing element.}
\end{align*}
\]

Throughout this paper, \(\phi\) denotes a function defined from \(L\) to \(L \cup \{\emptyset\}\). Since for an element \(a \in L\) with \(a \leq q\) but \(a \not\leq \phi(q)\) implies that \(a \not\leq q \land \phi(q)\), there is no loss generality in assuming that \(\phi(q) \leq q\). We henceforth make this assumption. For any two functions \(\psi_1, \psi_2 : L \to L \cup \{\emptyset\}\), we say \(\psi_1 \leq \psi_2\) if \(\psi_1(a) \leq \psi_2(a)\) for each \(a \in L\). Thus clearly we have the following order:

\[
\begin{align*}
\phi_0 &\leq \phi_0 \leq \phi_\infty \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1.
\end{align*}
\]

**Lemma 2.2.** Let \(q\) be a proper element of \(L\) and \(\psi_1, \psi_2 : L \to L \cup \{\emptyset\}\) be two functions with \(\psi_1 \leq \psi_2\). If \(q\) is a \(\psi_1\)-2-absorbing element of \(L\), then \(q\) is a \(\psi_2\)-2-absorbing element of \(L\).

**Proof.** Suppose that \(q\) is a \(\psi_1\)-2-absorbing element of \(L\) and \(a, b, c \in L\) such that \(abc \leq q\) and \(abc \not\leq \psi_1(q)\). Since \(abc \leq q\) and \(abc \not\leq \psi_1(q)\), we are done.

Hence we have the following relations among the concepts mentioned in Definition 2.1:

**Theorem 2.3.** Let \(q\) be a proper element of \(L\). Then

(i) \(q\) is a 2-absorbing element of \(L\) \(\Rightarrow q\) is a weakly 2-absorbing element of \(L\) \(\Rightarrow q\) is a \(\phi\)-2-absorbing element of \(L\) \(\Rightarrow q\) is an \((n + 1)\)-almost 2-absorbing element of \(L\) \(\Rightarrow q\) is an \(n\)-almost 2-absorbing element of \(L\) for all \(n \geq 2\) \(\Rightarrow q\) is an almost 2-absorbing element of \(L\).

(ii) \(q\) is a \(\phi\)-prime element of \(L\) \(\Rightarrow q\) is a \(\phi\)-2-absorbing element of \(L\).

(iii) A proper element \(q\) of \(L\) is an idempotent element \(\Rightarrow q\) is a \(\omega\)-2-absorbing element of \(L\) and \(q\) is an \(n\)-almost 2-absorbing element of \(L\) for all \(n \geq 2\).

(iv) \(q\) is an \(n\)-almost 2-absorbing element of \(L\) for all \(n \geq 2\) \(\Leftrightarrow q\) is a \(\omega\)-2-absorbing element of \(L\).

**Proof.** (i) It is clear from Lemma 2.2.

(ii) Suppose that \(a, b, c \in L\) with \(abc \leq q\), \(abc \not\leq \phi(q)\) and \(ab \not\leq q\). Hence \(c \leq q\) as \(q\) is a \(\phi\)-prime element of \(L\). Thus \(ac \leq q\) or \(bc \leq q\), we are done.

(iii) Suppose that \(q\) is an idempotent element of \(L\). Then \(q = q^n\) for all \(n > 0\), and so \(\phi_n(q) = \bigwedge_{n=1}^{\infty} q^n = q\). Thus \(q\) is a \(\omega\)-2-absorbing element of \(L\). Finally, \(q\) is an \(n\)-almost 2-absorbing element for all \(n \geq 2\) from (i).
(iv) Let $a, b, c \in L$ with $abc \leq q$ but $abc \not\leq \wedge_{n=1}^{\infty} q^n$. Hence $abc \leq q$ but $abc \not\leq q^n$ for some $m \geq 2$. Since $q$ is $n$-almost 2-absorbing for all $n \geq 2$, this implies either $ab \leq q$ or $bc \leq q$ or $ac \leq q$, we are done. The converse is clear from (i).

**Theorem 2.4.** Let $q$ be a $\phi$-2-absorbing element of $L$. If $\phi(q)$ is a 2-absorbing element of $L$, then $q$ is 2-absorbing.

**Proof.** Let $abc \leq q$ for some $a, b, c \in L$. If $abc \not\leq \phi(q)$, then we have either $ab \leq q$ or $ac \leq q$ or $bc \leq q$ as $q$ is $\phi$-2-absorbing. Suppose that $abc \leq \phi(q)$. Hence we conclude that either $ab \leq \phi(q)$ or $ac \leq \phi(q)$ or $bc \leq \phi(q)$. Since $\phi(q) \leq q$, we are done.

**Definition 2.5.** Let $q$ be a $\phi$-2-absorbing element of $L$ and $a, b, c \in L$. If $abc \leq \phi(q)$ but $ab \not\leq q$, $bc \not\leq q$, $ac \not\leq q$, then $(a, b, c)$ is called a $\phi$-triple zero of $q$.

**Remark 2.6.** If $q$ is a $\phi$-2-absorbing element of $L$ which is not 2-absorbing, then there exists $(a, b, c)$ a $\phi$-triple zero of $q$ for some $a, b, c \in L$.

**Lemma 2.7.** Let $q$ be a $\phi$-2-absorbing element of $L$ and suppose that $(a, b, c)$ is a $\phi$-triple zero of $q$ for some $a, b, c \in L$. Then

(i) $abq$, $bcq$, $acq \leq \phi(q)$.

(ii) $aq^2$, $bq^2$, $cq^2 \leq \phi(q)$.

(iii) $q^3 \leq \phi(q)$.

**Proof.** (i) Suppose that $abq \not\leq \phi(q)$. Then $ab(c \lor q) \not\leq \phi(q)$. Since $ab \not\leq q$ and $q$ is $\phi$-2-absorbing, we have $a(c \lor q) \leq q$ or $b(c \lor q) \leq q$. So $ac \leq q$ or $bc \leq q$, which contradicts with our hypothesis. Thus $abq \leq \phi(q)$. Similarly one can easily show that $bcq \leq \phi(q)$ and $acq \leq \phi(q)$.

(ii) Suppose that $aq^2 \not\leq \phi(q)$. Hence we have $a(b \lor q)(c \lor q) \not\leq \phi(q)$ by (i). So we conclude either $a(b \lor q) \leq q$ or $a(c \lor q) \leq q$ or $(b \lor q)(c \lor q) \leq q$. Thus either $ab \leq q$ or $ac \leq q$ or $bc \leq q$, a contradiction. Therefore $aq^2 \leq \phi(q)$. Similarly it can be easily verified that $bq^2$, $cq^2 \leq \phi(q)$.

(iii) Assume that $q^3 \not\leq \phi(q)$. Then we have $(a \lor q)(b \lor q)(c \lor q) \leq q$ but $(a \lor q)(b \lor q)(c \lor q) \not\leq \phi(q)$ by (i) and (ii). Since $q$ is $\phi$-2-absorbing, $(a \lor q)(b \lor q) \leq q$ or $(a \lor q)(c \lor q) \leq q$ or $(b \lor q)(c \lor q) \leq q$, so we conclude $ab \leq q$ or $ac \leq q$ or $bc \leq q$, a contradiction. Thus $q^3 \leq \phi(q)$.

Now we can give a condition for a $\phi$-2-absorbing element to be a 2-absorbing element of $L$.

**Corollary 2.8.** Let $q$ be a proper element of $L$. Then the following statements hold:

(i) If $q$ is a $\phi$-2-absorbing element of $L$ such that $q^3 \not\leq \phi(q)$, then $q$ is a 2-absorbing element of $L$.

(ii) Let $L$ be a $C$-lattice. If $q$ is a $\phi$-2-absorbing element of $L$ that is not a 2-absorbing, then $\sqrt{q} = \sqrt{\phi(q)}$.

**Proof.** (i) The proof is clear by Remark 2.6 and Lemma 2.7 (iii).

(ii) Since $q$ is not a 2-absorbing element of $L$, $q^3 \leq \phi(q)$ by Lemma 2.7 (iii). Hence $\sqrt{q} \leq \sqrt{\phi(q)}$. Since $\phi(q) \leq q$ is always hold, we get $\sqrt{q} = \sqrt{\phi(q)}$.

Recall from [13] that an element $e \in L$ is said to be principal, if it satisfies the dual identities (i) $a \land be = ((a : e) \land b : e)$ and (ii) $((a \lor b : e) = (b : e) \lor a$. Elements satisfying the identity (i) are called meet principal and elements satisfying the identity (ii) are called join principal. If the both identities are satisfied, then $e$ is said to be a principal element of $L$. Note that by [13, Lemma 3.3 and Lemma 3.4], a finite product of principal elements of $L$ is again principal. If every element of $L$ can be written as a join of some principal elements of $L$, then $L$ is said to be join principally generated lattice.

**Theorem 2.9.** Let $L$ be a join principally generated $C$-lattice and $a, b, c$ be proper join principal elements of $L$. Then $abc$ is a $\phi$-2-absorbing element of $L$ if and only if $abc = \phi(abc)$.
Proof. Suppose that \( abc \) is a \( \phi \)-2-absorbing element of \( L \). Assume that \( abc \neq \phi(abc) \). Then we have either \( ab \leq abc \) or \( ac \leq abc \) or \( bc \leq abc \). Without loss generality we may assume that \( ab \leq abc \). Since \( ab \) is principal, we conclude that \( 1_L = (abc : ab) = c \vee (0_L : ab) \). Observe that \( (0_L : ab) \neq 1_L \). Indeed, if \( (0_L : ab) = 1_L \), then \( abc = 0_L \leq \phi(abc) \), a contradiction. Since \( (0_L : ab) \neq 1_L, c \leq J(L) \) we conclude that \( 1_L \neq c \vee (0_L : ab) \), a contradiction. Thus \( abc = \phi(abc) \). The converse part is clear. □

Theorem 2.10. Let \( L \) be a local Noether domain. If \( q \) is an \( \phi_n \)-2-absorbing element of \( L \) for all \( n \geq 2 \), then \( q \) is a 2-absorbing element of \( L \).

Proof. Let \( abc \leq q \) for some \( a, b, c \in L \). If \( abc \notin \phi_n(q) \), then we have either \( ab \leq q \) or \( bc \leq q \) or \( ac \leq q \) as \( q \) is an \( \phi_n \)-2-absorbing. So suppose that \( abc \leq \phi_n(q) \). Since \( \cap_{n=1}^{\infty} q^n = 0_L \), from Corollary 3.3 of [13], we conclude that \( abc \leq 0_L \). Thus \( a \leq 0_L \) or \( b \leq 0_L \) or \( c \leq 0_L \) as \( L \) is a domain, so clearly \( ab \leq q \) or \( bc \leq q \) or \( ac \leq q \).

We remind to the reader that for any \( a \in L, L/a = \{b \in L : a \leq b \} \) is a multiplicative lattice with multiplication \( c \circ d = cd \vee a \).

Theorem 2.11. Let \( q \) be a proper element of \( L \). Then the following statements hold:

(i) \( q \) is a \( \phi \)-2-absorbing element of \( L \) if and only if \( q \) is a weakly 2-absorbing element of \( L/\phi(q) \).

(ii) \( q \) is a \( \phi \)-prime element of \( L \) if and only if \( q \) is a weakly prime element of \( L/\phi(q) \).

(iii) \( q \) is a \( \phi \)-primary element of \( L \) if and only if \( q \) is a weakly primary element of \( L/\phi(q) \).

Proof. (i) If \( \phi(q) = 0 \), then there is nothing to prove. Thus assume that \( \phi(q) \neq 0 \). Let \( \phi(q) \neq (a \circ \phi(q)) \circ (b \circ \phi(q)) \circ (c \circ \phi(q)) = abc \circ \phi(q) \leq q \) for some \( a, b, c \in L \). Then \( abc \leq q \), but \( abc \notin \phi(q) \). Hence either \( ab \leq q \) or \( bc \leq q \) or \( ac \leq q \). So \( (a \circ \phi(q)) \circ (b \circ \phi(q)) \leq q \) or \( (b \circ \phi(q)) \circ (c \circ \phi(q)) \leq q \) or \( (a \circ \phi(q)) \circ (c \circ \phi(q)) \leq q \). Therefore \( q \) is a weakly 2-absorbing element of \( L/\phi(q) \).

Conversely, let \( abc \leq q \) and \( abc \notin \phi(q) \) for some \( a, b, c \in L \). Then \( \phi(q) \neq (a \circ \phi(q)) \circ (b \circ \phi(q)) \circ (c \circ \phi(q)) \leq q \). Hence \( (a \circ \phi(q)) \circ (b \circ \phi(q)) \leq q \) or \( (b \circ \phi(q)) \circ (c \circ \phi(q)) \leq q \) or \( (a \circ \phi(q)) \circ (c \circ \phi(q)) \leq q \). Thus \( ab \leq q \) or \( bc \leq q \) or \( ac \leq q \). Similarly one can easily prove (ii) and (iii).

Corollary 2.12. Let \( q \) be a proper element of \( L \) and \( n \geq 2 \). Then

(i) \( q \) is a \( \phi_n \)-2-absorbing element of \( L \) if and only if \( q \) is a weakly 2-absorbing element of \( L/q^n \).

(ii) \( q \) is a \( \phi_n \)-prime element of \( L \) if and only if \( q \) is a weakly prime element of \( L/q^n \).

(iii) \( q \) is a \( \phi_n \)-primary element of \( L \) if and only if \( q \) is a weakly primary element of \( L/q^n \).

Proof. Since \( \phi_n(q) = q^n \), the proof is clear by Theorem 2.11. □

Corollary 2.13. Let \( q \) be a \( \phi \)-2-absorbing element of \( L \) such that \( \phi \leq \phi_3 \). Then

(i) \( q \) is a \( \phi_n \)-2-absorbing element of \( L \) for every \( n \geq 3 \).

(ii) \( q \) is a \( \phi_n \)-2-absorbing element of \( L \).

Proof. Suppose that \( q \) is a 2-absorbing element of \( L \). Hence (i) and (ii) are clear.

(i) Assume that \( q \) is not a 2-absorbing element of \( L \). Thus \( q^3 \leq \phi(q) \) by Lemma 2.7 (iii). Then we have \( q^3 \leq \phi(q) \leq q^3 \) as \( \phi \leq \phi_3 \). This follows \( q^3 = q^n = \phi(q) \) for every \( n \geq 3 \), so we are done.

(ii) Let \( abc \leq q \) and \( abc \notin \cap_{n=3}^{\infty} q^n \). Then \( abc \notin q^n \) for some \( n \geq 2 \). If \( n \geq 3 \), then it is clear from (i). So suppose that \( n = 2 \). Hence \( abc \notin q^2 \) which implies that \( abc \notin q^2 \), so from (i) the result is obtained. □

Theorem 2.14. Let \( x \) and \( y \) be two proper elements of \( L \) such that \( x \leq y \) and let \( n \geq 2 \). If \( y \) is a \( \phi_n \)-2-absorbing element of \( L \), then \( y \) is a \( \phi_n \)-2-absorbing element of \( L/x \).
Proof. Suppose that \( y \) is a \( \phi_n \)-2-absorbing element of \( L \). Assume that \( (a \lor x) \circ (b \lor x) \circ (c \lor x) = ab \lor x \leq y \) and \( (a \lor x) \circ (b \lor x) \circ (c \lor x) = ab \lor x \leq y^n \) for some \( a, b, c \in L \). As \( y \in L/x \), then \( y^n = y \circ y \circ \ldots \circ y = y^n \lor x \). Since \( x \leq y \) and \( ab \lor x \leq y^n \lor x \), then we have \( ab \leq y \) and \( abc \leq y^n \). Thus \( ab \leq y \) or \( ac \leq y \) or \( bc \leq y \). Hence \( (a \lor x) \circ (b \lor x) \leq y \) or \( (a \lor x) \circ (c \lor x) \leq y \) or \( (b \lor x) \circ (c \lor x) \leq y \) which means that \( y \) is a \( \phi_n \)-2-absorbing primary element of \( L/x \).

Corollary 2.15. Let \( x \) and \( y \) be two proper elements of \( L \) such that \( x \leq y \). If \( y \) is a \( \phi_n \)-2-absorbing element of \( L \), then \( y \) is a \( \phi_{n^2} \)-absorbing element of \( L/x \).

Proof. Similar to the proof of Theorem 2.14.

Definition 2.16. Let \( x \) be a proper element of \( L/q \) such that \( q \leq x \). Then \( x \) is called a \( \phi_q \)-2-absorbing element of \( L/q \) if whenever \( a, b, c \in L \), then \( ab \leq x \) or \( ac \leq x \) or \( bc \leq x \).

Theorem 2.17. Let \( p \) and \( q \) be two elements of \( L \) with \( q \leq p < 1 \). If \( p \) is a \( \phi \)-2-absorbing element of \( L \), then \( p \) is a \( \phi_p \)-2-absorbing element of \( L/q \).

Proof. Let \( (a \lor q) \circ (b \lor q) \circ (c \lor q) \leq p \) and \( ab \lor c \leq (a \lor q) \circ (b \lor q) \circ (c \lor q) \leq \phi(p) \lor q \) for some \( a, b, c \in L \). Hence \( ab \leq x \) or \( ac \leq x \) or \( bc \leq x \). So we get \((a \lor q) \circ (b \lor q) \leq p \) or \((a \lor q) \circ (c \lor q) \leq p \) or \((b \lor q) \circ (c \lor q) \leq p \).

Theorem 2.18. Let \( p \) and \( q \) be two proper elements of \( L \) such that \( q \leq \phi(p) \). Then the following statements are equivalent:

(i) \( p \) is a \( \phi \)-2-absorbing element of \( L \).

(ii) \( p \) is a \( \phi_p \)-2-absorbing element of \( L/q \).

(iii) \( p \) is a \( \phi_{p^n} \)-2-absorbing element of \( L/q^n \).

Proof. (i)\(\Rightarrow\)(ii): It is clear by Theorem 2.17.

(ii)\(\Rightarrow\)(iii): Let \( n \geq 1 \). Since \( q \leq \phi(p) \), we have \( q^n \leq q \leq \phi(p) \). Suppose that \( (a \lor q^n) \circ (b \lor q^n) \circ (c \lor q^n) \leq q \) for some \( a, b, c \in L \). Hence \( ab \leq \phi(p) \). Since \( q \leq \phi(p) \) and \( ab \leq \phi(p) \), we have \( ab \leq q \). Thus \( (a \lor q) \circ (b \lor q) \circ (c \lor q) \leq p \) and \((a \lor q) \circ (b \lor q) \circ (c \lor q) \leq \phi(p) \lor q \). Since \( p \) is a \( \phi_q \)-2-absorbing element of \( L/q \), one can conclude that \( ab \leq p \) or \( ac \leq p \) or \( bc \leq p \). Thus \( ab \lor q^n \leq p \) or \( ac \lor q^n \leq p \) or \( bc \lor q^n \leq p \) in \( L/q^n \).

(iii)\(\Rightarrow\)(i): Suppose that \( ab \leq p \) and \( ab \leq \phi(p) \) for some \( a, b, c \in L \). Since \( q^n \leq \phi(p) \), we have \( abc \leq q^n \). As \( q^n \leq \phi(p) \leq p \), we get \((a \lor q^n) \circ (b \lor q^n) \circ (c \lor q^n) = ab \lor q^n \leq p \) and \((a \lor q^n) \circ (b \lor q^n) \circ (c \lor q^n) \leq \phi(p) \lor q^n \). Since \( p \) is a \( \phi_{p^n} \)-2-absorbing element of \( L/q^n \), one can conclude that \( ab \leq p \) or \( ac \leq p \) or \( bc \leq p \).

Corollary 2.19. Let \( p \) and \( q \) be two proper elements of \( L \). Suppose that \( q \) is not a weakly 2-absorbing element of \( L \). The following statements are equivalent:

(i) \( p \) is a \( \phi \)-2-absorbing element of \( L \).

(ii) \( p \) is a \( \phi_p \)-2-absorbing element of \( L/p^3 \).

(iii) \( p \) is a \( \phi_{p^n} \)-2-absorbing element of \( L/p^n \) for every \( n \geq 3 \).

Proof. Suppose that \( p \) is not a weakly 2-absorbing element of \( L \). Hence \( p \) is not a 2-absorbing element of \( L \). So we conclude \( p^3 \leq \phi(q) \) by Lemma 2.7. Thus we are done by Theorem 2.18.

Definition 2.20. Let \( q \) be a proper element of \( L \) and \( n \geq 2 \). We call \( q \) as an \( n \)-potent 2-absorbing if whenever \( a, b, c \in L \) with \( abc \leq q^n \), then \( ab \leq q \) or \( bc \leq q \) or \( ac \leq q \).

Theorem 2.21. Let \( q \) be an \( n \)-almost 2-absorbing element for some \( n \geq 2 \). If \( q \) is \( k \)-potent 2-absorbing for some \( k \leq n \), then \( q \) is a 2-absorbing element.
Proof. Suppose that $q$ is an $n$-almost 2-absorbing element. Let $abc \leq q$ for some $a, b, c \in L$. If $abc \not\subseteq q^k$, then $abc \not\subseteq q^n$. It implies either $ab \leq q$ or $bc \leq q$ or $ac \leq q$ as $q$ is an $n$-almost 2-absorbing element. If $abc \subseteq q^k$, then we obtain the same result as $q$ is $k$-potent 2-absorbing, so we are done. \hfill\Box

In the following theorems, we obtain some conditions under which a $\phi$-2-absorbing element of $L$ is a 2-absorbing element of $L$.

Let $J(L) = \bigwedge \{m \in L : m$ is a maximal element of $L\}.

**Theorem 2.22.** Let $L$ be a Noether domain. Then an element $q$ of $L$ with $q \leq J(L)$ is a 2-absorbing element of $L$ if and only if $q$ is a $\phi_n$-2-absorbing element of $L$ for all $n \geq 2$.

**Proof.** If $q$ is 2-absorbing, then $q$ is a $\phi_n$-2-absorbing by Theorem 2.3. Conversely, suppose that $q$ is a $\phi_n$-2-absorbing for all $n \geq 2$ and let $a, b, c \in L$ with $abc \leq q$. If $abc \not\subseteq q^k$ for some $k \geq 2$, we have either $ab \leq q$ or $bc \leq q$ or $ac \leq q$. So suppose that $abc \subseteq q^n$ for all $n \geq 2$. Since $L$ is a Noether domain, we conclude $abc \leq \bigwedge_{n=1}^{\infty} q^n = 0_L$ by Corollary 1.4 in [4]. Since $0_L$ is prime, we get either $a = 0_L$ or $b = 0_L$ or $c = 0_L$. Without loss generality suppose that $a = 0_L$. This implies that $ab = 0_L \leq q$ which completes the proof. \hfill\Box

**Theorem 2.23.** Let $L$ be a Noether lattice. Let $q$ be a non-zero non-nilpotent proper element of $L$ satisfying the restricted cancellation law. Then $q$ is a 2-absorbing element of $L$ for some $\phi \leq \phi_n$ and for all $n \geq 2$ if and only if $q$ is a 2-absorbing element of $L$.

**Proof.** Assume that $q$ is a 2-absorbing element of $L$. Then $q$ is a $\phi$-2-absorbing element of $L$ for all $\phi$. Thus $q$ is a $\phi$-2-absorbing for some $\phi \leq \phi_n$ and for all $n \geq 2$.

Conversely assume that $q$ is a $\phi$-2-absorbing element of $L$ for some $\phi \leq \phi_n$ for all $n \geq 2$. Hence $q$ is a $\phi_n$-2-absorbing element of $L$ for all $n \geq 2$ by Lemma 2.2. Let $abc \leq q$ for some $a, b, c \in L$. Here there are two cases:

Case 1: Let $abc \not\subseteq q^n$ for some $n \geq 2$. Then by hypothesis we get $ab \leq q$ or $bc \leq q$ or $ac \leq q$.

Case 2: Let $abc \subseteq q^n$ for all $n \geq 2$. We have that $a(b \lor q) c \lor q = abc \lor abq \lor acq \lor aq^2 \leq q$.

If $(a \lor b \lor q) c \lor q \not\subseteq q^n$, then $(a \lor b \lor q) c \lor q \not\subseteq q$ or $(a \lor b \lor q) c \lor q \subseteq q$. It follows that either $ab \leq q$ or $bc \leq q$ or $ac \leq q$. If $(a \lor b \lor q) c \lor q \subseteq q^n$, then $(a \lor b \lor q) c \lor q = abc \lor abq \lor acq \lor aq^2 \subseteq q^n \leq q^2$. By [18, Lemma 1.11], we get $ab \leq q$ and $ac \leq q$. Thus $q$ is a 2-absorbing element of $L$.

**Remark 2.24.** Let $L = L_1 \times L_2 \times \ldots L_n$ where $L_1, L_2, \ldots, L_n$ are multiplicative lattices ($n \geq 1$) and let $\psi = \psi_1 \times \psi_2 \times \ldots \times \psi_n$ where $\psi_i : L_i \rightarrow L_i \cup \{\emptyset\}$ ($i = 1, \ldots, n$) be a function. Let $a = (a_1, a_2, \ldots, a_n)$ be an element of $L$. Observe that if $\psi_i(a_i) = \emptyset$ for some $i = 1, \ldots, n$, then there is no element of $\phi(a)$ and vice versa. Thus $\phi(a) = \emptyset$ if and only if $\psi_i(a_i) = \emptyset$ for some $i = 1, \ldots, n$.

**Lemma 2.25.** Let $L = L_1 \times L_2$ where $L_1, L_2$ are two multiplicative lattices. Let $\phi = \psi_1 \times \psi_2$, where $\psi_i : L_i \rightarrow L_i \cup \{\emptyset\}$ ($i = 1, 2$) is a function. Then $q_1$ is a 2-absorbing element of $L_1$ if and only if $q = (q_1, 1_{L_2})$ is a 2-absorbing element of $L$.

**Proof.** Suppose that $q_1$ is a 2-absorbing element of $L_1$ and $(a, 1_{L_2})(b, 1_{L_2})(c, 1_{L_2}) \leq q$ for some elements $(a, 1_{L_2}), (b, 1_{L_2}), (c, 1_{L_2})$ of $L$. Then $abc \leq q_1$ which implies that either $ab \leq q_1$ or $bc \leq q_1$ or $ac \leq q_1$. It follows $(a, 1_{L_2})(b, 1_{L_2}) \leq q$ or $(b, 1_{L_2})(c, 1_{L_2}) \leq q$ or $(a, 1_{L_2})(c, 1_{L_2}) \leq q$. Thus $q$ is a 2-absorbing element of $L$.

Conversely suppose that $q = (q_1, 1_{L_2})$ is a 2-absorbing element of $L$ but assume that $q_1$ is not a 2-absorbing element of $L_1$. Hence there exists $a, b, c \in L_1$ with $abc \leq q_1$ but neither $ab \leq q_1$ nor $bc \leq q_1$ nor $ac \leq q_1$. Thus we conclude $(a, 1_{L_2})(b, 1_{L_2})(c, 1_{L_2}) \leq q$ but $(a, 1_{L_2})(b, 1_{L_2}) \not\subseteq q$ and $(b, 1_{L_2})(c, 1_{L_2}) \not\subseteq q$ and $(a, 1_{L_2})(c, 1_{L_2}) \not\subseteq q$, a contradiction. \hfill\Box

**Theorem 2.26.** Let $L = L_1 \times L_2$ where $L_1, L_2$ are two multiplicative lattices. Let $\phi = \psi_1 \times \psi_2$, where $\psi_i : L_i \rightarrow L_i \cup \{\emptyset\}$ ($i = 1, 2$) is a function. Then the following statements hold:

(i) If $q_i$ is a proper element of $L_i$ with $\psi_i(q_i) = q_i$ ($i = 1, 2$), then $q = (q_1, q_2)$ is a 2-absorbing element of $L$. 

(ii) If \( q_1 \) is a \( \psi_1 \)-2-absorbing element of \( L_1 \), and \( \psi_2(1_{L_2}) = 1_{L_2} \), then \( q = (q_1, 1_{L_2}) \) is a \( \phi \)-2-absorbing element of \( L \).

(iii) If \( q_2 \) is a \( \psi_2 \)-2-absorbing element of \( L_2 \) and \( \psi_1(1_{L_1}) = 1_{L_1} \), then \( q = (1_{L_1}, q_2) \) is a \( \phi \)-2-absorbing element of \( L \).

**Proof.** (i) If \( \psi_1(q_1) = q_1 \) and \( \psi_2(q_2) = q_2 \), then there is no such an element \((a, b)\) which satisfies \((a, b) \leq (q_1, q_2)\) and \((a, b) \not\leq \phi(q_1, q_2) = (q_1, q_2)\), so we are done.

(ii) Suppose that \( \psi_1(q) = \emptyset \). Then \( q = (q_1, 1_{L_2}) \) is a \( \phi \)-2-absorbing element of \( L \) by **Lemma 2.25**. So assume that \( \psi_1(q) \neq \emptyset \) and \( q_1 \) is a \( \psi_1 \)-2-absorbing element of \( L_1 \). Let \( a = (a_1, a_2), b = (b_1, b_2) \) and \( c = (c_1, c_2) \) such that \( ab \leq q \) and \( abc \not\leq \phi(q) \). Hence \( a_1b_1c_1 \leq q_1 \) and \( a_1b_1c_1 \not\leq \psi_1(q_1) \), this implies that either \( a_1b_1 \leq q_1 \) or \( b_1c_1 \leq q_1 \) or \( a_1c_1 \leq q_1 \). Thus either \( ab \leq q \) or \( bc \leq q \) or \( ac \leq q \). (iii) This can be easily obtained similar to (ii).

**Theorem 2.27.** Let \( L = L_1 \times L_2 \) where \( L_1, L_2 \) are two multiplicative lattices and \( \phi = \psi_1 \times \psi_2 \), where \( \psi_i : L_i \rightarrow L_i \cup \{ \emptyset \} \) \((i = 1, 2)\) is a function such that \( \psi_2(1_{L_2}) \neq 1_{L_2} \). Let \( q_1 \) be a proper element of \( L_1 \) and \( q = (q_1, 1_{L_2}) \). Then the following statements are equivalent:

(i) \((q_1, 1_{L_2})\) is a \( \phi \)-2-absorbing element of \( L \).

(ii) \((q_1, 1_{L_2})\) is a 2-absorbing element of \( L \).

(iii) \( q_1 \) is a 2-absorbing element of \( L_1 \).

**Proof.** If \( \psi_1(q_1) = \emptyset \) and \( \psi_2(1_{L_2}) = \emptyset \), then \( \phi(q) = \emptyset \) by **Remark 2.24**. So we are done from **Lemma 2.25**. Thus assume that \( \psi_1(q_1) \neq \emptyset \) or \( \psi_2(1_{L_2}) \neq \emptyset \).

(i) \(\Rightarrow\) (ii): Assume that \( q = (q_1, 1_{L_2}) \) is a \( \phi \)-2-absorbing element of \( L \). Then \( q_1 \) is a \( \psi_1 \)-2-absorbing element of \( L_1 \). Indeed, if \( q_1 \) is not a \( \psi_1 \)-2-absorbing element of \( L_1 \), then there exist \( a, b, c \) in \( L_1 \) such that \( abc \leq q_1 \) and \( abc \not\leq \psi_1(q) \) but \( ab \not\leq q_1 \) and \( bc \not\leq q_1 \) and \( ac \not\leq q_1 \). Then \((abc, 1_{L_2}) = (a, 1_{L_2})(b, 1_{L_2})(c, 1_{L_2}) \leq q \) and \((abc, 1_{L_2}) = (a, 1_{L_2})(b, 1_{L_2})(c, 1_{L_2}) \not\leq \psi_1(q_1, 1_{L_2}) = \phi(q) \). This implies \((ab, 1_{L_2}) = (a, 1_{L_2})(b, 1_{L_2}) \leq q \) or \((bc, 1_{L_2}) = (b, 1_{L_2})(c, 1_{L_2}) \leq q \) or \((ac, 1_{L_2}) = (a, 1_{L_2})(c, 1_{L_2}) \leq q \), which means \( ab \leq q_1 \) or \( bc \leq q_1 \) or \( ac \leq q_1 \), a contradiction. Thus \( q_1 \) is a \( \psi_1 \)-2-absorbing element of \( L_1 \).

If \( q_1 \) is a 2-absorbing element of \( L_1 \), then \( q \) is a \( \phi \)-absorbing element of \( L_1 \). Hence \( q_1 \) has a \( \psi_1 \)-triangular \((x, y, z)\) for some \( x, y, z \) in \( L_1 \) by **Remark 2.6**.

Since \( \psi_2(1_{L_2}) \neq 1_{L_2} \), then we get \((xyz, 1_{L_2}) = (x, 1_{L_2})(y, 1_{L_2})(z, 1_{L_2}) \leq q \) and \((xyz, 1_{L_2}) = (x, 1_{L_2})(y, 1_{L_2})(z, 1_{L_2}) \not\leq \phi(q) \). Therefore \((x, 1_{L_2})(y, 1_{L_2}) \leq q \) or \((y, 1_{L_2})(z, 1_{L_2}) \leq q \) or \((x, 1_{L_2})(z, 1_{L_2}) \leq q \). So we get \( xy \leq q_1 \) or \( yz \leq q_1 \) or \( xz \leq q_1 \), a contradiction. Thus \( q_1 \) is a 2-absorbing element of \( L_1 \). Consequently, \((q_1, 1_{L_2})\) is a 2-absorbing element of \( L \).

(ii) \(\Rightarrow\) (iii): It can be easily shown similar to the argument in (i) \(\Rightarrow\) (ii).

(iii) \(\Rightarrow\) (i): It is clear.

**Lemma 2.28.** Let \( L = L_1 \times L_2 \times L_3 \) where \( L_1, L_2, L_3 \) are \( C \)-lattices. Let \( \phi = \psi_1 \times \psi_2 \times \psi_3 \), where \( \psi_i : L_i \rightarrow L_i \cup \{ \emptyset \} \) \((i = 1, 2, 3)\) is a function with \( \psi_i(1_{L_i}) \neq 1_{L_i} \). If \( q = (q_1, q_2, q_3) \) is a \( \phi \)-2-absorbing element of \( L \), then either \( q = \phi(q) \) or \( q \) is a 2-absorbing element of \( L \).

**Proof.** If \( \phi(q) = \emptyset \), then we are done. So assume \( \phi(q) \neq \emptyset \). Suppose that \( q \neq \phi(q) \). Hence there is an element \((a, b, c) \in L \) with \((a, b, c) \leq q \) but \((a, b, c) \not\leq \phi(q) \). So \((a, b, c) = (a, 1_{L_1})(1_{L_2}, b, 1_{L_3})(1_{L_1}, 1_{L_2}, c) \leq q \) implies that either \((a, 1_{L_1})(1_{L_2}, b, 1_{L_3}) \leq q \) or \((1_{L_1}, 1_{L_2}, c) \leq q \) or \((a, 1_{L_1})(1_{L_2}, 1_{L_3}, c) \leq q \). Without loss of generality assume that \((a, 1_{L_1})(1_{L_2}, b, 1_{L_3}) \leq q \). Then \( q_3 = 1_{L_3} \), which means that \( q_3 \not\leq \phi(q) \). Thus \( q \) is a 2-absorbing element of \( L \) by **Corollary 2.8**.

**Theorem 2.29.** Let \( L = L_1 \times L_2 \times L_3 \) where \( L_1, L_2, L_3 \) are \( C \)-lattices. Let \( \phi = \psi_1 \times \psi_2 \times \psi_3 \), where \( \psi_i : L_i \rightarrow L_i \cup \{ \emptyset \} \) \((i = 1, 2, 3)\) is a function with \( \psi_i(1_{L_i}) \neq 1_{L_i} \). If \( q \neq \phi(q) \), then the followings are equivalent:

(i) \( q \) is a \( \phi \)-2-absorbing element of \( L \).

(ii) \( q \) is a 2-absorbing element of \( L \).
(iii) \( q \) is in one of the following type:

I) \( q = (1_{L_1}, q_2, q_3) \), where \( q_2 \) is a prime element of \( L_2 \) and \( q_3 \) is a prime element of \( L_3 \).

II) \( q = (q_1, 1_{L_2}, q_3) \), where \( q_1 \) is a prime element of \( L_1 \) and \( q_3 \) is a prime element of \( L_3 \).

III) \( q = (q_1, q_2, 1_{L_3}) \), where \( q_1 \) is a prime element of \( L_1 \) and \( q_2 \) is a prime element of \( L_2 \).

IV) For some \( i \in \{1, 2, 3\} \), \( q_i \) is a 2-absorbing element of \( L_i \) and \( q_j = 1_i \) for every \( j \in \{1, 2, 3\} \setminus \{i\} \).

**Proof.** (i) \( \Rightarrow \) (ii): If \( \phi(q) = \emptyset \) and \( q \) is a \( \phi \)-2-absorbing element, then obviously \( q \) is a 2-absorbing element of \( L \). So assume that \( \phi(q) \neq \emptyset \). Let \( q = (q_1, q_2, q_3) \) be a \( \phi \)-2-absorbing element of \( L \), then \( q \) is a 2-absorbing element of \( L \) by Lemma 2.28.

(ii) \( \Rightarrow \) (iii): Suppose that \( q \) is a 2-absorbing element of \( L \). Since \( q \neq \phi(q) \), there is a compact element of \( L \) such that \( (a_1, a_2, a_3) \leq q \) and \( (a_1, a_2, a_3) \not\in \phi(q) \). Since \( (a_1, a_2, a_3) = (a_1, 1_{L_2}, 1_{L_3})(1_{L_1}, a_2, 1_{L_3})(1_{L_1}, 1_{L_2}, a_3) \) and \( q \) is \( \phi \)-2-absorbing, we have \( (a_1, a_2, a_3) \leq q \) or \( (1_{L_1}, a_2, a_3) \leq q \) or \( (a_1, 1_{L_2}, a_3) \leq q \). This means that either \( q_1 = 1_{L_1} \) or \( q_2 = 1_{L_2} \) or \( q_3 = 1_{L_3} \).

Case I. Suppose that \( q = (1_{L_1}, q_2, q_3) \) where \( q_2 \neq 1_{L_2} \) and \( q_3 \neq 1_{L_3} \). We show that \( q_2 \) is a prime element of \( L_2 \). Let \( xy \leq q_2 \). Hence \( (1_{L_1}, x, 1_{L_3})(1_{L_1}, 1_{L_2}, q_3)(1_{L_1}, y, 1_{L_3}) \leq q \) and it implies that \( (1_{L_1}, x, 1_{L_3})(1_{L_1}, 1_{L_2}, q_3)(1_{L_1}, y, 1_{L_3}) \leq q \) or \( (1_{L_1}, 1_{L_2}, q_3)(1_{L_1}, y, 1_{L_3}) \leq q \). Since \( q_3 \) is proper, we get \( (1_{L_1}, x, 1_{L_3})(1_{L_1}, 1_{L_2}, q_3)(1_{L_1}, y, 1_{L_3}) \not\leq q \). Thus \( x \leq q_2 \) or \( y \leq q_2 \), which shows that \( q_2 \) is prime. By the similar argument one can easily show that \( q_3 \) is a prime element of \( L_3 \).

Case II. \( q = (q_1, 1_{L_2}, q_3) \), where \( q_1 \neq 1_{L_1} \) and \( q_3 \neq 1_{L_3} \) and Case III. \( q = (q_1, q_2, 1_{L_3}) \), where \( q_1 \neq 1_{L_1} \) and \( q_2 \neq 1_{L_2} \) can be easily obtained similar to Case I.

Case IV. Without loss of generality suppose that \( q = (q_1, 1_{L_2}, 1_{L_3}) \) where \( q_1 \) is a proper element of \( L_1 \). Let \( x, y, z \leq 1_{L_1} \) for some \( x, y, z \in L_1 \). Then \( x, y, z \leq 1_{L_1} \). Hence \( (x, y, z) \leq (x, y, z) \). Thus \( x, y, z \) is a function. If a proper \( a = (a_1, a_2, a_3) \in L \) is a \( \phi \)-2-absorbing element, then \( \psi_i(a_i) \neq \emptyset \) or \( \psi_i(a_i) = a_i \) (i = 1, 2, 3) for every proper element \( a_i \) of \( L_k \).

**Theorem 2.30.** Let \( L = L_1 \times L_2 \times L_3 \) where \( L_1, L_2, L_3 \) are multiplicative lattices. Let \( \phi = \psi_1 \times \psi_2 \times \psi_3 \), where \( \psi_i : L_i \rightarrow L_i \cup \{\emptyset\} \) (i = 1, 2, 3) is a function. If a proper \( a = (a_1, a_2, a_3) \in L \) is a \( \phi \)-2-absorbing element, then \( \psi_i(a_i) \neq \emptyset \) or \( \psi_i(a_i) = a_i \) (i = 1, 2, 3) for every proper element \( a_i \) of \( L_i \).

**Proof.** Assume on the contrary that \( \psi_i(a_i) \neq a_i \) and \( \psi_i(a_i) \neq \emptyset \) for some proper element \( a_i \in L_i \). Put \( a = (a_1, 0_{L_2}, 0_{L_3}) \). Hence \( (a_1, 1_{L_2}, 1_{L_3})(1_{L_1}, 0_{L_2}, 1_{L_3})(1_{L_1}, 0_{L_2}, 1_{L_3}) \leq a \), but \( (a_1, 1_{L_2}, 1_{L_3})(1_{L_1}, 0_{L_2}, 1_{L_3})(1_{L_1}, 1_{L_2}, 0_{L_3}) \not\leq \phi(a) \). Since \( a \) is a \( \phi \)-2-absorbing element, we conclude either \( (a_1, 1_{L_2}, 1_{L_3})(1_{L_1}, 1_{L_2}, 0_{L_3}) \leq \phi(a) \) or \( (a_1, 1_{L_2}, 1_{L_3})(1_{L_1}, 1_{L_2}, 0_{L_3}) \leq \phi(a) \). It follows \( 1_{L_3} \leq a_3 \) or \( 1_{L_2} \leq a_2 \) or \( 1_{L_2} \leq a_2 \), which are contradictions. Thus \( \psi_i(a_i) = a_i \) (i = 1, 2, 3) for every proper element \( a_i \) of \( L_i \).

**References**


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