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# Solving Nonlinear Equations Using Steffensen-Type Methods With Optimal Order of Convergence

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**Abstract.** In this paper some families of derivative-free methods, with optimal and non-optimal order of convergence, for solving nonlinear equations are suggested. In the proposed methods, several linear combinations of divided differences are used in order to get a good estimation of the derivative of the given function at the different steps of the iteration. The efficiency indices of the members of this family are equal to 1.587. The convergence and error analysis are given. Numerical comparisons are made with other existing methods to show the performance of the presented methods.

## 1. Introduction

In this paper we are going to develop efficient methods to find approximations of the root $r$ of nonlinear equations $f(x) = 0$, without evaluate its derivatives. A number of ways are considered by many researchers to improve the local order convergence of Newton's method by the expense of additional evaluations of the functions, derivatives and changes in the points of iterations see [1-9]. There are several different methods in the literature for the computation of the root of the nonlinear equation. The most famous of these methods is the classical Newton's method (NM):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

which is a well-known basic method and possesses quadratic order of convergence in the neighborhood of simple root $r$. This method is not applicable when the derivative of any function is not defined in any interval. Therefore the Newton's method was modified by Steffensen, who replaced the first derivative $f'(x_n)$ by the forward difference approximation

$$f'(x_n) = \frac{f(x_n+\beta\, f(x_n))-f(x_n)}{\beta\, f(x_n)} = P_0(x_n). \tag{1}$$

and obtained the famous Steffensen's method [10]:

$$x_{n+1} = x_n - \frac{\beta\, f(x_n)^2}{f(x_n+\beta\, f(x_n)) - f(x_n)},$$

where the parameter $\beta$ to be freely chosen in $R - \{0\}$ and used to generate a class of Steffensen's methods provided that the denominator is not equal to zero. Newton and Steffensen's methods are of second order converges, both require two functional evaluations per step, but in contrast

to Newton's method, Steffensen's method is free from any derivative of the function, because sometimes the applications of the iterative methods which depend upon derivatives are restricted in engineering.
A family of Steffensen like methods was derived in [10-13] free from derivatives, which uses three functional evaluations per step and has cubic convergence. Recently, Cordero et al. [14] proposed a derivative free iterative method by replacing the forward-difference approximation in Ostrowski's method [15] by the central-difference approximation. However, it is still a method of third order and requires four functional evaluations per iteration. Therefore, these methods have efficiency index $3^{1/4} \approx 1.1316$ which is less than $2^{1/2} \approx 1.4142$ of the Newton and Steffensen [16]. However, it is still a method of second order and requires three function evaluation per iteration. We consider the definition of efficiency index as $p^{1/w}$where $p$ is the order of the method and $w$ is the number of function evaluations per iteration required by the method [15]. However, the purpose of this paper is to establish new derivative-free methods with optimal order, i.e., we aim to increase the convergence rate from three to four without any additional evaluations of the function.
The following sections of the paper are organized as follows: In Section 2, we describe our family of methods. In Section 3, we show the order of convergence of these methods. In Section 4, different numerical test confirm the theoretical results and allow us to compare this family with other known methods mentioned in this section.

## 2. Description of the methods

To construct a novel class of two-step methods with optimal fourth-order, we first consider the following two-step third-order iteration. For a given $x_0$, compute approximate solution $x_{n+1}$ by following the iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2}$$

$$x_{n+1} = x_n - \frac{f(y_n)}{f'(x_n)}. \tag{3}$$

Now we can modify the methods in (2) and (3) by removing the derivatives in the following two-step method.

**HM1:** For a given $x_0$, compute approximate solution $x_{n+1}$ by the iterative scheme

$$\begin{matrix} y_n = x_n - \frac{\beta\, f(x_n)^2}{f(w)-f(x_n)}, \\ w = x_n + \beta\ f(x_n) \end{matrix} \quad , x_{n+1} = x_n - \frac{f(x_n)}{P_0(x_n)}.$$

where $P_0(x_n)$ is defined by equation (1). Let us remark that, in terms of computational cost, this method requires only three functional evaluations per step and it has convergence of order three. So, it has efficiency indices $3^{1/3} \approx 1.442$, which is higher than $2^{1/2} \approx 1.4142$ of the Newton and Steffensen's methods [16]. Therefore, these methods do not produce an optimal order of convergence. However, the purpose of this paper is to establish new derivative-free methods with optimal order, i.e., we aim to increase the convergence rate to four without the need to compute any additional function evaluations. So, we shall use weight functions $W_1$ , $W_2$ and $W_3$ which are expressed as:

$$W_1 = \left\{ \frac{4}{1+\left(f\left[x_n,y_n\right]\ f\left[w_n,y_n\right]\right)/P_0(x_n)^2} - 1 \right\} \tag{4}$$

$$W_2 = \frac{f[w_n,x_n]^2}{f[x_n,y_n]\ f[w_n,y_n]} \left\{ 1 + \frac{f(y_n)\ f[w_n,x_n]^2 (f[w_n,x_n]-f[x_n,y_n])}{f(x_n)(\ f[x_n,y_n]\ f[w_n,y_n])^2} \right\} \tag{5}$$

$$W_3 = \frac{f(x_n)\ f[x_n,y_n]\ f[w_n,y_n]\ f[w_n,x_n]^2}{f(x_n)\ f[x_n,y_n]^2 f[w_n,y_n]^2 - f(y_n)(\ f[w_n,x_n]-f[x_n,y_n])\ f[w_n,x_n]^3} \tag{6}$$

where $f[x_i, x_j] = (f(x_i) - f(x_j))/(x_i - x_j),\ \forall i, j \in N, i \neq j$. Taking into account these weight functions, we can suggest the following new two-step optimal methods, which will denote by Hafiz Methods (HM2-HM5).

**HM2:** For a given $x_0$, compute approximate solution $x_{n+1}$ by the iterative schemes

$$y_n = x_n - \frac{\beta\ f(x_n)^2}{f(w) - f(x_n)}, \qquad w = x_n + \beta\ f(x_n), x_{n+1} = y_n + \frac{f(y_n)}{P_0(x_n)}\ W_1.$$

**HM3:** For a given $x_0$, compute approximate solution $x_{n+1}$ by the iterative schemes

$$y_n = x_n - \frac{\beta\ f(x_n)^2}{f(w) - f(x_n)}, \qquad w = x_n + \beta\ f(x_n), x_{n+1} = y_n - \frac{f(y_n)}{P_0(x_n)}\ W_2$$

**HM4:** For a given $x_0$, compute approximate solution $x_{n+1}$ by the iterative schemes

$$y_n = x_n - \frac{\beta\ f(x_n)^2}{f(w) - f(x_n)}, \qquad w = x_n + \beta\ f(x_n), x_{n+1} = y_n - \frac{f(y_n)}{P_0(x_n)}\ W_3$$

these methods (HM2-HM4) have convergence of order four and require only three functional evaluations per step. So, they have efficiency indices $4^{1/3} \approx 1.5874$, that is, the order of convergence of these methods become four which is the optimal order conjectured by Kung and Traub [18].

## 3. Convergence analysis

Let us now discuss the convergence analysis of the above algorithms.

**Theorem 3.1:** Let $r$ be a simple zero of sufficient differentiable function $f :\subseteq R \to R$ for an open interval $I$. If $x_0$is sufficiently close to $r$, then the two-step method defined by HM2 has third-order of convergence.

**The Proof**. Consider

$$y_n = x_n - \frac{\beta\ f(x_n)^2}{f(w)-f(x_n)}, \qquad w = x_n + \beta\ f(x_n) \tag{7}$$

$$x_{n+1} = y_n - \frac{f(y_n)}{P_0(x_n)} \left\{ \frac{4}{1+f[x_n,y_n]\ f[w_n,y_n]/P_0(x_n)^2} - 1 \right\} \tag{8}$$

Let $r$be a simple zero of $f$. Since $f$is sufficiently differentiable, by expanding $f(x_n)$ about $r$ we get

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + \cdots \tag{9}$$

Again by using Taylor's expansion we can get

$$w_n = e_n + (c_1 + e_n(c_2 + e_n(c_3 + e_n c_4)))\beta + \cdots \tag{10}$$

$$f(w_n) = (1 + c_1\beta)\ c_1 e_n + (3\beta\ c_1 c_2 + \beta^2 c_1^2 c_2 + c_2)\ e_n^2 + \cdots \tag{11}$$

where $c_k = \frac{f^{(k)}(r)}{k!}$, $k = 1, 2, 3, \ldots$ and $e_n = x_n - r$.

Now by substituting (9) and (11) in (7), we have

$$y_n = r + (1 + \beta\ c_1)\frac{c_2}{c_1}\ e_n^2 + \cdots$$

By using Taylor's theorem, we have

$$f(y_n) = (1 + \beta\ c_1)\ c_2 e_n^2 + (\beta^2\ c_1^3 c_3 - 2\ c_2^2 + 3\beta\ c_1^2 c_3 + 2\ c_1 c_3 - \beta^2 c_1^2 c_2^2 - 2\beta\ c_1 c_2^2)\frac{1}{c_1}\ e_n^3 + \cdots \tag{12}$$

$$f[w_n, x_n] = P_0(x_n) = c_1 + (2 + \beta\ c_1)\ c_2\ e_n + (3\ c_3 + 3\beta\ c_1 c_3 + \beta^2 c_1^2 c_3 + \beta\ c_2^2)\ e_n^2 + \cdots$$

$$f[w_n, y_n] = c_1 + (1 + \beta\ c_1)\ c_2\ e_n + (2\beta\, c_1 c_2^2 + \ c_1 c_3 + 2\beta\ c_1^2 c_3 + \beta^2 c_1^3 c_3 + \ c_2^2)\ \frac{1}{c_1}\ e_n^2 + \cdots$$

$$f[x_n, y_n] = c_1 + c_2\, e_n + (c_1\, c_3 + c_2^2 + \beta\, c_1 c_2^2)\frac{e_n^2}{c_1} + \cdots$$

$$\frac{f\,[x_n, y_n]\ \ f\,[w_n, y_n]}{P_0(x_n)^2} = 1 - c_2\,(2 + \beta\, c_1)\frac{e_n}{c_1} + [(7 + c_1\beta(6 + c_1\beta))c_2^2 - \beta c_1\, c_3\,(2 + c_1\beta)^2]\frac{e_n^2}{c_1^2} + \cdots$$

$$\begin{aligned}\frac{4}{f[x_n,y_n]f[w_n,y_n]/P_0(x_n)^2+1} &= 2 + (\beta\, c_1 + 2)\tfrac{c_2\, e_n}{c_1} + [\beta^2 c_1^3 c_3 + \\ &\beta\, c_1^2(4c_3 - \tfrac{1}{2}\beta\, c_2^2) + c_1(5\, c_3 - 4\beta\, c_2^2) - 5\, c_2^2]\tfrac{e_n^2}{c_1^2} + \cdots\end{aligned} \tag{13}$$

$$\tfrac{f(y_n)}{P_0(x_n)} = (1 + \beta\, c_1)\tfrac{c_2}{c_1}\, e_n^2 + [\, c_1 c_3(1 + \beta\, c_1)(2 + \beta\, c_1) - c_2^2(4 + \beta\, c_1(5 + 2\beta\, c_1\,))]\ \tfrac{e_n^3}{c_1^2} + \cdots \tag{14}$$

Now by substituting (12), (13) and (14) in (8), we have

$$e_{n+1} = [\, c_2^2(8 + \beta\, c_1(8 + \beta\, c_1)) - 2c_1 c_3(1 + \beta\, c_1)](1 + \beta\, c_1)\frac{c_2}{2c_1^3}\ e_n^4 + O(e_n^5)$$

from which it follows that (HM2) has convergence of fourth-order.

**Theorem 3.2:** Let $r$ be a simple zero of sufficient differentiable function $f :\subseteq R \to R$ for an open interval $I$. If $x_0$is sufficiently close to $r$, then the two-step method defined by (HM3) has fourth-order convergence.

**The Proof**. Consider

$$y_n = x_n - \tfrac{\beta\, f(x_n)^2}{f(w) - f(x_n)}, \qquad w = x_n + \beta\, f(x_n) \tag{15}$$

$$x_{n+1} = y_n - \tfrac{f(y_n)}{P_0(x_n)}\ W_2 \tag{16}$$

Now by substituting from (5), (12), (13) and (14) in (16), we have

$$e_{n+1} = [(1 + (\beta - 2)c_1)c_2^2 + c_3 c_1^2)](1 + \beta\, c_1)^2\tfrac{c_2}{c_1^4}e_n^4 + O(e_n^5) \tag{17}$$

from which it follows that (HM3) has four-order convergence.

In similar way, we observe that the method HM4 has also fourth order of convergence, but HM1 has three order of convergence as follows

$$\text{(HM4)} \quad e_{n+1} = (2 + \beta\, c_1)\,(1 + \beta\, c_1)\ \tfrac{c_2^2}{c_1^2}\ e_n^3 + O(e_n^4) \tag{18}$$

$$\text{(HM4)} \quad e_{n+1} = -[\, c_1 c_3 + (\beta\, c_1 - 1)\, c_2^2](1 + \beta\, c_1)^2\ \tfrac{c_2}{c_1^3}\ e_n^4 + O(e_n^5) \tag{19}$$

## 4. Numerical examples

For comparisons, we will use the fourth-order Ren's method (RM) [18] defined by:

$$y_n = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)},$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f\,[x_n, y_n] + f\,[w_n, y_n] - f\,[x_n, w_n] - (x_n - y_n)(w_n - y_n)}.$$

and Cordero method (CM) [20]

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}, \\ x_{n+1} &= x_n - \frac{f(y_n)}{\frac{a\, f(y_n) - b\, f(w_n)}{y_n - w_n} + \frac{c\, f(x_n) - d\, f(y_n)}{x_n - y_n}}, \end{aligned}$$

$$a = c = 1, \ \ b + d = 1.$$

We consider here some numerical examples to demonstrate the performance of the new modified two-step iterative methods, namely algorithms (HM2) - (HM4). We compare the Ren's method (RM), Cordero method (CM) with (HM2) - (HM4). The computational order of convergence ($COC$) [21], can be approximated using the formula

$$COC \approx \frac{\ln |(x_{n+1} - x_n)/(x_n - x_{n-1})|}{\ln |(x_n - x_{n-1})/(x_{n-1} - x_{n-2})|}.$$

Our examples are tested with precision $\varepsilon = 10^{-250}$ as tolerance and $\beta = 1$. All the computations are performed using Maple 15 with 300 significant digits, but only two digits are displayed for ($COC$). Results are summarized in Tables 1 and 2 as it shown, the new algorithms are comparable with all methods and in most cases it gives better or equal results.

## 5. Conclusions

The present study suggests a family of new derivative-free iterative methods for solving nonlinear equations. The efficiency indices of the members of this family are equal to 1.587. In addition, these methods are particularly suited to those problems in which derivatives require lengthy. In the sequel, numerical examples have used in order to show the efficiency and accuracy of the novel methods from our suggested derivative-free class. Finally, it should be noted that, like all other iterative methods, the new methods from the class (HM2)-(HM4) have their own domains of validity and in certain circumstances should not be used.

**Table 1:** Test functions and their roots.

| Functions | Roots |
|---|---|
| $f_1(x) = \sin^2 x - x^2 + 1,$ | 1.40449164821534 |
| $f_2(x) = x^2 - e^x - 3x + 2,$ | 0.25753028543986 |
| $f_3(x) = \cos x - x,$ | 0.73908513321516 |
| $f_4(x) = x - 3\ \log x,$ | 1.85718386020784 |
| $f_5(x) = e^{-x} + \cos x,$ | 1.74613953040801 |
| $f_6(x) = x - \sin(\cos x) + 1$ | 0.1660390510510 |
| $f_7(x) = \frac{1}{x} - \lvert x\rvert,$ | 1 |
| $f_8(x) = \sin x - x/100,$ | 0 |
| $f_9(x) = e^x - 1.5 - \tan^{-1} x,$ | 0.85945269083105 |
| $f_{10}(x) = \sqrt{x} - \cos x$ | 1.29799774328037 |

**Table 2.** Comparison of some derivative-free methods for the number of iterations

| *Iterations* | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ | $f_8$ | $f_9$ | $f_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $x_0$ | 1.3 | 1.0 | 1.7 | 0.5 | 3.0 | 1.6 | -0.5 | 0.7 | 0.1 | 1.6 |
| RM | 5 | 5 | 19 | 8 | 5 | 18 | 5 | 8 | 6 | 6 |
| CM | 4 | 5 | 5 | 6 | 5 | 6 | 5 | 5 | 5 | 5 |
| HM2 | 5 | 5 | 5 | 6 | 5 | 6 | 4 | 6 | 5 | 5 |
| HM3 | 4 | 5 | 5 | 7 | 5 | 6 | 5 | 5 | 5 | 5 |
| HM4 | 5 | 5 | 5 | 8 | 5 | 6 | 5 | 5 | 5 | 5 |

**Table 3.** Comparison of the computational order of convergence (*COC*) for various methods.

| *Iterations* | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ | $f_8$ | $f_9$ | $f_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $x_0$ | 1.3 | 1.0 | 1.7 | 0.5 | 3.0 | 1.6 | -0.5 | 0.7 | 0.1 | 1.6 |
| RM | 4 | 3.99 | 4 | 4 | 4 | 4 | 4 | 5 | 3.99 | 3.99 |
| CM | 4 | 3.99 | 3.99 | 3.99 | 4 | 3.99 | 3.99 | 5 | 4 | 4 |
| HM2 | 4 | 3.99 | 3.99 | 3.99 | 4 | 3.99 | 5.99 | 5 | 4 | 4 |
| HM3 | 4 | 3.99 | 3.99 | 4 | 4 | 3.99 | 3.99 | 5 | 4 | 4 |
| HM4 | 4 | 3.99 | 4 | 3.99 | 4 | 3.99 | 4 | 5 | 4 | 4 |

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