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# On Generalization Ostrowski Type Inequalities for Functions of Two Variables with Bounded Variation

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**Abstract** In this paper, we establish a generalization of the Ostrowski type integral inequalities for functions of two independent variables with bounded variation and we give some applications for general quadrature formulae.

## 1 Introduction

Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $(a,b)$ whoose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on $(a,b)$, i.e. $\|f'\|_\infty := \sup\limits_{t\in(a,b)} |f'(t)| < \infty$. Then we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty , \tag{1.1}$$

for all $x \in [a,b]$[18]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.*

In [10], Dragomir proved following Ostrowski type inequalities related functions of bounded variation:

**Theorem 1.1.** *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left| \int_a^b f(t)dt - (b-a) f(x) \right| \leq \left[ \frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (f)$$

*holds for all $x \in [a,b]$. The constant $\frac{1}{2}$ is the best possible.*

In a recent years, many authors studied the well-known Ostrowski inequality in one variable for variant types of functions such as, Lipschitzian, absolutely continuous and $n$-differentiable functions as well as the functions of bounded variation. However, a small attention and a few works have been considered for functions of two variables with bounded variation (see, [3], [6], [7], [16] ). Among others, in particular, Dragomir and his group studied a very interesting inequalities for functions of one variable. For more information and recent developments on inequalities for mappings of bounded variation, please refer to([1], [2], [5], [8]-[14], [17], [19]-[24]).

## 2 Preliminaries and Lemmas

In 1910, Fréchet [15] has given the following characterization for the double Riemann-Stieltjes integral. Assume that $f(x,y)$ and $\alpha(x,y)$ are defined over the rectangle $Q = [a,b] \times [c,d]$; let $R$ be the divided into rectangular subdivisions, or cells, by the net of straight lines $x = x_i$, $y = y_i$,

$$a = x_0 < x_1 < ... < x_n = b, \text{ and } c = y_0 < y_1 < ... < y_m = d;$$

let $\xi_i, \eta_j$ be any numbers satisfying $\xi_i \in [x_{i-1}, x_i]$, $\eta_j \in [y_{j-1}, y_j]$, :($i = 1,2,...,n$; $j = 1,2,...,m$); and for all $i,j$ let

$$\Delta_{11}\alpha(x_i,y_j) = \alpha(x_{i-1},y_{j-1}) - \alpha(x_{i-1},y_j) - \alpha(x_i,y_{j-1}) + \alpha(x_i,y_j).$$

Then if the sum

$$S = \sum_{i=1}^{n}\sum_{j=1}^{m} f(\xi_i,\eta_j)\,\Delta_{11}\alpha(x_i,y_j)$$

tends to a finite limit as the norm of the subdivisions approaches zero, the integral of $f$ with respect to $\alpha$ is said to exist. We call this limit the restricted integral, and designate it by the symbol

$$\int_a^b \int_c^d f(x,y) d_y d_x \alpha(x,y). \tag{2.1}$$

If in the above formulation $S$ is replaced by the sum

$$S^* = \sum_{i=1}^{n} \sum_{j=1}^{m} f(\xi_{ij}, \eta_{ij}) \Delta_{11} \alpha(x_i, y_j),$$

where $\xi_{ij}, \eta_{ij}$ are numbers satisfying $\xi_{ij} \in [x_{i-1}, x_i]$, $\eta_{ij} \in [y_{j-1}, y_j]$, we call the limit, when it exist, the unrestricted integral, and designate it by the symbol

$$\int_a^b \int_c^d f(x,y) d_y d_x \alpha(x,y). \tag{2.2}$$

Clearly, the existence of (2.2) implies both the existence of (2.1) and its equality (2.2). On the other hand, Clarkson ([7]) has shown that the existence of (2.1) does not imply the existence of (2.2).

In [6], Clarkson and Adams gave the following definitions of bounded variation for functions of two variables:

### 2.1 Definitions

The function $f(x,y)$ is assumed to be defined in rectangle $R(a \leq x \leq b,\, c \leq y \leq d)$. By the term $net$ we shall, unless otherwise specified mean a set of parallels to the axes:

$$\begin{aligned} x &= x_i(i = 0,1,2,...,m),\ a = x_0 < x_1 < ... < x_m = b; \\ y &= y_j(j = 0,1,2,...,n),\ c = y_0 < y_1 < ... < y_n = d. \end{aligned}$$

Each of the smaller rectangles into which $R$ is devided by a net will be called a *cell*. We employ the notation

$$\Delta_{11} f(x_i, y_j) = f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j),$$

$$\Delta f(x_i, y_j) = f(x_{i+1}, y_{j+1}) - f(x_i, y_j).$$

The total variation function, $\phi(\overline{x})$ $[\psi(\overline{y})]$, is defined as the total variation of $f(\overline{x}, y)$ $[f(x, \overline{y})]$ considered as a function of $y$ $[x]$ alone in interval $(c,d)$ $[(a,b)]$, or as $+\infty$ if $f(\overline{x}, y)$ $[f(x, \overline{y})]$ is of unbounded variation.

**Definition 2.1.** (Vitali-Lebesque-Fréchet-de la Vallée Poussin). The function $f(x,y)$ is said tobe of bounded variation if the sum

$$\sum_{i=0\,,\,j=0}^{m-1\,,\,n-1} |\Delta_{11} f(x_i, y_j)|$$

is bounded for all nets.

**Definition 2.2.** (Fréchet). The function $f(x,y)$ is said tobe of bounded variation if the sum

$$\sum_{i=0\,,\,j=0}^{m-1\,,\,n-1} \epsilon_i \overline{\epsilon_j} |\Delta_{11} f(x_i, y_j)|$$

is bounded for all nets and all possible choices of $\epsilon_i = \pm 1$ and $\overline{\epsilon_j} = \pm 1$.

**Definition 2.3.** (Hardy-Krause). The function $f(x,y)$ is said tobe of bounded variation if it satisfies the condition of Definition 2.1 and if in addition $f(\overline{x}, y)$ is of bounded variation in $y$ (i.e. $\phi(\overline{x})$ is finite) for at least one $\overline{x}$ and $f(x, \overline{y})$ is of bounded variation in $y$ (i.e. $\psi(\overline{y})$ is finite) for at least one $\overline{y}$.

**Definition 2.4.** (Arzelà). Let $(x_i, y_i)$ $(i = 0, 1, 2, ..., m)$ be any set of points satisfiying the conditions

$$\begin{aligned} a &= x_0 < x_1 < ... < x_m = b; \\ c &= y_0 < y_1 < ... < y_m = d. \end{aligned}$$

Then $f(x, y)$ is said tobe of bounded variation if the sum

$$\sum_{i=1}^{m} |\Delta f(x_i, y_i)|$$

is bounded for all such sets of points.

Therefore, one can define the consept of total variation of a function of two variables, as follows:

Let $f$ be of bounded variation on $Q = [a, b] \times [c, d]$, and let $\sum (P)$ denote the sum $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} |\Delta_{11} f(x_i, y_j)|$ corresponding to the partition $P$ of $Q$. The number

$$\bigvee_Q (f) := \bigvee_c^d \bigvee_a^b (f) := \sup \left\{ \sum (P) : P \in P(Q) \right\},$$

is called the total variation of $f$ on $Q$. Here $P([a, b])$ denotes the family of partitions of $[a, b]$.

In [16], authors proved following Lemmas related double Riemann-Stieltjes integral:

**Lemma 2.5.** *(Integrating by parts) If $f \in RS(\alpha)$ on $Q$, then $\alpha \in RS(f)$ on $Q$, and we have*

$$\begin{aligned} &\int_c^d \int_a^b f(t, s) d_t d_s \alpha(t, s) + \int_c^d \int_a^b \alpha(t, s) d_t d_s f(t, s) \qquad (2.3) \\ &= f(b, d)\alpha(b, d) - f(b, c)\alpha(b, c) - f(a, d)\alpha(a, d) + f(a, c)\alpha(a, c). \end{aligned}$$

**Lemma 2.6.** *Assume that $g \in RS(\alpha)$ on $Q$ and $\alpha$ is of bounded variation on $Q$, then*

$$\left| \int_c^d \int_a^b g(x, y) d_x d_y \alpha(x, y) \right| \leq \sup_{(x,y) \in Q} |g(x, y)| \bigvee_Q (\alpha). \qquad (2.4)$$

In [16], Jawarneh and Noorani obtained following Ostrowski type inequality for functions of two variables with bounded variation:

**Theorem 2.7.** *Let $f : Q \to\to \mathbb{R}$ be a mapping of bounded variation on $Q$. Then for all $(x, y) \in Q$, we have inequality*

$$\begin{aligned} &\left| (b - a)(d - c) f(x, y) - \int_c^d \int_a^b f(t, s) dt ds \right| \qquad (2.5) \\ &\leq \left[ \frac{1}{2}(b - a) + \left| x - \frac{a + b}{2} \right| \right] \left[ \frac{1}{2}(d - c) + \left| y - \frac{c + d}{2} \right| \right] \bigvee_Q (f) \end{aligned}$$

*where $\bigvee_Q (f)$ denotes the total (double) variation of $f$ on $Q$.*

The aim of this paper is to establish a generalization of the Ostrowski type integral inequalities for functions of two independent variables with bounded variation and we give some applications for general quadrature formulae.

## 3 Main Results

We first prove the following theorem:

**Theorem 3.1.** *Let $f : Q \to \mathbb{R}$ be a mapping of bounded variation on $Q$. Then for all $(x, y) \in Q$, we have inequality*

$$\Bigg|(b-a)(d-c)\Bigg[(1-\lambda)(1-\eta)f(x,y)+\frac{(1-\lambda)\eta}{2}[f(a,y)+f(b,y)] \tag{3.1}$$

$$+\frac{\lambda(1-\eta)}{2}[f(x,c)+f(x,d)]+\frac{\lambda\eta}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)]\Bigg]-\int_a^b\int_c^d f(t,s)dsdt\Bigg|$$

$$\leq \max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right),\left(\frac{(2-\lambda)b+\lambda a}{2}-x\right)\right\}$$

$$\times \max\left\{\eta\frac{d-c}{2},\left(y-\frac{(2-\eta)c+\eta d}{2}\right),\left(\frac{(2-\eta)d+\eta c}{2}-y\right)\right\}\bigvee_a^b\bigvee_c^d(f)$$

*for any $\lambda, \eta \in [0, 1]$ and $a + \lambda\frac{b-a}{2} \leq x \leq b - \lambda\frac{b-a}{2}$, $c + \eta\frac{d-c}{2} \leq y \leq d - \eta\frac{d-c}{2}$, where $\bigvee_a^b\bigvee_c^d(f)$ denotes he total variation of $f$ on $Q$.*

*Proof.* Applying Lemma 2.5, we have

$$\int_a^x\int_c^y\left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(s-\left(c+\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s) \tag{3.2}$$

$$=\left(x-a-\lambda\frac{b-a}{2}\right)\left(y-c-\eta\frac{d-c}{2}\right)f(x,y)$$

$$+\left(x-a-\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(x,c)$$

$$+\left(\lambda\frac{b-a}{2}\right)\left(y-c-\eta\frac{d-c}{2}\right)f(a,y)$$

$$+\left(\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(a,c)-\int_a^x\int_c^y f(t,s)dsdt,$$

and similarly

$$\int_a^x\int_y^d\left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(s-\left(d-\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s) \tag{3.3}$$

$$=\left(x-a-\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(x,d)$$

$$+\left(x-a-\lambda\frac{b-a}{2}\right)\left(d-y-\eta\frac{d-c}{2}\right)f(x,y)$$

$$+\left(\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(a,d)$$

$$+\left(\lambda\frac{b-a}{2}\right)\left(d-y-\eta\frac{d-c}{2}\right)f(a,y)-\int_a^x\int_y^d f(t,s)dsdt,$$

$$\int_x^b\int_c^y \left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(s-\left(c+\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s) \tag{3.4}$$
$$\begin{aligned}
&= \left(\lambda\frac{b-a}{2}\right)\left(y-c-\eta\frac{d-c}{2}\right)f(b,y)\\
&+\left(\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(a,c)\\
&+\left(b-x-\lambda\frac{b-a}{2}\right)\left(y-c-\eta\frac{d-c}{2}\right)f(x,y)\\
&+\left(b-x-\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(x,c)-\int_x^b\int_c^y f(t,s)dsdt,
\end{aligned}$$

$$\int_x^b\int_y^d \left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(s-\left(d-\eta\frac{d-c}{2}\right)\right)d_sd_tf(t,s) \tag{3.5}$$
$$\begin{aligned}
&= \left(\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(b,d)\\
&+\left(\lambda\frac{b-a}{2}\right)\left(d-y-\eta\frac{d-c}{2}\right)f(b,y)\\
&+\left(b-x-\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(x,d)\\
&+\left(b-x-\lambda\frac{b-a}{2}\right)\left(d-y-\eta\frac{d-c}{2}\right)f(x,y)-\int_x^b\int_y^d f(t,s)dsdt.
\end{aligned}$$

Summing (3.2)-(3.5), we have

$$\int_a^b\int_c^d P(x,t;y,s)d_sd_tf(t,s) \tag{3.6}$$
$$\begin{aligned}
&= (b-a)(d-c)\left[(1-\lambda)(1-\eta)f(x,y)+\frac{(1-\lambda)\eta}{2}[f(a,y)+f(b,y)]\right.\\
&\left.+\frac{\lambda(1-\eta)}{2}[f(x,c)+f(x,d)]+\frac{\lambda\eta}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)]\right]-\int_a^b\int_c^d f(t,s)dsdt
\end{aligned}$$

where

$$P(x,t;y,s)=\begin{cases}\left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(s-\left(c+\eta\frac{d-c}{2}\right)\right) & ,(t,s)\in[a,x]\times[c,y]\\ \left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(s-\left(d-\eta\frac{d-c}{2}\right)\right) & ,(t,s)\in[a,x]\times(y,d]\\ \left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(s-\left(c+\eta\frac{d-c}{2}\right)\right) & ,(t,s)\in(x,b]\times[c,y]\\ \left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(s-\left(d-\eta\frac{d-c}{2}\right)\right) & ,(t,s)\in(x,b]\times(y,d]\end{cases}$$

for any $\lambda,\eta\in[0,1]$ and $a+\lambda\frac{b-a}{2}\le x\le b-\lambda\frac{b-a}{2}$, $c+\eta\frac{d-c}{2}\le y\le d-\eta\frac{d-c}{2}$.

Now, taking the modulus in (3.6), we have

$$\left|\int_a^b\int_c^d P(x,t;y,s)d_sd_tf(t,s)\right|$$

$$= \left|(b-a)(d-c)\left[(1-\lambda)(1-\eta)f(x,y)+\frac{(1-\lambda)\eta}{2}[f(a,y)+f(b,y)]\right.\right.$$

$$\left.\left.+\frac{\lambda(1-\eta)}{2}[f(x,c)+f(x,d)]+\frac{\lambda\eta}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)]\right]-\int_a^b\int_c^d f(t,s)dsdt\right|.$$

On the other hand, using Lemma 2.6 it follows that

$$\left|\int_a^b\int_c^d P(x,t;y,s)d_sd_tf(t,s)\right|$$

$$\leq \sup_{(t,s)\in Q}|P(x,y;t,s)|\bigvee_a^b\bigvee_c^d(f)$$

$$= \max\left\{\sup_{(t,s)\in[a,x]\times[c,y]}|P(x,y;t,s)|,\sup_{(t,s)\in[a,x]\times(y,d]}|P(x,y;t,s)|,\right.$$

$$\left.\sup_{(t,s)\in(x,b]\times[c,y]}|P(x,y;t,s)|,\sup_{(t,s)\in(x,b]\times(y,d]}|P(x,y;t,s)|\right\}\bigvee_a^b\bigvee_c^d(f)$$

$$= \max\left\{\max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right)\right\}\max\left\{\eta\frac{d-c}{2},\left(y-\frac{(2-\eta)c+\eta d}{2}\right)\right\},\right.$$

$$\max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right)\right\}\max\left\{\eta\frac{d-c}{2},\left(\frac{(2-\eta)d+\eta c}{2}-y\right)\right\},$$

$$\max\left\{\lambda\frac{b-a}{2},\left(\frac{(2-\lambda)b+\lambda a}{2}-x\right)\right\}\max\left\{\eta\frac{d-c}{2},\left(y-\frac{(2-\eta)c+\eta d}{2}\right)\right\},$$

$$\left.\max\left\{\lambda\frac{b-a}{2},\left(\frac{(2-\lambda)b+\lambda a}{2}-x\right)\right\}\max\left\{\eta\frac{d-c}{2},\left(\frac{(2-\eta)d+\eta c}{2}-y\right)\right\}\right\}\bigvee_a^b\bigvee_c^d(f)$$

$$\leq \max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right),\left(\frac{(2-\lambda)b+\lambda a}{2}-x\right)\right\}$$

$$\times\max\left\{\eta\frac{d-c}{2},\left(y-\frac{(2-\eta)c+\eta d}{2}\right),\left(\frac{(2-\eta)d+\eta c}{2}-y\right)\right\}\bigvee_a^b\bigvee_c^d(f).$$

This completes the proof of Theorem. □

**Remark 3.2.** Under the assumptions of Theorem 3.1 with $\lambda=0$ and $\eta=0$, the inequality (3.1) reduces inequality (2.5).

**Remark 3.3.** If we take $\lambda=1$ and $\eta=1$ in Theorem 3.1, we have the trapezoid inequality

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\leq\frac{1}{4}\bigvee_a^b\bigvee_c^d(f) \quad (3.7)$$

which proved by Jawarneh and Noorani in [16]. The constant $\frac{1}{4}$ is the best possible.

*Proof.* For proof of the sharpness of the constant, assume that (3.7) holds with a constant $A > 0$,that is,

$$\left|\frac{f(b,d)+f(a,d)+f(b,c)+f(a,c)}{4}-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right| \leq A\bigvee_a^b\bigvee_c^d(f). \tag{3.8}$$

If we choose $f : Q \to \mathbb{R}$ with

$$f(x,y)=\begin{cases} 1 \text{ if } & x=a,b \text{ and } y=c,d \\ 0 \text{ if } & (x,y)\in(a,b)\times(c,d)\end{cases}$$

then $f$ is of bounded variation on $Q$, and

$$\frac{f(b,d)+f(a,d)+f(b,c)+f(a,c)}{4}=1, \int_a^b\int_c^d f(t,s)dsdt=0, \text{ and } \bigvee_Q(f)=4,$$

giving in (3.8), $1 \leq 4A$, thus $A \geq \frac{1}{4}$. □

**Corollary 3.4.** *Under the assumptions of Theorem 3.1 with $\lambda = \frac{1}{3}$ and $\eta = \frac{1}{3}$, we have the inequality*

$$\begin{aligned} &\left|(b-a)(d-c)\left[\frac{4}{9}f(x,y)+\frac{f(a,y)+f(b,y)+f(x,c)+f(x,d)}{9}\right.\right. \\ &\left.\left.++\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{36}\right]-\int_a^b\int_c^d f(t,s)dsdt\right| \\ \leq\ & \max\left\{\frac{b-a}{6},\left(x-\frac{5a+b}{6}\right),\left(\frac{5b+a}{6}-x\right)\right\} \\ &\times\max\left\{\frac{d-c}{6},\left(y-\frac{5c+d}{6}\right),\left(\frac{5d+c}{6}-y\right)\right\}\bigvee_a^b\bigvee_c^d(f)\end{aligned} \tag{3.9}$$

*for $\frac{5a+b}{6} \leq x \leq \frac{5b+a}{6}$ and $\frac{5c+d}{6} \leq y \leq \frac{5d+c}{6}$.*

**Remark 3.5.** If we choose $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in Corollary 3.4, then we have the "Simpson's rule inequality "

$$\begin{aligned} &\left|(b-a)(d-c)\left[\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{36}\right.\right. \\ &+\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{9} \\ &\left.\left.+\frac{4}{9}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]-\int_a^b\int_c^d f(t,s)dsdt\right| \\ \leq\ & \frac{1}{9}(b-a)(d-c)\bigvee_a^b\bigvee_c^d(f)\end{aligned}$$

which is proved by Jawarneh and Noorani in [16].

**Corollary 3.6.** *Under the assumptions of Theorem 3.1 with $\lambda = \frac{1}{2}$ and $\eta = \frac{1}{2}$, we have the* *inequality*

$$\left|\frac{(b-a)(d-c)}{4}\left[f(x,y)+\frac{f(a,y)+f(b,y)+f(x,c)+f(x,d)}{2}\right.\right. \tag{3.10}$$

$$\left.\left.++\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right]-\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\leq \max\left\{\frac{b-a}{4},\left(x-\frac{3a+b}{4}\right),\left(\frac{3b+a}{4}-x\right)\right\}$$

$$\times\max\left\{\frac{d-c}{4},\left(y-\frac{3c+d}{4}\right),\left(\frac{3d+c}{4}-y\right)\right\}\bigvee_a^b\bigvee_c^d(f)$$

*for* $\frac{3a+b}{4}\leq x\leq\frac{3b+a}{4}$ *and* $\frac{3c+d}{4}\leq y\leq\frac{3d+c}{4}$.

**Corollary 3.7.** *If we take* $x=\frac{a+b}{2}$ *and* $y=\frac{c+d}{2}$ *in Corollary 3.6, then we get*

$$\left|\frac{(b-a)(d-c)}{4}\left[\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{4}\right.\right.$$

$$+\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{2}$$

$$\left.\left.+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]-\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\leq \frac{(b-a)(d-c)}{16}\bigvee_a^b\bigvee_c^d(f).$$

*The constant* $\frac{1}{16}$ *is the best possible.*

*Proof.* Assume that (3.10) holds with a constant $C>0$, i.e.,

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{16}\right. \tag{3.11}$$

$$+\frac{f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)}{8}$$

$$\left.+\frac{1}{4}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\leq C\bigvee_Q(f).$$

Define the set

$$E := \left\{(a,c),(a,d),(b,c),(b,d),\left(a,\frac{c+d}{2}\right),\left(b,\frac{c+d}{2}\right),\right.$$

$$\left.\left(\frac{a+b}{2},c\right),\left(\frac{a+b}{2},d\right),\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right\}.$$

If we choose $f:Q\to\mathbb{R}$ with

$$f(x,y)=\begin{cases}1 \text{ if} & (x,y)\in E\\ 0 \text{ if} & (x,y)\in[a,b]\times[c,d]\setminus E\end{cases}$$

then $f$ is of bounded variation on $Q$, and

$$\begin{aligned}
\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{16} &= \frac{1}{4}, \\
\frac{f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)}{8} &= \frac{1}{2}, \\
\frac{1}{4}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &= \frac{1}{4}, \\
\int\limits_a^b\int\limits_c^d f(t,s)dsdt &= 0, \\
\text{and } \bigvee_Q(f) &= 16.
\end{aligned}$$

Therefore, we get in (3.11), $1 \leq 16C$, thus $C \geq \frac{1}{16}$, which implies the constant $\frac{1}{16}$ is the best possible. This completes the proof. □

## 4 Some Composite Quadrature Formula

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_n = b$, and $J_m : c = y_0 < y_1 < ... < y_m = d$, $h_i := x_{i+1} - x_i$, and $l_j := y_{j+1} - y_j$,

$$\upsilon(h) := \max\left\{h_i |\ i = 0, ..., n-1\right\},$$

$$\upsilon(l) := \max\left\{l_j |\ j = 0, ..., m-1\right\}.$$

Then the following Theorem holds.

**Theorem 4.1.** *Let $f : Q \to \mathbb{R}$ is of bounded variatin on Q and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, $\tau_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$. Then we have the quadrature formula:*

$$\begin{aligned}
&\int\limits_a^b\int\limits_c^d f(t,s)dsdt \\
&= \frac{4}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(\xi_i,\tau_j)h_i l_j \\
&\quad +\frac{1}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f(x_i,\tau_j)+f(x_{i+1},\tau_j)+f(\xi_i,y_j)+f(\xi_i,y_{j+1})\right]h_i l_j \\
&\quad +\frac{1}{36}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})\right]h_i l_j \\
&\quad +R(I_n,J_m,\xi,\tau,f)
\end{aligned}$$

*The remainder $R(I_n, J_m, \xi, \eta, f)$ satisfies*

$$\begin{aligned}
&|R(I_n,J_m,\xi,\eta,f)| \\
&\leq \max_{i\in\{0,...,n-1\}}\left\{\max\left\{\frac{h_i}{6},\left(\xi_i-\frac{5x_i+x_{i+1}}{6}\right),\left(\frac{5x_{i+1}+x_i}{6}-\xi_i\right)\right\}\right\} \\
&\quad \times \max_{j\in\{j=0,...,m-1\}}\left\{\max\left\{\frac{l_j}{6},\left(\tau_j-\frac{5y_j+y_{j+1}}{6}\right),\left(\frac{5y_{j+1}+y_j}{6}-\tau_j\right)\right\}\right\} \\
&\quad \times \bigvee_a^b\bigvee_c^d(f).
\end{aligned}$$

*Proof.* Applying Corollary 3.4 to bidimentional interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, $\tau_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$, we have the inequality

$$\Bigg|(b-a)(d-c)\left[\frac{4}{9}f(\xi_i, y) + \frac{f(x_i,\tau_j) + f(x_{i+1},\tau_j) + f(\xi_i, y_j) + f(\xi_i, y_{j+1})}{9}\right. \tag{4.1}$$

$$\left. + + \frac{f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})}{36}\right] - \int_a^b \int_c^d f(t,s)dsdt\Bigg|$$

$$\leq \max\left\{\frac{h_i}{6}, \left(\xi_i - \frac{5x_i + x_{i+1}}{6}\right), \left(\frac{5x_{i+1} + x_i}{6} - \xi_i\right)\right\}$$

$$\times \max\left\{\frac{l_j}{6}, \left(\tau_j - \frac{5y_j + y_{j+1}}{6}\right), \left(\frac{5y_{j+1} + y_j}{6} - \tau_j\right)\right\} \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f).$$

Summing the inequality (4.1) over $i$ from 0 to $n-1$ and $j$ from 0 to $m-1$, then we get

$$|R(I_n, J_m, \xi, \tau, f)|$$

$$\leq \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[\max\left\{\frac{h_i}{6}, \left(\xi_i - \frac{5x_i + x_{i+1}}{6}\right), \left(\frac{5x_{i+1} + x_i}{6} - \xi_i\right)\right\}\right.$$

$$\left.\times \max\left\{\frac{l_j}{6}, \left(\tau_j - \frac{5y_j + y_{j+1}}{6}\right), \left(\frac{5y_{j+1} + y_j}{6} - \tau_j\right)\right\} \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f)\right]$$

$$\leq \max_{i\in\{0,\ldots,n-1\}}\left\{\max\left\{\frac{h_i}{6}, \left(\xi_i - \frac{5x_i + x_{i+1}}{6}\right), \left(\frac{5x_{i+1} + x_i}{6} - \xi_i\right)\right\}\right\}$$

$$\times \max_{j\in\{j=0,\ldots,m-1\}}\left\{\max\left\{\frac{l_j}{6}, \left(\tau_j - \frac{5y_j + y_{j+1}}{6}\right), \left(\frac{5y_{j+1} + y_j}{6} - \tau_j\right)\right\}\right\}$$

$$\times \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \bigvee_{x_i}^{x_{i+1}} \bigvee_{y_j}^{y_{j+1}} (f)$$

$$= \max_{i\in\{0,\ldots,n-1\}}\left\{\max\left\{\frac{h_i}{6}, \left(\xi_i - \frac{5x_i + x_{i+1}}{6}\right), \left(\frac{5x_{i+1} + x_i}{6} - \xi_i\right)\right\}\right\}$$

$$\times \max_{j\in\{j=0,\ldots,m-1\}}\left\{\max\left\{\frac{l_j}{6}, \left(\tau_j - \frac{5y_j + y_{j+1}}{6}\right), \left(\frac{5y_{j+1} + y_j}{6} - \tau_j\right)\right\}\right\}$$

$$\times \bigvee_a^b \bigvee_c^d (f)$$

which is the required result. □

**Corollary 4.2.** *Let* $I_n, J_m$ *and* $f$ *be as above. If we choose* $\xi_i = \frac{x_i + x_{i+1}}{2}$ *and* $\tau_j = \frac{y_j + y_{j+1}}{2}$ *in*

*Theorem 4.1, then we have the "Simpson's rule"*

$$\begin{aligned}
&\int_a^b \int_c^d f(t,s)dsdt \\
&= \frac{4}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)h_i l_j \\
&+\frac{1}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f\left(\frac{x_i+x_{i+1}}{2},y_j\right)+f\left(\frac{x_i+x_{i+1}}{2},y_{j+1}\right)\right]h_i l_j \\
&+\frac{1}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f\left(x_i,\frac{y_j+y_{j+1}}{2}\right)+f\left(x_{i+1},\frac{y_j+y_{j+1}}{2}\right)\right]h_i l_j \\
&+\frac{1}{36}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})\right]h_i l_j \\
&+R_S(I_n,J_m,f).
\end{aligned}$$

*The remainder $R_S(I_n,J_m,f)$ satisfies*

$$|R_S(I_n,J_m,f)| \leq \frac{1}{9}\upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f).$$

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