A subclass of Meromorphically Multivalent Functions Associated with an Operator

Ping He and Defei Zhang

Communicated by Ayman Badawi

Abstract. In this paper, we introduce a new subclass of meromorphic functions and investigate inclusion relationships and properties of a certain class of meromorphically p-valent functions, which are defined by means of the familiar operator involving the generalized hypergeometric function in the paper.

1 Introduction

Let Σ_p denote the class of meromorphically multivalent functions $f(z)$ of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p} \ \ (n \in N = 1, 2, 3, ...), \tag{1.1}$$

which are analytic in the punctured open unit disk $\mathbb{U}^* = \{z : 0 < |z| < 1\} = \mathbb{U} \backslash \{0\}$.

For functions $f \in \Sigma_p$ given by (1.1) and $g \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{n-p},$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z^{-p} + \sum_{n=1}^{\infty} a_n b_n z^{n-p}. \tag{1.2}$$

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. We say that the function $g(z)$ is subordinate to $f(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, and such that $g(z) = f(w(z))$. In such a case, we write $g(z) \prec f(z)$. If the function f is univalent in $\mathbb{U}$, then $g(z) \prec f(z)$ if and only if $g(0) = f(0)$ and $g(\mathbb{U}) \subset f(\mathbb{U})$.

For complex parameters $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ $(\beta_j \neq \mathbb{Z}_0^- := \{0, -1, -2, ...\}; j = 1, ..., s)$, we now define the generalized hypergeometric function $_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ as follows

$$_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n, ..., (\alpha_q)_n}{(\beta_1)_n, ..., (\beta_s)_n} \frac{z^n}{n!} \tag{1.3}$$

$$(q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}),$$

where $(\lambda)_n$ is the Poehhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \left\{ \begin{array}{l} 1, (n = 0; \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda + 1)...(\lambda + n - 1), (n \in \mathbb{N}; \lambda \in \mathbb{C}). \end{array} \right. \tag{1.4}$$

Corresponding to a function $h_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$, given by

$$h_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z^{-p} \cdot_q F_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z),$$

we consider a linear operator

$$H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) : \Sigma_p \mapsto \Sigma_p,$$

which is defined by the following Hadamard product (convolution):

$$H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)f(z) = h_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * f(z) \tag{1.5}$$

$$(q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}^*),$$

so that, for a function f of form (1.1), we have

$$H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)f(z) = z^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n, ..., (\alpha_q)_n}{(\beta_1)_n, ..., (\beta_s)_n} a_n \frac{z^{n-p}}{n!}. \tag{1.6}$$

In order to make the notation simple, we write

$$H_p^{q,s}(\alpha_1)f(z) := H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z),$$

then one can easily verify from definition (1. 5) that

$$z(H_p^{q,s}(\alpha_1)f(z))' = \alpha_1 H_p^{q,s}(\alpha_1 + 1)f(z) - (\alpha_1 + p)H_p^{q,s}(\alpha_1)f(z). \tag{1.7}$$

In recent years, Raina and Srivastava [1], and Aouf [2] obtained many interesting results involving the linear operator $H_p^{q,s}(\alpha_1)$. Especially $q = 2$, $s = 1$, $\alpha_1 = a$, $\beta_1 = c$, and $\alpha_2 = 1$, we obtain the following linear operator:

$$\mathscr{L}_p(a, c)f(z) = H_p(\alpha_1, 1; \beta_1)f(z),$$

which was introduced and studied by Liu and Srivastava [3], and was further studied in a subsequent investigation by Srivastava [4]. Some interesting subclasses of analytic functions associated with the generalized hypergeometric function, were considered recently by Dziok and Srivastava [5,6].

Throughout this paper, we assume that $p, k \in \mathbb{N}, q, s \in \mathbb{N}_0, \varepsilon_k = exp(\frac{2\pi i}{k})$,

$$f_{p,k}^{q,s}(\alpha_1; z) = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^{jp}(H_p^{q,s}(\alpha_1)f)(\varepsilon_k^j z) = z^{-p} + \cdots (f \in \Sigma_p). \tag{1.8}$$

Let $\mathscr{P}$ denote the class of functions of the form:

$$p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n,$$

which are analytic and convex in $\mathbb{U}$ and satisfy the following condition : $\Re(p(z)) > 0, (z \in \mathbb{U})$.

Definition 1. A function $f(z) \in \Sigma_p$ is said to be in the class $\mathscr{M}(\lambda, \alpha, \alpha_1; h)$, if it satisfies the subordination condition

$$-\lambda \frac{z[(1+\alpha)(H_p^{q,s}(\alpha_1)f)'(z) + \alpha(H_p^{q,s}(\alpha_1+1)f)'(z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z) + \alpha f_{p,k}^{q,s}(\alpha_1+1;z)]} - (1-\lambda)\frac{z(H_p^{q,s}(\alpha_1)f)'(z)}{p f_{p,k}^{q,s}(\alpha_1;z)} \prec h(z).$$

(1.9)

For simplicity, we write $\mathscr{M}(0, \alpha, \alpha_1; h) = \mathscr{M}(\alpha, \alpha_1; h)$.

2 Some Lemmas

In order to prove our main results, we need the following lemmas.

Lemma 1 (see [7]). Let $\beta, \gamma \in \mathbb{C}$. Suppose also that $\phi(z)$ is convex and univalent in $\mathbb{U}$ with

$$\phi(0) = 1, \Re(\beta\phi(z) + \gamma) > 0, (z \in \mathbb{U}).$$

If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$, then the following subordination:

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \phi(z), (z \in \mathbb{U}),$$

implies that $p(z) \prec \phi(z), (z \in \mathbb{U})$.

Lemma 2 (see [8]). Let $\beta, \gamma \in \mathbb{C}$. Suppose that $\phi(z)$ is convex and univalent in $\mathbb{U}$ with

$$\phi(0) = 1, \Re(\beta\phi(z) + \gamma) > 0; (z \in \mathbb{U}).$$

Also let

$$q(z) \prec \phi(z); (z \in \mathbb{U}).$$

If $p(z) \in \mathscr{P}$ and satisfies the following subordination:

$$p(z) + \frac{zp'(z)}{\beta q(z) + \gamma} \prec \phi(z),$$

then $p(z) \prec \phi(z)$.

Lemma 3 Let $f \in \mathscr{M}(\lambda, \alpha, \alpha_1; \phi(z))$. Then

$$-\lambda \frac{z[(1+\alpha)(f_{p,k}^{q,s}(\alpha_1)f)'(z) + \alpha(f_{p,k}^{q,s}(\alpha_1+1)f)'(z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z) + \alpha f_{p,k}^{q,s}(\alpha_1+1;z)]} - (1-\lambda)\frac{z(f_{p,k}^{q,s}(\alpha_1)f)'(z)}{p f_{p,k}^{q,s}(\alpha_1;z)} \prec \phi(z).$$

(2.1)

Furthermore, if $\phi(z) \in \mathscr{P}$ with

$$\Re(\frac{1}{\lambda}(\frac{\alpha_1}{\alpha} + 2\alpha_1 + p - p\phi(z)) > 0, (\alpha > 0, z \in \mathbb{U})),$$

then

$$-\frac{z(f_{p,k}^{q,s}(\alpha_1;z))'}{p f_{p,k}^{q,s}(\alpha_1;z)} \prec \phi(z), (z \in \mathbb{U}).$$

Proof . By (1.8), we have

$$\begin{aligned}
f_{p,k}^{q,s}(\alpha_1; \varepsilon_k^j z) &= \frac{1}{k}\sum_{n=0}^{k-1} \varepsilon_k^{np}(H_p^{q,s}(\alpha_1)f)(\varepsilon_k^{n+j}z) \\
&= \varepsilon_k^{-jp}\frac{1}{k}\sum_{n=0}^{k-1} \varepsilon_k^{(n+j)p}(H_p^{q,s}(\alpha_1)f)(\varepsilon_k^{n+j}z) \\
&= \varepsilon_k^{-jp}f_{p,k}^{q,s}(\alpha_1;z),
\end{aligned}$$

(2.2)

$$(j \in \{0, 1, \cdots, k-1\}),$$

and

$$(f_{p,k}^{q,s}(\alpha_1; z))' = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^{j(p+1)} (H_p^{q,s}(\alpha_1)f)'(\varepsilon_k^j z). \tag{2.3}$$

Replacing α_1 by $\alpha_1 + 1$ in (2.2) and (2.3), respectively, we can get

$$f_{p,k}^{q,s}(\alpha_1 + 1; \varepsilon_k^j z) = \varepsilon_k^{-jp} f_{p,k}^{q,s}(\alpha_1 + 1; z), (j \in \{0, 1, \cdots, k-1\}), \tag{2.4}$$

and

$$(f_{p,k}^{q,s}(\alpha_1 + 1; z))' = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^{j(p+1)} (H_p^{q,s}(\alpha_1 + 1)f)'(\varepsilon_k^j z). \tag{2.5}$$

From (2.2) to (2.5), we can get

$$-\lambda \frac{z[(1+\alpha)(f_{p,k}^{q,s}(\alpha_1)f)'(z) + \alpha(f_{p,k}^{q,s}(\alpha_1+1)f)'(z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1; z) + \alpha f_{p,k}^{q,s}(\alpha_1+1; z)]} - (1-\lambda)\frac{z(f_{p,k}^{q,s}(\alpha_1)f)'(z)}{p f_{p,k}^{q,s}(\alpha_1; z)}$$

$$= -\frac{1}{k} \sum_{j=0}^{k-1} \{\lambda \frac{\varepsilon_k^{j(p+1)} z[(1+\alpha)(H_p^{q,s}(\alpha_1)f)'(\varepsilon_k^j z) + \alpha(H_p^{q,s}(\alpha_1+1)f)'(\varepsilon_k^j z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1; z) + \alpha f_{p,k}^{q,s}(\alpha_1+1; z)]} + (1-\lambda)\frac{\varepsilon_k^{j(p+1)} z(H_p^{q,s}(\alpha_1)f)'(\varepsilon_k^j z)}{p f_{p,k}^{q,s}(\alpha_1; z)}\}$$

$$= -\frac{1}{k} \sum_{j=0}^{k-1} \{\lambda \frac{\varepsilon_k^j z[(1+\alpha)(H_p^{q,s}(\alpha_1)f)'(\varepsilon_k^j z) + \alpha(H_p^{q,s}(\alpha_1+1)f)'(\varepsilon_k^j z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1; \varepsilon_k^j z) + \alpha f_{p,k}^{q,s}(\alpha_1+1; \varepsilon_k^j z)]} + (1-\lambda)\frac{\varepsilon_k^j z(H_p^{q,s}(\alpha_1)f)'(\varepsilon_k^j z)}{p f_{p,k}^{q,s}(\alpha_1; \varepsilon_k^j z)}\}. \tag{2.6}$$

Moreover, since $f \in \mathscr{M}(\lambda, \alpha, \alpha_1; h)$, it follows that

$$-\lambda \frac{\varepsilon_k^j z[(1+\alpha)(H_p^{q,s}(\alpha_1)f)'(\varepsilon_k^j z) + \alpha(H_p^{q,s}(\alpha_1+1)f)'(\varepsilon_k^j z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1; \varepsilon_k^j z) + \alpha f_{p,k}^{q,s}(\alpha_1+1; \varepsilon_k^j z)]} - (1-\lambda)\frac{\varepsilon_k^j z(H_p^{q,s}(\alpha_1)f)'(\varepsilon_k^j z)}{p f_{p,k}^{q,s}(\alpha_1; \varepsilon_k^j z)} \prec \phi(z). \tag{2.7}$$

By noting that $\phi(z)$ is convex and univalent in $\mathbb{U}$, we conclude from (2.6) and (2.7) that the assertion (2.1) of Lemma 3 holds true.

Next, making use of the relationships (1.7) and (1.8), we have

$$z(f_{p,k}^{q,s}(\alpha_1; z))' + (\alpha_1 + p)f_{p,k}^{q,s}(\alpha_1; z) = \frac{\alpha_1}{k} \sum_{j=0}^{k-1} \varepsilon_k^{jp}(H_p^{q,s}(\alpha_1 + 1)f)(\varepsilon_k^j z)$$

$$= \alpha_1 f_{p,k}^{q,s}(\alpha_1 + 1; z). \tag{2.8}$$

Let $f \in \mathscr{M}(\lambda, \alpha, \alpha_1; \phi)$ and suppose that

$$\psi(z) = -\frac{z(f_{p,k}^{q,s}(\alpha_1; z))'}{p f_{p,k}^{q,s}(\alpha_1; z)}, (z \in \mathbb{U}). \tag{2.9}$$

Then $\psi(z)$ is analytic in $\mathbb{U}$ and $\psi(0) = 1$. It follows from (2.8) and (2.9) that

$$\alpha_1 + p - p\psi(z) = \alpha_1 \frac{f_{p,k}^{q,s}(\alpha_1 + 1; z)}{f_{p,k}^{q,s}(\alpha_1; z)}. \tag{2.10}$$

By (2.9) and (19), we get

$$z(f_{p,k}^{q,s}(\alpha_1 + 1; z))' = -\frac{p}{\alpha_1}(z\psi'(z) + [\alpha_1 + p - p\psi(z)]\psi(z))f_{p,k}^{q,s}(\alpha_1; z), (z \in \mathbb{U}^*). \qquad (2.11)$$

It now follows from (2.1) and (2.9)-(2.11) that

$$-\lambda\frac{z[(1+\alpha)(f_{p,k}^{q,s}(\alpha_1)f)'(z) + \alpha(f_{p,k}^{q,s}(\alpha_1 + 1)f)'(z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1; z) + \alpha f_{p,k}^{q,s}(\alpha_1 + 1; z)]} - (1-\lambda)\frac{z(f_{p,k}^{q,s}(\alpha_1)f)'(z)}{pf_{p,k}^{q,s}(\alpha_1; z)}$$

$$= \lambda\frac{p(1+\alpha)\psi(z)f_{p,k}^{q,s}(\alpha_1; z) + \frac{\alpha}{\alpha_1}p(z\psi'(z) + [\alpha_1 + p - p\psi(z)]\psi(z))f_{p,k}^{q,s}(\alpha_1; z)}{p(1+\alpha)f_{p,k}^{q,s}(\alpha_1; z) + \frac{\alpha}{\alpha_1}p[\alpha_1 + p - p\psi(z)]f_{p,k}^{q,s}(\alpha_1; z)} + (1-\lambda)\psi(z)$$

$$= \lambda\frac{(1+\alpha)\psi(z) + \frac{\alpha}{\alpha_1}(z\psi'(z) + [\alpha_1 + p - p\psi(z)]\psi(z))}{(1+\alpha) + \frac{\alpha}{\alpha_1}[\alpha_1 + p - p\psi(z)]} + (1-\lambda)\psi(z)$$

$$= \psi(z) + \frac{z\psi'(z)}{\frac{1}{\lambda}(\frac{\alpha_1}{\alpha} + 2\alpha_1 + p - p\psi(z))} \prec \phi(z), (z \in \mathbb{U}). \qquad (2.12)$$

Since

$$\Re(\frac{1}{\lambda}(\frac{\alpha_1}{\alpha} + 2\alpha_1 + p - p\phi(z)) > 0, (\alpha > 0, z \in \mathbb{U})),$$

by means of (2.12) and Lemma 1, we find that

$$\psi(z) = -\frac{z(f_{p,k}^{q,s}(\alpha_1; z))'}{pf_{p,k}^{q,s}(\alpha_1; z)} \prec \phi(z), (z \in \mathbb{U}).$$

This completes the proof of Lemma 3 .

3 Main results

Theorem 1. Let $\phi(z) \in \mathscr{P}$ with

$$\Re(\frac{1}{\lambda}(\frac{\alpha_1}{\alpha} + 2\alpha_1 + p - p\phi(z))) > 0; (\alpha > 0; z \in \mathbb{U}).$$

Then $\mathscr{M}(\lambda, \alpha, \alpha_1; \phi(z)) \subset \mathscr{M}(\alpha, \alpha_1; \phi(z))$.

Proof . Let $f \in \mathscr{M}(\lambda, \alpha, \alpha_1; \phi(z))$ and suppose that

$$q(z) = -\frac{z(H_p^{q,s}(\alpha_1)f)'(z)}{pf_{p,k}^{q,s}(\alpha_1; z)}; (z \in \mathbb{U}). \qquad (3.1)$$

Then $q(z)$ is analytic in $\mathbb{U}$ and $q(0) = 1$, It follows from (1.7) and (3.1) that

$$q(z)f_{p,k}^{q,s}(\alpha_1; z) = -\frac{\alpha_1}{p}H_p^{q,s}(\alpha_1 + 1)f(z) + \frac{\alpha_1 + p}{p}H_p^{q,s}(\alpha_1)f(z). \qquad (3.2)$$

Differentiating both sides of (3.2) with respect to z and using (3.1), we have

$$zq'(z) + (\alpha_1 + p + \frac{z(f_{p,k}^{q,s}(\alpha_1; z))'}{f_{p,k}^{q,s}(\alpha_1; z)})q(z) = -\frac{\alpha_1}{p}\frac{z(H_p^{q,s}(\alpha_1 + 1)f)'(z)}{f_{p,k}^{q,s}(\alpha_1; z)}. \qquad (3.3)$$

It now follows from (1.9),(2.9),(2.10),(3.1)and (3.3) that

$$-\lambda\frac{z[(1+\alpha)(H_p^{q,s}(\alpha_1)f)'(z) + \alpha(H_p^{q,s}(\alpha_1 + 1)f)'(z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1; z) + \alpha f_{p,k}^{q,s}(\alpha_1 + 1; z)]} - (1-\lambda)\frac{z(H_p^{q,s}(\alpha_1)f)'(z)}{pf_{p,k}^{q,s}(\alpha_1; z)}$$

$$= \quad \lambda \frac{p(1+\alpha)q(z)f_{p,k}^{q,s}(\alpha_1;z) + \frac{\alpha}{\alpha_1}p[zq'(z) + [\alpha_1 + p - p\psi(z)]q(z)]f_{p,k}^{q,s}(\alpha_1;z)}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z)] + \frac{\alpha}{\alpha_1}[\alpha_1 + p - p\psi(z)]f_{p,k}^{q,s}(\alpha_1;z)} + (1-\lambda)q(z)$$

$$= \quad \lambda \frac{(1+\alpha)q(z) + \frac{\alpha}{\alpha_1}[zq'(z) + [\alpha_1 + p - p\psi(z)]q(z)]}{(1+\alpha) + \frac{\alpha}{\alpha_1}[\alpha_1 + p - p\psi(z)]} + (1-\lambda)q(z)$$

$$= \quad q(z) + \frac{zq'(z)}{\frac{1}{\lambda}(\frac{\alpha_1}{\alpha} + 2\alpha_1 + p - p\psi(z))} \prec \phi(z). \tag{3.4}$$

Moreover, since

$$\Re(\frac{1}{\lambda}(\frac{\alpha_1}{\alpha} + 2\alpha_1 + p - p\phi(z))) > 0; (\alpha > 0; z \in \mathbb{U}).$$

by Lemma 3, we have

$$\psi(z) = -\frac{z(f_{p,k}^{q,s}(\alpha_1;z))'}{pf_{p,k}^{q,s}(\alpha_1;z)} \prec \phi(z), (z \in \mathbb{U}).$$

Thus, by (3.4) and Lemma 2, we find that

$$q(z) \prec \phi(z), (z \in \mathbb{U}),$$

that is, that $f \in \mathscr{M}(\alpha, \alpha_1; \phi(z))$. This implies that

$$\mathscr{M}(\lambda, \alpha, \alpha_1; \phi(z)) \subset \mathscr{M}(\alpha, \alpha_1; \phi(z)).$$

The proof of Theorem 1 is completed.

Theorem 2. Let $h \in \mathscr{P}$, and $0 \le \lambda_1 < \lambda_2$,

$$\Re(\frac{1}{\lambda_2}(\frac{\alpha_1}{\alpha} + 2\alpha_1 + p - ph(z)) > 0, (\alpha > 0, z \in \mathbb{U})),$$

then

$$\mathscr{M}(\lambda_2, \alpha, \alpha_1; h) \subset \mathscr{M}(\lambda_1, \alpha, \alpha_1; h).$$

Proof . For $f \in \mathscr{M}(\lambda_2, \alpha, \alpha_1; h)$, we have

$$-\lambda_2 \frac{z[(1+\alpha)(H_p^{q,s}(\alpha_1)f)'(z) + \alpha(H_p^{q,s}(\alpha_1 + 1)f)'(z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z) + \alpha f_{p,k}^{q,s}(\alpha_1 + 1;z)]} - (1-\lambda_2)\frac{z(H_p^{q,s}(\alpha_1)f)'(z)}{pf_{p,k}^{q,s}(\alpha_1;z)} \prec h(z).$$
$$\tag{3.5}$$

Put

$$q(z) = -\frac{z(H_p^{q,s}(\alpha_1)f)'(z)}{pf_{p,k}^{q,s}(\alpha_1;z)}, (z \in \mathbb{U}).$$

By Theorem 1, we get

$$\mathscr{M}(\lambda_2, \alpha, \alpha_1; h(z)) \subset \mathscr{M}(\alpha, \alpha_1; h(z)).$$

Hence,

$$q(z) \prec h(z), (z \in \mathbb{U}). \tag{3.6}$$

Since $0 \le \frac{\lambda_1}{\lambda_2} < 1$, and since $h(z)$ is convex univalent in $\mathbb{U}$, we deduce from (3.5) and (3.6) that

$$-\lambda_1 \frac{z[(1+\alpha)(H_p^{q,s}(\alpha_1)f)'(z) + \alpha(H_p^{q,s}(\alpha_1 + 1)f)'(z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z) + \alpha f_{p,k}^{q,s}(\alpha_1 + 1;z)]} - (1-\lambda_1)\frac{z(H_p^{q,s}(\alpha_1)f)'(z)}{pf_{p,k}^{q,s}(\alpha_1;z)}$$

$$= \frac{\lambda_1}{\lambda_2}(-\lambda_2 \frac{z[(1+\alpha)(H_p^{q,s}(\alpha_1)f)'(z) + \alpha(H_p^{q,s}(\alpha_1+1)f)'(z)]}{p[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z) + \alpha f_{p,k}^{q,s}(\alpha_1+1;z)]} - (1-\lambda_2)\frac{z(H_p^{q,s}(\alpha_1)f)'(z)}{pf_{p,k}^{q,s}(\alpha_1;z)})$$

$$+ \quad (1-\frac{\lambda_1}{\lambda_2})q(z) \prec h(z), (z \in \mathbb{U}). \tag{3.7}$$

Thus $f \in \mathscr{M}(\lambda_1, \alpha, \alpha_1; h)$ and the proof of Theorem 2 is completed.

References

[1] R.K. Raina, H.M. Srivastava, A new class of meromorphically multivalent functions with applications to generalized hypergeometric functions, Math. Comput. Modelling. 43 (2006), 350-356.

[2] M.K. Aouf, Certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function, Comput. Math. Appl. 55 (2008), 494-509.

[3] J.-L. Liu, H.M. Srivastava, A linear operator and associated families of meromorphically multivalent functions, J.Math. Anal. Appl. 259 (2001), 566-581.

[4] H.M. Srivastava, D.-G. Yang, N.E. Xu, Some subclasses of meromorphically multivalent functions associatedwith a linear operator, Appl. Math. Comput. 195 (2008), 11-23.

[5] J. Dziok, H.M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput. 103 (1999), 1-13.

[6] J. Dziok, H.M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, Integral Transforms Spec. Funct. 14(2003), 7-18.

[7] S.S. Miller, P.T. Mocanu, On some classes of first order differential subordinations, Michigan Math. J. 32 (1985),185-195.

[8] K.S. Padmanabhan, R. Parvatham, Some applications of differential subordination, Bull. Austral. Math. Soc. 32 (1985), 321-330.

Author information

Ping He and Defei Zhang, Department of Mathematics, Honghe University, Mengzi, 661100, Yunnan, China.
E-mail: hepingky@163.com   hdefei@163.com