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MODULES THAT HAVE A WEAK SUPPLEMENT IN EVERY COFINITE EXTENSION

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Abstract We say that over an arbitrary ring a module M has *the property* (CWE) (respectively, $(CWEE)$) if M has a weak supplement (respectively, ample weak supplements) in every cofinite extension. We show that if every submodule of a module M has the property (CWE) then M has the property $(CWEE)$. A ring R is semilocal iff every left R-module has the property (CWE). We also prove that over a commutative Von-Neumann regular ring R, an R-module M has the property (CWE) iff M is cofinitely injective.

1 Introduction

Throughout this paper, R is an associative ring with identity and all modules are unital left R-modules, unless otherwise stated. Let M be an R-module. The notation $(U << M)$ $U \leq M$ means that U is a *(small)* submodule of M. A submodule U of M is called small in M, if $M \neq U + L$ for every proper submodule L of M. By $Rad(M)$ we denote the intersection of all maximal submodules of M, equivalently the sum of all small submodules of M (see [10]). A module M is called *radical* if M has no maximal submodules, that is, $M = Rad(M)$. A submodule U of M is called *cofinite* (in M) if the factor module $\frac{M}{U}$ is finitely generated. Let U and K be submodules of M. K is called *supplement* of U in M if it is minimal with respect to $M = U + K$. A module M is called *supplemented* if every submodule of M has a supplement in M. K is a supplement of U in M if and only if $M = U + K$ and $U \cap K \ll K$ [10]. If $M = U + K$ and $U \cap K \ll M$, then K is called a *weak supplement* of U in M, and clearly in this case, U is a weak supplement of K in M, too. A submodule U of M has *ample (weak) supplements* in M if, whenever $M = U + L$, L contains a (weak) supplement of U in M. M is a *weakly supplemented* module if every submodule of M has a weak supplement in M (see [6], [10], [11]). M is called a *cofinitely supplemented* module if every cofinite submodule of M has a supplement in M(see [1]). In [2], Alizade and Büyükaşık introduced *cofinitely weak supplemented* module (or briefly a *cws-module*), every cofinite submodule of which has a weak supplement in M.

Let R be a ring and let M and N be R-modules. N is called *(cofinite) extension* of M in case $M \subseteq N$ (and $\frac{N}{M}$ is finitely generated). A module M is said to be *cofinitely injective* if it is a direct summand in its every cofinite extension (see [4]).

In [12], Zöschinger initiated the study of the modules with the following properties defined as:

(E) A module M has a supplement in every extension.

(EE) A module M has ample supplements in every extension.

The author showed in the same paper some properties of these modules. Modifying his concepts, Çalışıcı and Türkmen say that a module M has *the property* (CE)(respectively, (CEE)) if M has a supplement (respectively, ample supplements) in every cofinite extension. They give a characterization of semiperfect rings via these modules in [4].

In this paper, we study the modules with the property (CWE) (respectively, $(CWEE)$) as a generalization of the modules with the property (CE) (respectively, (CEE)). We show that if a module M has the property (CWE) then every direct summand of M has the property (CWE). We give a characterization of semilocal rings via the modules with the property (CWE). Moreover, we give that for a ring R, every R-module is cofinitely weak supplemented if and only if every R-module has the property (CWE). Finally, we show that there exists a module which has the property (CWE) but not (CE).

2 Modules with the Properties (CWE) and $(CWEE)$

In this section, we define the notion of modules with the properties (CWE) and $(CWEE)$. We obtain various properties of these modules. We prove that every R-module has the property (CWE) if and only if the ring R is semilocal.

Definition 2.1. Let M be an R-module. We say that:
M has *the property* (CWE) if M has a weak supplement in every cofinite extension, and
M has *the property* $(CWEE)$ if M has ample weak supplements in every cofinite extension.

It is shown in [12, Lemma 1.3.(a)] that the property (E) is preserved by direct summands. Now we give an analogue of this fact for the modules with the property (CWE).

Proposition 2.2. *Let M be a module. If M has the property (CWE), then every direct summand of M has the property (CWE).*

Proof. Let M_1 be a direct summand of M. Then there exists a submodule M_2 of M such that $M = M_1 \oplus M_2$. Let N be any cofinite extension of M_1, K be the external direct sum $N \oplus M_2$ and $\varphi : M \to K$ be the canonical embedding. Then $M \cong \varphi(M)$ has the property (CWE). We have that

$$\frac{N}{M_1} \cong \frac{N\oplus M_2}{\varphi(M)} = \frac{K}{\varphi(M)}$$

is finitely generated. Since $\varphi(M)$ has the property (CWE), there exists a submodule U of K such that $K = \varphi(M) + U$ and $\varphi(M) \cap U \ll K$. Consider the projection $\pi : K \longrightarrow N$. We have $N = M_1 + \pi(U)$. Also since $Ker(\pi) \subseteq \varphi(M)$, $\pi(\varphi(M) \cap U) = \pi(\varphi(M)) \cap \pi(U) = M_1 \cap \pi(U) \ll N$. Hence $\pi(U)$ is a weak supplement of M_1 in N. □

Now we give the relation between the modules with the property (CWE) and $(CWEE)$.

Proposition 2.3. *If every submodule of a module M has the property (CWE), then M has the property $(CWEE)$.*

Proof. Let N be any cofinite extension of M and $N = M + V$ for a submodule V of N. Note that

$$\frac{N}{M} \cong \frac{V}{M\cap V}$$

is finitely generated. By the hyphotesis, $M\cap V$ has a weak supplement K in V that is $(M\cap V)+K = V$, $(M\cap V)\cap K = M\cap K \ll V$. Also we have that $N = M+V = M+(M\cap V)+K = M + K$. Consequently, K is an ample weak supplement of M in N. □

Now we show that every simple module has the property (CWE). Firstly, we give the following well known fact for completeness.

Lemma 2.4. *Every simple submodule S of a module M is either a direct summand of M or small in M.*

Proof. Suppose that S is not small in M, then there exists a proper submodule K of M such that $S + K = M$. Since S is simple and $K \neq M$, $S \cap K = 0$. Thus $M = S \oplus K$. □

Proposition 2.5. *Every simple module has the property (CWE).*

Proof. Let S be a simple module and N be any cofinite extension of S. Then by Lemma 2.4, $S \ll N$ or $S \oplus K = N$ for a submodule $K \leq N$. In the first case N is a weak supplement of S in N. In the second case K is a weak supplement of S in N. Hence, S has the property (CWE). □

Let M be a module and U be a submodule of M. If the factor module $\frac{M}{U}$ has the property (CWE), M does not need to have the property (CWE). For example, for the ring $R = \mathbb{Z}$, the R-module $M = \frac{3\mathbb{Z}}{9\mathbb{Z}}$ has a weak supplement in every cofinite extension since it is simple. But $3\mathbb{Z}$ does not have a weak supplement in its cofinite extension $\mathbb{Z}$.

Now we obtain that the statement mentioned at the beginning of the last paragraph is true under a special condition.

Proposition 2.6. *Let M be a module and U be a submodule of M. If $U \ll M$ and the factor module $\frac{M}{U}$ has the property (CWE), then M also has the property (CWE).*

Proof. Let N be any cofinite extension of M. Then we obtain that

$$\frac{N}{M} \cong \frac{\frac{N}{U}}{\frac{M}{U}}$$

is finitely generated. Since $\frac{M}{U}$ has the property (CWE), there exists a submodule $\frac{V}{U}$ of $\frac{N}{U}$ such that $\frac{M}{U} + \frac{V}{U} = \frac{N}{U}$ and $\frac{M}{U} \cap \frac{V}{U} = \frac{M \cap V}{U} \ll \frac{N}{U}$. Note that $M + V = N$. Suppose that $(M \cap V) + S = N$ for a submodule S of N. Then we obtain $\frac{M\cap V}{U} + \frac{S+U}{U} = \frac{N}{U}$. Since $\frac{M\cap V}{U} \ll \frac{N}{U}$, we have that $\frac{S+U}{U} = \frac{N}{U}$. By hypothesis, it follows that $N = S + U = S$. Hence $M \cap V \ll N$. □

The following corollary is immediate consequence of the last proposition.

Corollary 2.7. *Let M be a finitely generated module. If $\frac{M}{Rad(M)}$ has the property (CWE), then M has the property (CWE).*

Proposition 2.8. *Let M be an R-module and N be a cofinite extension of M. If M has a weak supplement V in N, then M has a finitely generated weak supplement W in N such that $W \leq V$.*

Proof. Let V be a weak supplement of M in N. Then we have that $N = M+V$ and $M\cap V \ll N$. Since N is a cofinite extension of M,

$$\frac{N}{M} = \frac{M+V}{M} \cong \frac{V}{M\cap V}$$

is finitely generated. Suppose that for some positive integer n and elements $x_i \in V(1 \leq i \leq n)$, elements $x_1 + M \cap V$, $x_2 + M \cap V$, ... , $x_n + M \cap V$ generate the factor module $\frac{V}{M\cap V}$. Then for a finitely generated submodule $W = Rx_1 + Rx_2 + ... + Rx_n$ of V, we have that $W + M = W + (V \cap M) + M = V + M = N$. Consequently, W is a finitely generated weak supplement of M in N. □

Recall from [2] the ring R is semilocal if and only if every R-module is cofinitely weak supplemented. Now we give a characterization for semilocal rings via the modules with the property (CWE).

Theorem 2.9. *Let R be a ring. R is semilocal if and only if every R-module has the property (CWE).*

Proof. Let R be a semilocal ring, M be an R-module and N be a cofinite extension of M. Since R is semilocal, N is cofinitely weak supplemented module from [2, Corollary 2.22]. Therefore, M has a weak supplement in N.

Conversely, let M be an R-module and U be any cofinite submodule of M. By hyphothesis, U has the property (CWE). Then U has a weak supplement in M, so that M is cofinitely weak supplemented. Hence R is semilocal by [2, Corollary 2.22]. □

The following corollary is immediate result of [2, Corollary 2.22] and Theorem 2.9.

Corollary 2.10. *Let R be a ring. Every R-module is cofinitely weak supplemented if and only if every R-module has the property (CWE).*

In [10] a ring R is said to be *Von-Neumann regular* if every element a of R is regular, i.e., there exists an element $b \in R$ such that $aba = a$. In [4], a module M is called *cofinitely injective* if it is a direct summand in every cofinite extension.

Theorem 2.11. *Let R be a commutative Von-Neumann regular ring. An R-module M has the property (CWE) if and only if M is cofinitely injective.*

Proof. Let M be an R-module with the property (CWE) and N be any cofinite extension of M. Then there is a submodule K of N such that $N = M + K$ and $M \cap K \ll N$. From [10] we have $Rad(N) = 0$. Hence M is a direct summand of N.

Conversely, let N be any cofinite extension of M. By hyphothesis, there is a submodule K of N such that $N = M \oplus K$. Hence K is a weak supplement of M in N. □

It is clear that every module with the property (CE) has the property (CWE), but the converse can not be true in general. The following example shows that there exists a module that has the property (CWE) but not (CE).

Example 2.12. (see [6, Remark 3.3]) For primes p and q, consider the following ring:

$$R := \mathbb{Z}_{p,q} := \{\frac{a}{b} | a, b \in \mathbb{Z}, b \neq 0, (p, b) = (q, b) = 1\}.$$

R is a semilocal ring that is not semiperfect. Then there exists an R-module M that has not the property (CE) from [4, Theorem 2.12]. Since R is semilocal ring, by Theorem 2.9 M has the property (CWE).

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