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RELATIVE ORDER CONCERNING ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Abstract. In this paper we discuss some growth rates of entire functions of several complex variables on the basis of the definition of relative order and relative lower order of entire functions of several complex variables as introduced by Dutta in [4].

1 Introduction, Definitions and Notations

Let f be an entire function of two complex variables holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : \ |z_i| \leq r_i, \ i = 1, 2 \text{ for all } \ r_1 \geq 0, r_2 \geq 0\}$$

and $M_f(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_i| \leq r_i, \ i = 1, 2\}$. Then in view of maximum principal and Hartogs theorem {[7], p. 2, p. 51}, $M_f(r_1, r_2)$ is an increasing functions of r_1, r_2.
The following definition is well known:

Definition 1.1. {[7], p. 339, (see also [1])} The order ${}_{v_2}\rho_f$ and the lower order ${}_{v_2}\lambda_f$ of an entire function f of two complex variables are defined as

$${}_{v_2}\rho_f = \limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_f(r_1, r_2)}{\log(r_1 r_2)} \text{ and } {}_{v_2}\lambda_f = \liminf_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_f(r_1, r_2)}{\log(r_1 r_2)},$$

where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$, $k = 1, 2, 3, \ldots$ *and* $\log^{[0]} x = x$.

If we consider the above definition for single variable, then the definition coincides with the classical definition of order (see [13]) which is as follows:

Definition 1.2. [13] The order ρ_f and the lower order λ_f of an entire function f are defined in the following way:

$$\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r},$$

where $M_f(r) = \max\{|f(z)| : |z| = r\}$.

If f is non-constant then $M_f(r)$ is strictly increasing and continuous, and its inverse $M_f{}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s \to \infty} M_f{}^{-1}(s) = \infty$. Bernal {[2], [3]} introduced the definition of relative order of g with respect to f, denoted by $\rho_f(g)$ as follows :

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r \to \infty} \frac{\log M_g^{-1} M_f(r)}{\log r} . \end{aligned}$$

The definition coincides with the classical one [13] if $g(z) = \exp z$.
During the past decades, several authors (see [5],[8],[9],[10],[11],[12]) made close investigations on the properties of relative order of entire functions of single variable. In the case of relative order, it was then natural for Banerjee and Dutta [4] to define the relative order of entire functions of two complex variables as follows:

Definition 1.3. [4] The relative order between two entire functions of two complex variables denoted by ${}_{v_2}\rho_g(f)$ is defined as:

$$
\begin{aligned}
{}_{v_2}\rho_g(f) &= \inf\left\{\mu > 0 : M_f(r_1, r_2) < M_g\left(r_1^{\mu}, r_2^{\mu}\right); r_1 \geq R(\mu), r_2 \geq R(\mu)\right\} \\
&= \limsup_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)}
\end{aligned}
$$

where f and g are entire functions holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : \ |z_i| \leq r_i, \ i = 1, 2 \text{ for all } \ r_1 \geq 0, r_2 \geq 0\}$$

and the definition coincides with Definition 1.1 {see [4]} if $g(z) = \exp(z_1 z_2)$.

Extending this notion, Dutta [6] introduced the idea of relative order of entire functions of several complex variables in the following way:

Definition 1.4. [6] Let $f(z_1, z_2, ..., z_n)$ and $g(z_1, z_2, ..., z_n)$ be any two entire functions of n complex variables $z_1, z_2, ..., z_n$ with maximum modulus functions $M_f(r_1, r_2, ..., r_n)$ and $M_g(r_1, r_2, ..., r_n)$ respectively then the relative order of f with respect to g, denoted by ${}_{v_n}\rho_g(f)$ is defined by

$$
\begin{aligned}
{}_{v_n}\rho_g(f) = \inf\{\mu > 0 : M_f(r_1, r_2, ..., r_n) < M_g\left(r_1^{\mu}, r_2^{\mu}, ..., r_n^{\mu}\right); \\
\text{for } r_i \geq R(\mu), i = 1, 2, ..., n\} \ .
\end{aligned}
$$

The above definition can equivalently be written as

$$
{}_{v_n}\rho_g(f) = \limsup_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} \ .
$$

Similarly, one can define the relative lower order of f with respect to g denoted by ${}_{v_n}\lambda_g(f)$ as follows :

$$
{}_{v_n}\lambda_g(f) = \liminf_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} \ .
$$

If we consider $g(z_1, z_2, ..., z_n) = \exp(z_1 z_2 ... z_n)$, then Definition 1.4 reduces to the following classical definition of order and lower order in connection with several complex variables:

Definition 1.5. The order ${}_{v_n}\rho_f$ and the lower order ${}_{v_n}\lambda_f$ of an entire function f of several complex variables are defined as

$$
\begin{aligned}
{}_{v_n}\rho_f &= \limsup_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[2]} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} \text{ and} \\
{}_{v_n}\lambda_f &= \liminf_{r_1, r_2, \ldots, r_n \to \infty} \frac{\log^{[2]} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} \ .
\end{aligned}
$$

Also an entire function of several complex variables for which order and lower order are the same is said to be of regular growth. The function $\exp(z_1 z_2 ... z_n)$ is an example of regular growth of entire function of several complex variables. Further the functions which are not of regular growth are said to be of irregular growth.

In this paper we wish to prove some results related to the growth properties of composite entire functions of several complex variables on the basis of relative order and relative lower order of entire functions of several complex variables. We do not explain the standard definitions and notations in the theory of entire function of two complex variables as those are available in [7].

2 Main Results

In this section we present the main results of the paper.

Theorem 2.1. *Let f and g be any two entire functions of several complex variables. Then*

$$
\frac{{}_{v_n}\lambda_f}{{}_{v_n}\rho_g} \leq {}_{v_n}\lambda_g(f) \leq \min\left\{\frac{{}_{v_n}\lambda_f}{{}_{v_n}\lambda_g}, \frac{{}_{v_n}\rho_f}{{}_{v_n}\rho_g}\right\} \leq \max\left\{\frac{{}_{v_n}\lambda_f}{{}_{v_n}\lambda_g}, \frac{{}_{v_n}\rho_f}{{}_{v_n}\rho_g}\right\} \leq {}_{v_n}\rho_g(f) \leq \frac{{}_{v_n}\rho_f}{{}_{v_n}\lambda_g} \ .
$$

Proof. From the definitions of ${}_{v_n}\rho_f$ and ${}_{v_n}\lambda_f$ we have for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_f(r_1, r_2, ..., r_n) \leq \exp^{[2]}\{({}_{v_n}\rho_f + \varepsilon)\log(r_1 r_2 ... r_n)\}, \tag{2.1}$$

$$M_f(r_1, r_2, ..., r_n) \geq \exp^{[2]}\{({}_{v_n}\lambda_f - \varepsilon)\log(r_1 r_2 ... r_n)\} \tag{2.2}$$

and also for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity, we get that

$$M_f(r_1, r_2, ..., r_n) \geq \exp^{[2]}\{({}_{v_n}\rho_f - \varepsilon)\log(r_1 r_2 ... r_n)\}, \tag{2.3}$$

$$M_f(r_1, r_2, ..., r_n) \leq \exp^{[2]}\{({}_{v_n}\lambda_f + \varepsilon)\log(r_1 r_2 ... r_n)\}. \tag{2.4}$$

Similarly from the definitions of ${}_{v_n}\rho_g$ and ${}_{v_n}\lambda_g$, it follows for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_g(r_1, r_2, ..., r_n) \leq \exp^{[2]}\{({}_{v_n}\rho_g + \varepsilon)\log(r_1 r_2 ... r_n)\}$$

$$i.e.,\ (r_1 r_2 ... r_n) \leq M_g^{-1}\left[\exp^{[2]}\{({}_{v_n}\rho_g + \varepsilon)\log(r_1 r_2 ... r_n)\}\right]$$

$$i.e.,\ M_g^{-1}(r_1, r_2, ..., r_n) \geq \exp\left[\frac{\log^{[2]}(r_1 r_2 ... r_n)}{({}_{v_n}\rho_g + \varepsilon)}\right], \tag{2.5}$$

$$M_g(r_1, r_2, ..., r_n) \geq \exp^{[2]}\{({}_{v_n}\lambda_g - \varepsilon)\log(r_1 r_2 ... r_n)\}$$

$$i.e.,\ (r_1 r_2 ... r_n) \geq M_g^{-1}\left[\exp^{[2]}\{({}_{v_n}\lambda_g - \varepsilon)\log(r_1 r_2 ... r_n)\}\right]$$

$$i.e.,\ M_g^{-1}(r_1, r_2, ..., r_n) \leq \exp\left[\frac{\log^{[2]}(r_1 r_2 ... r_n)}{({}_{v_n}\lambda_g - \varepsilon)}\right] \tag{2.6}$$

and for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity we obtain that

$$M_g(r_1, r_2, ..., r_n) \geq \exp^{[2]}\{({}_{v_n}\rho_g - \varepsilon)\log(r_1 r_2 ... r_n)\}$$

$$i.e.,\ (r_1 r_2 ... r_n) \geq M_g^{-1}\left[\exp^{[2]}\{({}_{v_n}\rho_g - \varepsilon)\log(r_1 r_2 ... r_n)\}\right]$$

$$i.e.,\ M_g^{-1}(r_1, r_2, ..., r_n) \leq \exp\left[\frac{\log^{[2]}(r_1 r_2 ... r_n)}{({}_{v_n}\rho_g - \varepsilon)}\right], \tag{2.7}$$

$$M_g(r_1, r_2, ..., r_n) \leq \exp^{[2]}\{({}_{v_n}\lambda_g + \varepsilon)\log(r_1 r_2 ... r_n)\}$$

$$i.e.,\ (r_1 r_2 ... r_n) \leq M_g^{-1}\left[\exp^{[2]}\{({}_{v_n}\lambda_g + \varepsilon)\log(r_1 r_2 ... r_n)\}\right]$$

$$i.e.,\ M_g^{-1}(r_1, r_2, ..., r_n) \geq \exp\left[\frac{\log^{[2]}(r_1 r_2 ... r_n)}{({}_{v_n}\lambda_g + \varepsilon)}\right]. \tag{2.8}$$

Now from (2.3) and in view of (2.5), we get for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$\log M_g^{-1} M_f(r_1, r_2, ..., r_n) \geq \log M_g^{-1}\left[\exp^{[2]}\{({}_{v_n}\rho_f - \varepsilon)\log(r_1 r_2 ... r_n)\}\right]$$

$$i.e.,\ \log M_g^{-1} M_f(r_1, r_2, ..., r_n) \geq \log\exp\left[\frac{\log^{[2]}\exp^{[2]}\{({}_{v_n}\rho_f - \varepsilon)\log(r_1 r_2 ... r_n)\}}{({}_{v_n}\rho_g + \varepsilon)}\right]$$

$$i.e.,\ \log M_g^{-1} M_f(r_1, r_2, ..., r_n) \geq \frac{({}_{v_n}\rho_f - \varepsilon)}{({}_{v_n}\rho_g + \varepsilon)}\log(r_1 r_2 ... r_n)$$

$$i.e.,\ \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} \geq \frac{({}_{v_n}\rho_f - \varepsilon)}{({}_{v_n}\rho_g + \varepsilon)}.$$

As $\varepsilon\,(> 0)$ is arbitrary, it follows that

$$\limsup_{r_1, r_2, ..., r_n \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} \geq \frac{{}_{v_n}\rho_f}{{}_{v_n}\rho_g}$$

$$i.e.,\ {}_{v_n}\rho_g(f) \geq \frac{{}_{v_n}\rho_f}{{}_{v_n}\rho_g}. \tag{2.9}$$

Analogously from (2.2) and in view of (2.8) , it follows for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$\log M_g^{-1} M_f(r_1, r_2, ..., r_n) \geq \log M_g^{-1}\left[\exp^{[2]}\left\{({}_{v_n}\lambda_f - \varepsilon)\log(r_1 r_2 ... r_n)\right\}\right]$$

$$i.e.,\ \log M_g^{-1} M_f(r_1, r_2, ..., r_n) \geq \log\exp\left[\frac{\log^{[2]}\exp^{[2]}\left\{({}_{v_n}\lambda_f - \varepsilon)\log(r_1 r_2 ... r_n)\right\}}{({}_{v_n}\lambda_g + \varepsilon)}\right]$$

$$\begin{aligned} i.e.,\ \log M_g^{-1} M_f(r_1, r_2, ..., r_n) &\geq \frac{({}_{v_n}\lambda_f - \varepsilon)}{({}_{v_n}\lambda_g + \varepsilon)}\log(r_1 r_2 ... r_n) \\ i.e.,\ \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} &\geq \frac{({}_{v_n}\lambda_f - \varepsilon)}{({}_{v_n}\lambda_g + \varepsilon)}. \end{aligned}$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\begin{aligned} \limsup_{r_1, r_2, ..., r_n \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} &\geq \frac{{}_{v_n}\lambda_f}{{}_{v_n}\lambda_g} \\ i.e.,\ {}_{v_n}\rho_g(f) &\geq \frac{{}_{v_n}\lambda_f}{{}_{v_n}\lambda_g}. \end{aligned} \tag{2.10}$$

Again in view of (2.6) , we have from (2.1) for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$\log M_g^{-1} M_f(r_1, r_2, ..., r_n) \leq \log M_g^{-1}\left[\exp^{[2]}\left\{({}_{v_n}\rho_f + \varepsilon)\log(r_1 r_2 ... r_n)\right\}\right]$$

$$i.e.,\ \log M_g^{-1} M_f(r_1, r_2, ..., r_n) \leq \log\exp\left[\frac{\log^{[2]}\exp^{[2]}\left\{({}_{v_n}\rho_f + \varepsilon)\log(r_1 r_2 ... r_n)\right\}}{({}_{v_n}\lambda_g - \varepsilon)}\right]$$

$$\begin{aligned} i.e.,\ \log M_g^{-1} M_f(r_1, r_2, ..., r_n) &\leq \frac{({}_{v_n}\rho_f + \varepsilon)}{({}_{v_n}\lambda_g - \varepsilon)}\log(r_1 r_2 ... r_n) \\ i.e.,\ \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} &\leq \frac{({}_{v_n}\rho_f + \varepsilon)}{({}_{v_n}\lambda_g - \varepsilon)}. \end{aligned}$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\begin{aligned} \limsup_{r_1, r_2, ..., r_n \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} &\leq \frac{{}_{v_n}\rho_f}{{}_{v_n}\lambda_g} \\ i.e.,\ \rho_g(f) &\leq \frac{{}_{v_n}\rho_f}{{}_{v_n}\lambda_g}. \end{aligned} \tag{2.11}$$

Again from (2.2) and in view of (2.5) , we get for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$\log M_g^{-1} M_f(r_1, r_2, ..., r_n) \geq \log M_g^{-1}\left[\exp^{[2]}\left\{({}_{v_n}\lambda_f - \varepsilon)\log(r_1 r_2 ... r_n)\right\}\right]$$

$$i.e.,\ \log M_g^{-1} M_f(r_1, r_2, ..., r_n) \geq \log\exp\left[\frac{\log^{[2]}\exp^{[2]}\left\{({}_{v_n}\lambda_f - \varepsilon)\log(r_1 r_2 ... r_n)\right\}}{({}_{v_n}\rho_g + \varepsilon)}\right]$$

$$\begin{aligned} i.e.,\ \log M_g^{-1} M_f(r_1, r_2, ..., r_n) &\geq \frac{({}_{v_n}\lambda_f - \varepsilon)}{({}_{v_n}\rho_g + \varepsilon)}\log(r_1 r_2 ... r_n) \\ i.e.,\ \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} &\geq \frac{({}_{v_n}\lambda_f - \varepsilon)}{({}_{v_n}\rho_g + \varepsilon)}. \end{aligned}$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\begin{aligned} \liminf_{r_1, r_2, ..., r_n \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} &\geq \frac{{}_{v_n}\lambda_f}{{}_{v_n}\rho_g} \\ i.e.,\ {}_{v_n}\lambda_g(f) &\geq \frac{{}_{v_n}\lambda_f}{{}_{v_n}\rho_g}. \end{aligned} \tag{2.12}$$

Also in view of (2.7), we get from (2.1) for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$\log M_g^{-1} M_f(r_1, r_2, ..., r_n) \le \log M_g^{-1}\left[\exp^{[2]}\{({}_{v_n}\rho_f + \varepsilon)\log(r_1 r_2 ... r_n)\}\right]$$

$$i.e.,\ \log M_g^{-1} M_f(r_1, r_2, ..., r_n) \le \log\exp\left[\frac{\log^{[2]}\exp^{[2]}\{({}_{v_n}\rho_f + \varepsilon)\log(r_1 r_2 ... r_n)\}}{({}_{v_n}\rho_g - \varepsilon)}\right]$$

$$i.e.,\ \log M_g^{-1} M_f(r_1, r_2, ..., r_n) \le \frac{({}_{v_n}\rho_f + \varepsilon)}{({}_{v_n}\rho_g - \varepsilon)}\log(r_1 r_2 ... r_n)$$

$$i.e.,\ \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} \le \frac{({}_{v_n}\rho_f + \varepsilon)}{({}_{v_n}\rho_g - \varepsilon)}\ .$$

Since $\varepsilon\,(> 0)$ is arbitrary, we get from above that

$$\liminf_{r_1, r_2, ..., r_n \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} \le \frac{{}_{v_n}\rho_f}{{}_{v_n}\rho_g}$$

$$i.e.,\ {}_{v_n}\lambda_g(f) \le \frac{{}_{v_n}\rho_f}{{}_{v_n}\rho_g}\ . \tag{2.13}$$

Similarly from (2.4) and in view of (2.6), it follows for a sequence of values of $r_1, r_2, ..., r_n$ tending to infinity that

$$\log M_g^{-1} M_f(r_1, r_2, ..., r_n) \le \log M_g^{-1}\left[\exp^{[2]}\{({}_{v_n}\lambda_f + \varepsilon)\log(r_1 r_2 ... r_n)\}\right]$$

$$i.e.,\ \log M_g^{-1} M_f(r_1, r_2, ..., r_n) \le \log\exp\left[\frac{\log^{[2]}\exp^{[2]}\{({}_{v_n}\lambda_f + \varepsilon)\log(r_1 r_2 ... r_n)\}}{({}_{v_n}\lambda_g - \varepsilon)}\right]$$

$$i.e.,\ \log M_g^{-1} M_f(r_1, r_2, ..., r_n) \le \frac{({}_{v_n}\lambda_f + \varepsilon)}{({}_{v_n}\lambda_g - \varepsilon)}\log(r_1 r_2 ... r_n)$$

$$i.e.,\ \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} \le \frac{({}_{v_n}\lambda_f + \varepsilon)}{({}_{v_n}\lambda_g - \varepsilon)}\ .$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain from above that

$$\liminf_{r_1, r_2, ..., r_n \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2, ..., r_n)}{\log(r_1 r_2 ... r_n)} \le \frac{{}_{v_n}\lambda_f}{{}_{v_2}\lambda_g}$$

$$i.e.,\ {}_{v_n}\lambda_g(f) \le \frac{{}_{v_n}\lambda_f}{{}_{v_n}\lambda_g}\ . \tag{2.14}$$

Thus the theorem follows from (2.9), (2.10), (2.11), (2.12), (2.13) and (2.14). □

Corollary 2.2. *Let f and g be any two entire functions of several complex variables such that g is of regular growth. Then*

$${}_{v_n}\lambda_g(f) = \frac{{}_{v_n}\lambda_f}{{}_{v_n}\rho_g} \quad \textit{and} \quad {}_{v_n}\rho_g(f) = \frac{{}_{v_n}\rho_f}{{}_{v_n}\rho_g}\ .$$

In addition, if ${}_{v_n}\rho_f = {}_{v_n}\rho_g$, then

$${}_{v_n}\rho_g(f) = {}_{v_n}\lambda_f(g) = 1\ .$$

Corollary 2.3. *Let f and g be any two entire functions of several complex variables with regular growth respectively . Then*

$${}_{v_n}\lambda_g(f) = {}_{v_n}\rho_g(f) = \frac{{}_{v_n}\rho_f}{{}_{v_n}\rho_g}\ .$$

Corollary 2.4. *Let f and g be any two entire functions of several complex variables with regular growth respectively. Also suppose that ${}_{v_n}\rho_f = {}_{v_n}\rho_g$. Then*

$${}_{v_n}\lambda_g(f) = {}_{v_n}\rho_g(f) = {}_{v_n}\lambda_f(g) = {}_{v_n}\rho_f(g) = 1\ .$$

Corollary 2.5. *Let f and g be any two entire functions of several complex variables with regular growth respectively. Then*

$$_{v_n}\rho_g(f) \cdot {}_{v_n}\rho_f(g) = {}_{v_n}\lambda_g(f) \cdot {}_{v_n}\lambda_f(g) = 1 .$$

Corollary 2.6. *Let f and g be any two entire functions of several complex variables such that either f is not of regular growth or g is not of regular growth. Then*

$$_{v_n}\lambda_g(f) \cdot {}_{v_n}\lambda_f(g) < 1 < {}_{v_n}\rho_g(f) \cdot {}_{v_n}\rho_f(g) .$$

Corollary 2.7. *Let f and g be any two entire functions of several complex variables. Then*

$$\begin{aligned}(i)\ {}_{v_n}\lambda_g(f) &= \infty \text{ when } {}_{v_n}\rho_g = 0 ,\\ (ii)\ {}_{v_n}\rho_g(f) &= \infty \text{ when } {}_{v_n}\lambda_g = 0 ,\\ (iii)\ {}_{v_n}\lambda_g(f) &= 0 \text{ when } {}_{v_n}\rho_g = \infty\end{aligned}$$

and

$$(iv)\ {}_{v_n}\rho_g(f) = 0 \text{ when } {}_{v_n}\lambda_g = \infty .$$

Corollary 2.8. *Let f and g be any two entire functions of several complex variables. Then*

$$\begin{aligned}(i)\ {}_{v_n}\rho_g(f) &= 0 \text{ when } {}_{v_n}\rho_f = 0 ,\\ (ii)\ {}_{v_n}\lambda_g(f) &= 0 \text{ when } {}_{v_n}\lambda_f = 0 ,\\ (iii)\ {}_{v_n}\rho_g(f) &= \infty \text{ when } {}_{v_n}\rho_f = \infty\end{aligned}$$

and

$$(iv)\ {}_{v_n}\lambda_g(f) = \infty \text{ when } {}_{v_n}\lambda_f = \infty .$$

Theorem 2.9. *Let f, g and h be any three entire functions of several complex variables such that ${}_{v_n}\rho_h(f) < \infty$ and ${}_{v_n}\lambda_h(f \circ g) = \infty$. Then*

$$\lim_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r_1,r_2,\ldots,r_n)}{\log M_h^{-1} M_f(r_1,r_2,\ldots,r_n)} = \infty .$$

Proof. Let us suppose that the conclusion of the theorem do not hold. Then we can find a constant $\beta > 0$ such that for a sequence of values of $r_1, r_2, \ldots, r_n$ tending to infinity,

$$\log M_h^{-1} M_{f\circ g}(r_1,r_2,\ldots,r_n) \le \beta \log M_h^{-1} M_f(r_1,r_2,\ldots,r_n) . \tag{2.15}$$

Again from the definition of ${}_{v_2}\rho_h(f)$, it follows for all sufficiently large values of $r_1, r_2, \ldots, r_n$ that

$$\log M_h^{-1} M_f(r_1,r_2,\ldots,r_n) \le ({}_{v_n}\rho_h(f) + \epsilon) \log(r_1 r_2 \ldots r_n) . \tag{2.16}$$

Thus from (2.15) and (2.16), we have for a sequence of values of $r_1, r_2, \ldots, r_n$ tending to infinity that

$$\log M_h^{-1} M_{f\circ g}(r_1,r_2,\ldots,r_n) \le \beta ({}_{v_n}\rho_h(f) + \epsilon) \log(r_1 r_2 \ldots r_n)$$

$$i.e.,\ \frac{\log M_h^{-1} M_{f\circ g}(r_1,r_2,\ldots,r_n)}{\log(r_1 r_2 \ldots r_n)} \le \frac{\beta({}_{v_n}\rho_h(f) + \epsilon)\log(r_1 r_2 \ldots r_n)}{\log(r_1 r_2 \ldots r_n)}$$

$$i.e.,\ \liminf_{r_1,r_2,\ldots,r_n\to\infty} \frac{\log M_h^{-1} M_{f\circ g}(r_1,r_2,\ldots,r_n)}{\log(r_1 r_2 \ldots r_n)} = {}_{v_n}\lambda_h(f\circ g) < \infty .$$

This is a contradiction.

Thus the theorem follows. □

Remark 2.10. Theorem 2.9 is also valid with "limit superior" instead of "limit" if ${}_{v_n}\lambda_h(f\circ g) = \infty$ is replaced by ${}_{v_n}\rho_h(f\circ g) = \infty$ and the other conditions remain the same.

Corollary 2.11. *Under the assumptions of Theorem 2.9 and Remark 2.10,*

$$\lim_{r_1,r_2,\ldots,r_n\to\infty} \frac{M_h^{-1} M_{f\circ g}(r_1,r_2,\ldots,r_n)}{M_h^{-1} M_f(r_1,r_2,\ldots,r_n)} = \infty \text{ and } \limsup_{r_1,r_2,\ldots,r_n\to\infty} \frac{M_h^{-1} M_{f\circ g}(r_1,r_2,\ldots,r_n)}{M_h^{-1} M_f(r_1,r_2,\ldots,r_n)} = \infty$$

respectively hold.

The proof is omitted.

Analogously one may also state the following theorem, remark and corollary without their proofs as those may be carried out in the line of Remark 2.10, Theorem 2.9 and Corollary 2.11 respectively.

Theorem 2.12. *Let f, g and h be any three entire functions of several complex variables with ${}_{v_n}\rho_h(g) < \infty$ and ${}_{v_n}\rho_h(f \circ g) = \infty$. Then*

$$\limsup_{r_1,r_2,\ldots,r_n \to \infty} \frac{\log M_h^{-1} M_{f\circ g}(r_1, r_2, ..., r_n)}{\log M_h^{-1} M_g(r_1, r_2, ..., r_n)} = \infty \ .$$

Remark 2.13. Theorem 2.12 is also valid with "limit" instead of "limit superior" if ${}_{v_n}\rho_h\ (f \circ g) = \infty$ is replaced by ${}_{v_n}\lambda_h\ (f \circ g) = \infty$ and the other conditions remain the same.

Corollary 2.14. *Under the assumptions of Theorem 2.12 and Remark 2.13,*

$$\limsup_{r_1,r_2,\ldots,r_n \to \infty} \frac{M_h^{-1} M_{f\circ g}(r_1, r_2, ..., r_n)}{M_h^{-1} M_g(r_1, r_2, ..., r_n)} = \infty \text{ and } \lim_{r_1,r_2,\ldots,r_n \to \infty} \frac{M_h^{-1} M_{f\circ g}(r_1, r_2, ..., r_n)}{M_h^{-1} M_g(r_1, r_2, ..., r_n)} = \infty$$

respectively hold.

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