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Semigroup ideals with semiderivations in 3-prime near-rings

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Abstract. The purpose of this paper is to obtain the structure of certain near-rings satisfying the following conditions:

(i) $d(I) \subseteq Z(N)$, **(ii)** $d(-I) \subseteq Z(N)$, **(iii)** $d([x, y]) = 0$,
(iv) $d([x, y]) = [x, y]$, **(v)** $d(x \circ y) = 0$, **(vi)** $d(x \circ y) = x \circ y$

for all $x, y \in I$, with I is a semigroup ideal and d is a semiderivation associated with an automorphism. Furthermore; an example is given to illustrate that the 3-primeness hypothesis is not superfluous.

Definitions and terminology

In this paper N will denote a zero symmetric left near-ring. For any $x, y \in N$ the symbol $[x, y]$ will denote the commutator $xy - yx$, while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. The symbol $Z(N)$ will represent the multiplicative center of N, that is, $Z(N) = \{x \in N \mid xy = yx \text{ for all } y \in N\}$. Unless specified, we will use the word near-ring to mean zero symmetric left near-ring. A near-ring N is said to be 3-prime if $xNy = \{0\}$ for all $x, y \in N$ implies $x = 0$ or $y = 0$. A nonempty subset I of N is called a semigroup right ideal (resp. semigroup left ideal) if $IN \subset I$ (resp. $NI \subset I$); and if I is both a semigroup right and a semigroup left ideal, then I is said to be a semigroup ideal. N is said to be 2-torsion free if $x \in N$ and $2x = 0$ implies $x = 0$. An additive mapping $\delta : N \to N$ is called a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ holds for all $x, y \in N$. Let g be an additive mapping of N, an additive mapping $d : N \to N$ is called a semiderivation of N associated with g if $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and $d(g(x)) = g(d(x))$ for all $x, y \in N$, or equivalently, as noted in [10], that $d(xy) = xd(y) + d(x)g(y) = g(x)d(y) + d(x)y$ and $d(g(x)) = g(d(x))$ for all $x, y \in N$. In the case of rings, semiderivations have received significant attention in recent years. We prove that some theorems in the setting of a semigroup ideal of a 3-prime near ring admitting a semiderivation, and thereby extend some known results [4, Theorem 2.1], [8, Theorem 2.6] and [8, Theorem 2.9].

1 Main Results

In this paper, the semiderivations used are associated with automorphisms. To prove our main theorems, we need the following lemmas.

Lemma 1. [4, Lemma 1.4(i)] Let N be a 3-prime near-ring, and I a nonzero semigroup ideal of N. If $x, y \in N$ and $xIy = \{0\}$, then $x = 0$ or $y = 0$.

Lemma 2. Let N be a 3-prime near-ring.
(i) [4, Lemma 1.2 (iii)] If $z \in Z(N)\backslash\{0\}$ and $xz \in Z(N)$, then $x \in Z(N)$.
(ii) [2, Lemma 1.5] If $N \subseteq Z(N)$, then N is a commutative ring.

Lemma 3. Let N be a near-ring and d is a semiderivation of N. Then N satisfies the following partial distributive law

i) $\big(d(x)y + g(x)d(y)\big)z = d(x)yz + g(x)d(y)z$ for all $x, y, z \in N$.

ii) $\big(xd(y) + d(x)g(y)\big)z = xd(y)z + d(x)g(y)z$ for all $x, y, z \in N$.

Theorem 1. Let N be a 3-prime near-ring and I be a nonzero semigroup ideal of N. If N admits a nonzero semiderivation d, then the following assertions are equivalent

i) $d(I) \subseteq Z(N)$

ii) N is a commutative ring.

Proof. $ii) \Rightarrow i)$ is obvious.
$i) \Rightarrow ii)$ by the hypothesis given, we have

$$d(xy)z = zd(xy) \text{ for all } y \in I, x, z \in N.$$

Taking Lemma 3(i), we get

$$d(x)yz + g(x)d(y)z = zd(x)y + zg(x)d(y) \text{ for all } y \in I, x, z \in N.$$

So that,

$$d(x)yz + d(y)g(x)z = zd(x)y + d(y)zg(x) \text{ for all } y \in I, x, z \in N. \tag{1.1}$$

Substituting $g(x)$ for z in (1.1), we obtain

$$d(x)yg(x) = g(x)d(x)y \text{ for all } y \in I, x \in N. \tag{1.2}$$

Replacing y by yt in (1.2) and using this, we get

$$\begin{aligned} d(x)ytg(x) &= (g(x)d(x)y)t \\ &= d(x)yg(x)t \text{ for all } y \in I, x, t \in N. \end{aligned}$$

The last equation shows that

$$d(x)y[g(x), t] = 0 \text{ for all } y \in I, x, t \in N$$

this means that

$$d(x)I[g(x), t] = \{0\} \text{ for all } x, t \in N.$$

By Lemma 1, this implies that

$$d(x) = 0 \text{ or } g(x) \in Z(N) \text{ for all } x \in N. \tag{1.3}$$

Taking the fact that $d \neq 0$, then (1.3) shows that there is an element $x_0 \in N$ such as $g(x_0) \in Z(N)$ and $d(x_0) \neq 0$. In this case, equation (1.1) yields

$$d(x_0)yz = zd(x_0)y \text{ for all } y \in I, z \in N.$$

Again replacing y by yt, we get

$$\begin{aligned} d(x_0)ytz &= (zd(x_0)y)t \\ &= d(x_0)yzt \text{ for all } y \in I, z, t \in N. \end{aligned}$$

Hence,

$$d(x_0)y[z, t] = 0 \text{ for all } y \in I, z, t \in N$$

this is reduced to

$$d(x_0)I[z, t] = \{0\} \text{ for all } z, t \in N. \tag{1.4}$$

Taking Lemma 1, (1.4) implies that $d(x_0) = 0$ or $N \subset Z(N)$. And since the first of these conditions is impossible, the second must hold N a commutative ring by Lemma 2(ii). □

Corollary 1. [4, Theorem 2.1] Let N be a 3-prime near-ring, and let I be a nonzero semigroup ideal of N. If N admits a nonzero derivation d for which $d(I) \subseteq Z(N)$, then N is a commutative ring.

Theorem 2. Let N be a 2-torsion free 3-prime near-ring and I be a nonzero semigroup ideal of N. If N admits a nonzero semiderivation d, then the following assertions are equivalent

i) $d(-I) \subseteq Z(N)$

ii) N is a commutative ring.

Proof. For $ii) \Rightarrow i)$, the proof is obvious.
$i) \Rightarrow ii)$, we have $d(-x) \in Z(N)$ for all $x \in I$, then

$$d(-tx) = d(t(-x)) \in Z(N) \text{ for all } x \in I, t \in N. \quad (1.5)$$

In particular, for all $t \in Z(N)$ we have

$$d(t(-x)) = td(-x) + d(t)g(-x) \in Z(N) \text{ for all } x \in I.$$

by Lemma 3(ii), we obtain

$$d(t)g(-x) \in Z(N) \text{ for all } x \in I. \quad (1.6)$$

Since g is an automorphism, then $d(t) \in Z(N)$. By the application of Lemma 2(i), (1.6) yields

$$d(t) = 0 \text{ or } g(-x) \in Z(N) \text{ for all } x \in I, t \in Z(N). \quad (1.7)$$

If $d(Z(N)) = \{0\}$, taking (1.5) into account, we get

$$d\Big(d\big(t(-x)\big)\Big) = 0 \text{ for all } x \in I, t \in N.$$

So that,

$$d^2(t)(-x) + 2g(d(t))d(-x) = 0 \text{ for all } x \in I, t \in N. \quad (1.8)$$

Replacing t by $d(t)$ in (1.8), we get

$$d^3(t)(-x) + 2g(d^2(t))d(-x) = 0 \text{ for all } x \in I, t \in N. \quad (1.9)$$

on the other hand, applying d for (1.8), we find that

$$d^3(t)(-x) + 3g(d^2(t))d(-x) = 0 \text{ for all } x \in I, t \in N. \quad (1.10)$$

From (1.9) and (1.10), we conclude that $g(d^2(t))d(-x) = 0$ for all $x \in I, t \in N$. Taking the fact that $d(-x) \in Z(N)$, then

$$d^2\big(g(t)\big)Nd(-x) = \{0\} \text{ for all } x \in I, t \in N.$$

In the light of the 3-primeness of N, the last equation implies that

$$d^2 = 0 \text{ or } d = 0. \quad (1.11)$$

If $d^2 = 0$, then $d = 0$ (see prove of Theorem 2 in [10]), and therefore (1.11) shows that $d = 0$, a contradiction. Consequently $d(Z(N)) \neq \{0\}$ and (1.7) prove that $g(-x) \in Z(N)$ for all $x \in I$. Let $v \in N$ and $x \in I$, we have $g(-vx) = g(v)g(-x) \in Z(N)$, by Lemma 2$(i)$, we get

$$g(-x) = 0 \text{ or } g(v) \in Z(N) \text{ for all } x \in I, v \in N. \quad (1.12)$$

$i)$ If $g(-x) = 0$ for all $x \in I$, by this hypothesis we have

$$d(-yx) = d(y)g(-x) + yd(-x) \in Z(N) \text{ for all } x \in I, y \in N.$$

So that,

$$yd(-x) \in Z(N) \text{ for all } x \in I, y \in N.$$

Using Lemma 2(i) and taking the fact $d \neq 0$, we arrive at $N \subset Z(N)$. Applying Lemma 2(ii), we conclude that N is a commutative ring.
$ii)$ If there is an element $x_0 \in I$ such that $g(-x_0) \neq 0$, then equation (1.12) shows that $g(v) \in Z(N)$ for all $v \in N$. Since g is an automorphism we conclude that $N \subset Z(N)$. Thus N is a commutative ring. This completes the proof of our theorem. □

Corollary 2. [8, Lemma 2.4] Let N be a 2-torsion free 3-prime near-ring and I be a nonzero semigroup ideal of N. If N admits a nonzero derivation d for which $d(-I) \subseteq Z(N)$, then N is a commutative ring.

Theorem 3. Let N be a 3-prime near-ring and I be a nonzero semigroup ideal of N. If N admits a semiderivation d, then the following assertions are equivalent:

i) $d([x, y]) = 0$ for all $x, y \in I$.

ii) $d([x,y]) = [x,y]$ for all $x, y \in I$.

iii) N is a commutative ring.

Proof. iii) $\Rightarrow$ i) and iii) $\Rightarrow$ ii) are obvious.
Proving that i) $\Rightarrow$ iii). Suppose that

$$d([x,y]) = 0 \quad \text{for all } \ x, y \in I. \tag{1.13}$$

Substituting xy for y in (1.13), we have

$$d(x)[x,y] + g(x)d[x,y] = 0 \quad \text{for all } \ x, y \in I.$$

Hence,

$$d(x)xy = d(x)yx \ \text{ for all } \ x, y \in I. \tag{1.14}$$

Replacing y by yt in (1.14) and using this, we get

$$d(x)I[x,t] = \{0\} \ \text{ for all } \ x \in I, t \in N.$$

Taking into account the Lemma 1, we get

$$d(x) = 0 \ \text{ or } \ x \in Z(N) \ \text{ for all } \ x \in I. \tag{1.15}$$

Since d is associated with an automorphism, we have $d(x) \in Z(N)$ for each $x \in Z(N)$, then (1.15) illustrated $d(I) \subseteq Z(N)$. By the use of Theorem 1, we obtain N is a commutative ring.
Proving that ii) $\Rightarrow$ iii). By the hypothesis given, we have

$$d([x,y]) = [x,y] \ \text{ for all } \ x, y \in I. \tag{1.16}$$

Replacing y by xy in (1.16), we get

$$xd([x,y]) + d(x)g([x,y]) = x[x,y] \ \text{ for all } \ x, y \in I.$$

It follows that

$$d(x)g(x)g(y) = d(x)g(y)g(x) \ \text{ for all } \ x, y \in I. \tag{1.17}$$

Since g is an automorphism,(1.17) shows that

$$d(x)g(x)j = d(x)jg(x) \ \text{ for all } \ x \in I, j \in J \tag{1.18}$$

with $J = g(I)$, it is clear that J is a semigroup ideal of N. Substituting jz for j in (1.18) and using this, we obtain

$$d(x)j[g(x), z] = 0 \ \text{ for all } \ x \in I, j \in J, z \in N. \tag{1.19}$$

Thus,

$$d(x)J[g(x), z] = \{0\} \ \text{ for all } \ x \in I, z \in N. \tag{1.20}$$

By the application of Lemma 1, (1.20) yields that

$$d(x) = 0 \ \text{ or } \ g(x) \in Z(N) \ \text{ for all } \ x \in I.$$

Which implies that

$$d(g(x)) \in Z(N) \ \text{ for all } \ x \in I$$

Consequently, we deduce that $d(J) \subset Z(N)$. And therefore, Theorem 1 assures that N is a commutative ring. This completes the proof of our theorem. □

Corollary 3. [4, Theorem 4.1] Let N be a 3-prime near-ring, and U a nonzero semigroup ideal. If N admits a derivation d such that $d^2 \neq 0$ and $d(uv) = d(vu)$ for all $u, v \in U$, then N is a commutative ring.

Corollary 4. Let N be a 3-prime near-ring. If N admits a nonzero derivation d such that $d([x,y]) = 0$ for all $x, y \in N$, then N is a commutative ring.

Corollary 5. [8, Theorem 2.6] Let N be a 3-prime near-ring and I be a nonzero semigroup ideal of N. If N admits a nonzero derivation d such that $d([x,y]) = [x,y]$ for all $x, y \in I$, then N is a commutative ring.

Corollary 6. [7, Theorem 2.2] Let N be a 3-prime near-ring. If N admits a nonzero derivation d such that $d([x,y]) = [x,y]$ for all $x,y \in N$, then N is a commutative ring.

Now, replacing the commutator $[x,y]$ by the anti-commutator $x \circ y$, our aim is to study this issue and to see if the results are different.

Theorem 4. Let N be a 2-torsion free 3-prime near-ring and I be a semigroup ideal of N, then N admits no nonzero semiderivation d satisfying one of the assertions as the following:

i) $d(x \circ y) = 0$ for all $x,y \in I$.

ii) $d(x \circ y) = x \circ y$ for all $x,y \in I$.

Proof. i) Suppose that there is d which indicates the following

$$d(x \circ y) = 0 \text{ for all } x,y \in I. \tag{1.21}$$

Replacing y by xy in (1.21) and taking the fact that $x \circ xy = x(x \circ y)$, we get

$$d(x)(x \circ y) = 0 \text{ for all } x,y \in I.$$

So that,

$$d(x)xy = -d(x)yx \text{ for all } x,y \in I. \tag{1.22}$$

Substituting yt for y in (1.22), we obtain

$$d(x)y(-x)t = d(x)yt(-x) \text{ for all } x,y \in I, t \in N$$

which can be rewritten as

$$d(x)I[-x,t] = \{0\} \text{ for all } x \in I, t \in N.$$

By using the lemma 1, we have

$$d(x) = 0 \text{ or } -x \in Z(N) \text{ for all } x \in I.$$

Hence, $d(-x) \in Z(N)$ for all $x \in I$, it means that $d(-I) \subset Z(N)$. According to theorem 2, we get N is a commutative ring. In this case, returning to the hypothesis given, we have

$$d(xy) = 0 \text{ for all } x,y \in I.$$

It follows that

$$d(x)y + g(x)d(y) = 0 \text{ for all } x,y \in I \tag{1.23}$$

Taking yz instead of y in (1.23), we get

$$d(x)yz = 0 \text{ for all } x,y,z \in I.$$

Therefore,

$$d(x)Iz = \{0\} \text{ for all } x,z \in I.$$

By Lemma 1, the last expression shows that $d = 0$, a contradiction.
ii) Suppose there is d such that

$$d(x \circ y) = x \circ y \text{ for all } x,y \in I. \tag{1.24}$$

Putting xy instead of y in (1.24), we arrive at

$$d(x)g(x)g(y) = -d(x)g(y)g(x) \text{ for all } x,y \in I.$$

Which implies that

$$d(x)g(x)n = -d(x)ng(x) \text{ for all } x \in I, n \in J = g(I). \tag{1.25}$$

Writing nm instead of n in (1.25), we find that

$$d(x)n[g(-x),m] = 0 \text{ for all } x \in I, n \in J, m \in N$$

implying

$$d(x)J[g(-x),m] = \{0\} \text{ for all } x \in I, m \in N. \tag{1.26}$$

Applying Lemma 1, (1.26) shows that

$$d(x) = 0 \text{ or } g(-x) \in Z(N) \text{ for all } x \in I.$$

And therefore $d(g(-x)) \in Z(N)$ for all $x \in I$, then $d(-J) \subseteq Z(N)$. According to Theorem 2, we conclude that N is a commutative ring. In this case, returning to the hypothesis given, we have

$$d(xy) = xy \text{ for all } x, y \in I$$

it follows that

$$d(x)y + g(x)d(y) = xy \text{ for all } x, y \in I. \tag{1.27}$$

Substituting xz for x in (1.27), we obtain

$$g(x)g(z)d(y) = 0 \text{ for all } x, y, z \in I.$$

Which can be rewritten as

$$g(x)Jd(y) = \{0\} \text{ for all } x, y \in I. \tag{1.28}$$

By Lemma 1, (1.28) demonstrates $g(I) = \{0\}$ or $d = 0$, but each of these conditions yields a contradiction. □

Corollary 7. [8, Theorem 2.9] Let N be a 2-torsion free 3-prime near-ring, and I be a nonzero semigroup ideal of N. Then there is no derivation d such that $d(x \circ y) = x \circ y$ for all $x, y \in I$.

The following example shows that the primeness is necessary in the hypotheses of the above theorems.

Example Let S be a 2-torsion free noncommutative near-ring. Let us define N and $d, g : N \to N$ by:

$$N = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y \in S \right\}.$$

$$d\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, it is straightforward to check that N is not 3-prime left near-ring admitting a nonzero semiderivation d associated with g. Moreover; it is easy to verify that d satisfies the properties:

i) $d(N) \subseteq Z(N)$ **ii)** $d(-N) \subseteq Z(N)$ **iii)** $d([A, B]) = 0$

iv) $d([A, B]) = [A, B]$ **v)** $d(A \circ B) = 0$ **vi)** $d(A \circ B) = A \circ B$

for all $A, B \in N$. However, N is not commutative.

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