

# Mathematical Modelling Of Ecological Networks, Structure and Interaction Of Prey And Predator

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**Abstract** Mathematical models of population growth have been constructed to provide an abstract of some significant aspect of true ecological situation. In this paper, we put some models where the parameters of the biological growth model systematically change over time. The growth rate of the predator depends upon predation on the modeled and alternate prey. Two stable equilibrium separated by a saddle point potentially exist for the predator-prey system, and stochastic variability can lead to movement between equilibrium abundance levels. Predator abundance can increase when the modeled prey abundance is low, due to consumption of alternate prey. Here we have discussed about some important five growth models.

## 1 Introduction

The bases for analyzing the dynamics of complex ecological systems, such as food chains, are the interactions between two species, particularly the dynamical relationship between predators and their prey. From the seminal Lotka-Volterra model, several alternatives for modeling continuous time consumer-resource interactions have been proposed. Volterra (1926) first proposed a simple model for the predation of one species by another to explain the oscillatory levels of certain fish catches in the Adriatic, which is described in model 2. The increasing study of realistic and practically useful mathematical models in population biology, whether we are dealing with a human population with or without its age distribution, population of an endangered species, bacterial or viral growth and so on, is a reflection of their use in helping to understand the dynamic process involved and in making practical prediction.

## 2 Some Basic Definitions:

### 2.1 community

A community in ecology comprises all species populations interacting in an area. An example of a community is a coral reef, where numerous populations of fishes, crustaceans, and corals coexist and interact.

### 2.2 food web

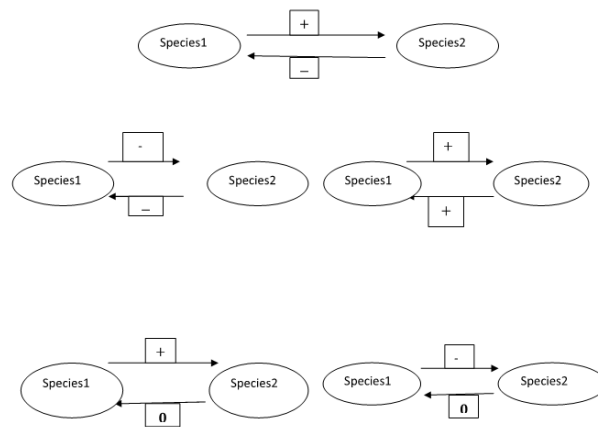
A food web in an ecosystem is an assemblage of various organisms that are interconnected with each other through their different life history processes, such as feeding and shelter

### 2.3 trophic level

A trophic level in a food web consists of all the species that prey on the same species and are also preyed upon by the same species [3].

### 2.4 Ecological interactions

Ecological interactions are the relationships between two species in an ecosystem [2]. Based on either effects, or on mechanisms, these relationships can be categorized into many different classes of interactions as shown in Fig. 1.



**Figure 1.** Fig: 1 Types of ecological interactions: (i) predation, (ii) competition, (iii) mutualism, (iv) commensalism, (v) ammensalism.

## 2.5 Predation

Predation is a biological interaction in which one species feeds on another. Most of the interactions in a food web are predatory. Figure 1(i) shows the network for this interaction, where species 2 preys on species 1. This interaction enhances the fitness of predators, but reduces the fitness of the prey species .



**Figure 2.** Example of Predation. Here the predator cat eats the prey rabbits.

## 2.6 Parasitism

Parasitism is similar to predation by mechanism, as it enhances the fitness of the parasite, but impairs the host.



**Figure 3.** Example of parasitism. In this picture mosquito is parasite and human body is the host. Here mosquito decreases the body's immunity by seeking our blood.

## 2.7 Competition

Competition between two species occurs when they share a limited resource and each tends to prevent the other from accessing it. This reduces the fitness of one or both species. This reduces the fitness of one or both species, as is shown by Fig. 1(ii).



**Figure 4.** Example of competition. Here in the picture we can see deadly hyena and the tiger fighting out for a limited resource.

## 2.8 mutualism

In mutualism or symbiosis, two species provide resources or services to each other. This enhances the fitness of both species (shown by Fig. 1(iii)).



**Figure 5.** Example of mutualism. Here we can see that the bird sitting on the deer's body eats insects from its body. So both are benefited.

## 2.9 Commensalism

Commensalism is an interaction, where one species receives a benefit from another species. This enhances the fitness of one species without any effect on fitness of the other species. (Shown by Fig. 1(iv)).



**Figure 6.** Example of commensalism. Here in the picture we see cows eating grasses meanwhile birds eating insects sharing the same spaces. That means common sharing and interaction of one species with another without harming each other.

## 2.10 Ammensalism

In ammensalism, one species impedes or restricts the success of the other without being affected positively or negatively by its presence. (Shown by Fig. 1(v)).



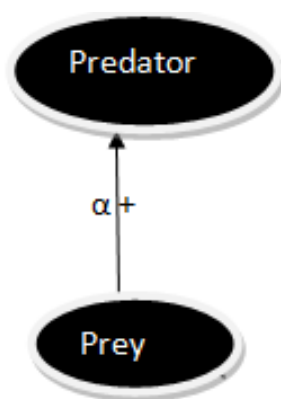
**Figure 7.** Example of ammensalism. In this picture we see that the species lion restricts the growth of buffalo by eating it where as both species can live without harming the other.

### 3 Model1

The below figure shows a prey-predator system where the predator species is commensally on the prey species [9]. In this simple model, in the absence of the predator ( $Y$ ), the prey species ( $X$ ) follows a density-dependent logistic growth with  $r$  as its intrinsic growth rate, and  $K$  is the carrying capacity of the environment. However, in the presence of the predator, the growth of the prey is reduced due to predation of  $Y$  on  $X$ . This interaction follows a hyperbolic function with  $\hat{S}$  denoting the half saturation coefficient of predation and  $\hat{I}$  deciding the strength of interaction, i.e., the per capita consumption rate. In the absence of prey, the predator species dies out exponentially at a rate  $d$ . On predation, the rate at which this food adds to the growth of the predator population is given by the conversion rate  $\beta$ . The rate of change of the prey ( $dX/dt$ ) and predator ( $dY/dt$ ) populations with time are governed by the following equations:

$$\frac{dX}{dt} = rx\left(1 - \frac{X}{k}\right) - \frac{\alpha xy}{\gamma + x}$$

$$\frac{dy}{dt} = y\left(-d + \frac{\beta \alpha x}{\gamma + x}\right)$$



**Figure 8.**Food web configurations of Model(1).

### 4 Model2

If  $N(t)$  is the prey population and  $P(t)$  that of the predator at time  $t$  then Where  $a$ ,  $b$ ,  $c$  and  $d$  are positive constants. This is known as Lotka-Volterra model[6, 10]. The assumptions in the model are: (i) The prey in the absence of any predation grows unboundedly in a Malthusian way; this is the  $aN$  term in equ. (1). (ii) The effect of the predation is to reduce the prey per capita growth rate by a term proportional to the prey and predator populations; this is the  $-bNP$  term. (iii) In the absence of any prey for sustenance the predators death rate results in exponential decay, that is, the  $dP$  term in equ. (2). (iv) The preys contribution to the predators growth rate is  $cNP$ ; that

is, it is proportional to the available prey as well as to the size of the predator population. The NP terms can be to another: bNP is taken from the prey and cNP accrues to the predators.

$$\frac{dN}{dt} = N(a - bP)$$

$$\frac{dP}{dt} = P(cN - d)$$

## 5 Model3

In this model, the ruling and the opposition parties behave as predator species with the third party as the prey. Hence, third party members (prey) switch towards ruling and opposition parties (predator) which is vice-versa to previous models. In the predator-prey models, the predator is always harmful for the growth of the prey species, but here, when the ruling and the opposition party members enter into the third party, they strengthen it [9] Tansky considers a general Volterra type two prey-one predator model that may be expressed as follows:

$$\frac{dx_1}{dt} = (\delta_1 - \frac{b_1 y}{1 + (\frac{x_2}{x_1})^n})x_1$$

$$\frac{dx_2}{dt} = (\delta_2 - \frac{b_2 y}{1 + (\frac{x_2}{x_1})^n})x_2$$

$$\frac{dy}{dt} = (-d + \frac{b_1 x_1}{1 + (\frac{x_2}{x_1})^n} + \frac{b_2 x_2}{1 + (\frac{x_2}{x_1})^n})$$

Where  $x_1(t)$ ,  $x_2(t)$ , and  $y(t)$  denote the abundances of the prey species 1 and 2 and the predator species, respectively.  $\delta_1$  and  $\delta_2$  are the specific growth rates of the prey species in the absence of predation and  $d$  is the per capita death rate for the predator. The functions

$$\frac{b_1 y}{1 + (\frac{x_2}{x_1})^n}$$

and

$$\frac{b_2 y}{1 + (\frac{x_2}{x_1})^n}$$

possess the characteristic property of switching through functional response of relative abundance of the prey species. When  $n = 1$ , this is a simple multiplicative effect, where as for  $n > 1$ , the effect is stronger.

## 6 Model4

This model is Leslie-Gower predator-prey model [11] incorporating the Allee effect [12] phenomenon on prey is described by the Kolmogorov type differential equation system given by:

$$\frac{dx}{dt} = (r(1 - \frac{x}{k})(x - m) - qy)x$$

$$\frac{dy}{dt} = s(1 - \frac{y}{nx})y$$

With  $x = x(t)$  and  $y = y(t)$ , indicating the prey and predators population size, respectively and the parameters have the following biological meanings:  $r$  and  $s$  represent the intrinsic prey and predator growth rates, respectively,  $k$  is the prey environment carrying capacity,  $m$  is the Allee threshold or minimum of viable population,  $q$  is the maximal per capita consumption rate, i.e., the maximum number of prey that can be eaten by a predator in each time unit,  $n$  is a measure of food quality that the prey provides for conversion into predator births.

## 7 Model5

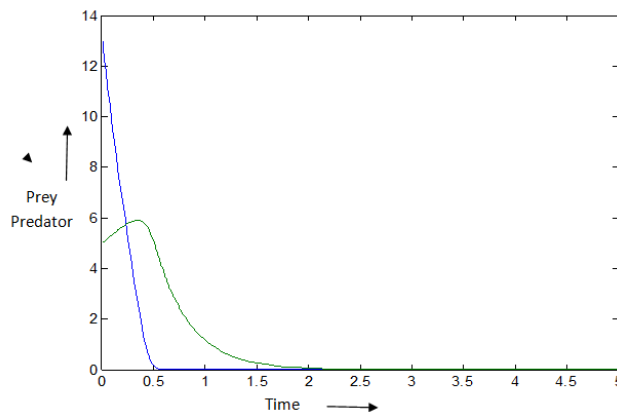
This model is called predator-prey situation. That means, a predator-prey situation occurs with two populations, when encounters between members of the populations benefit the members of one population (predators) but adversely affect the members of the other population (prey) [9]. For a simple model assume that the prey has an adequate food supply and that in the absence of interaction there will be a positive rate of increase proportional to the population size. Likewise, assume that in the absence of the prey, the predators food supply will be inadequate and that there will be a negative rate of increase proportional to the population size. The incidence of interaction will depend on the size of both populations and may be taken to be proportional to their product, and the effect will increase the rate of growth of the predator population and decrease the rate of growth of the prey population [6]. Letting  $X$ ,  $Y$  denotes the size of the prey and predator populations respectively, we can write down two simultaneous differential equations to represent the situation.

$$\frac{dX}{dt} = kX - \lambda XY = kX(1 - \alpha Y)$$

$$\frac{dY}{dt} = -lY + \gamma XY = -lY(1 - \beta X)$$

## 8 Result and Discussion

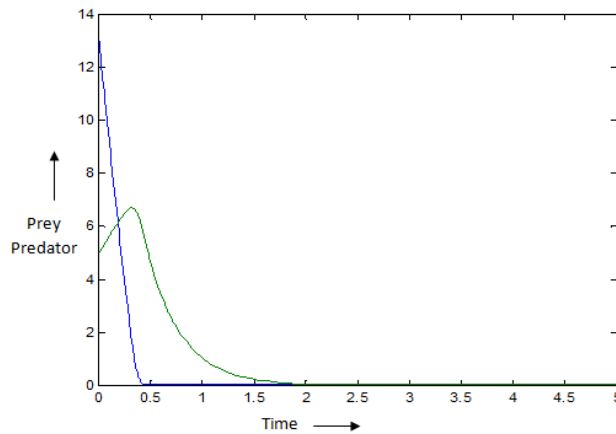
For this study In model 1, the parameter values are taken as  $r=0.5$ ,  $K=5$ ,  $d=3$ ,  $\gamma=0.8$ , and  $\beta=0.7$ . The main parameter,  $\alpha$ , which indicates the interaction strength of predation, is varied from 5.5 to 6.5 to describe the change in dynamics in this model network. Here let,  $X=13$ ,  $Y=5$  and  $t$  is varied from 0 to 5. Now for  $\alpha=5.5$ , we plot the figure 9. we can see from the figure 9. When time  $t=0$ , the number of prey is maximum and it is 13. On the other hand the number of predator at time  $t=0.4$  is maximum and it is becoming approximately 5.6. This shows that initially number of prey is higher than predator but gradually it decreases than no. of predators. Moreover predator is highest when prey is nearly minimum and gradually predator becomes less with the decrease in number of prey, and ultimately predator finishes as there is no prey left:



**Figure 9.** Population growth rate forms for prey and predator in case of  $\alpha=5.5$ . Here blue color denotes the growth rate of prey( $X$ ) and green denote the growth rate of predator( $Y$ ).

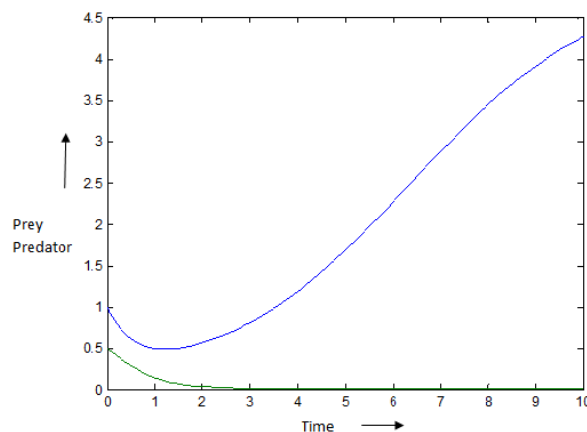
Now, for  $\alpha=6.4$  we plot this graph 10 and other values remaining same. Here in figure 10 the number of predator is maximum at time  $t=0.4$  and it is becoming approximately 6.8. Otherwise figure 9 and 10 almost similar, the only difference between the highest point of predator:





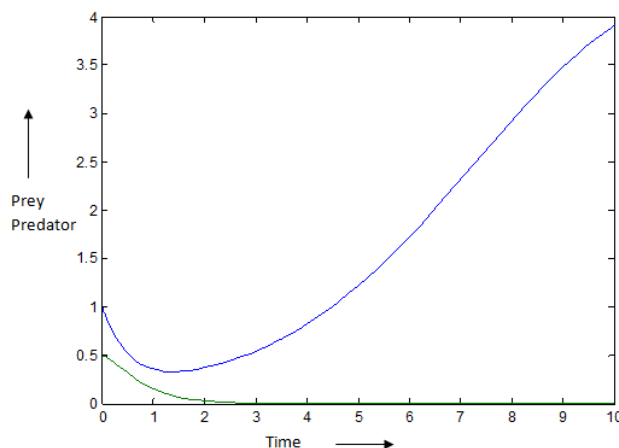
**Figure 10.** Population growth forms for prey and predator in case of  $\alpha = 6.4$ . Here blue color denotes the growth rate of prey(X) and green denotes the growth rate of predator(Y).

Now let,  $X=1$ ;  $Y=0.5$ ; and  $t$  is varied from 0 to 10. Now for  $\alpha = 5.5$  we plot the figure 11 and we can see from the figure 2(c) that at time  $t=10$  the number of prey is maximum and it is becoming approximately 4.2. On the other hand the number of Predator at time  $t=0$  is maximum and it is 0.5. This shows that there is competition between predator and prey.



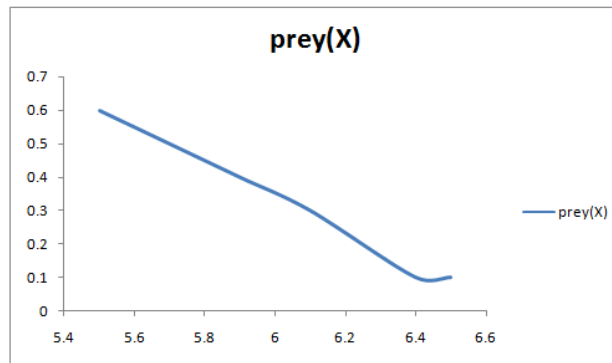
**Figure 11.** Population growth rate forms for prey and predator in case of  $\alpha = 5.5$ . Here blue color denotes the growth rate of prey(X) and green denotes the growth rate of predator(Y).

Now for  $\alpha = 6.4$  we plot this graph 12 and other values are remaining same. We can see from the figure 12 that at time  $t=10$  the number of prey is maximum and it is becoming approximately 3.8. On the other hand the number of Predator at time  $t=0$  is maximum and it is 0.5. This shows that there is competition between predator and prey.

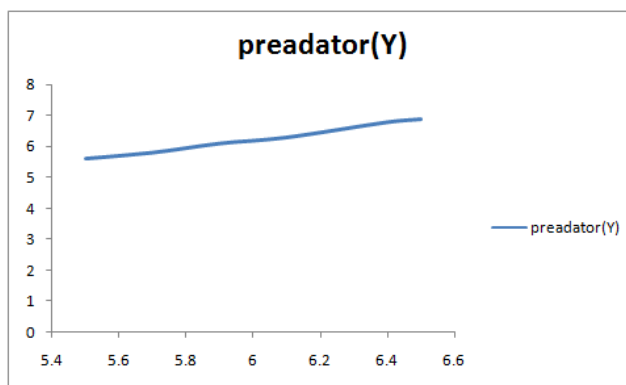




**Figure 12.** Population growth rate forms for prey and predator in case of  $\alpha = 6.4$ . Here blue color denotes the growth rate of prey(X) and green denote the growth rate of predator(Y).

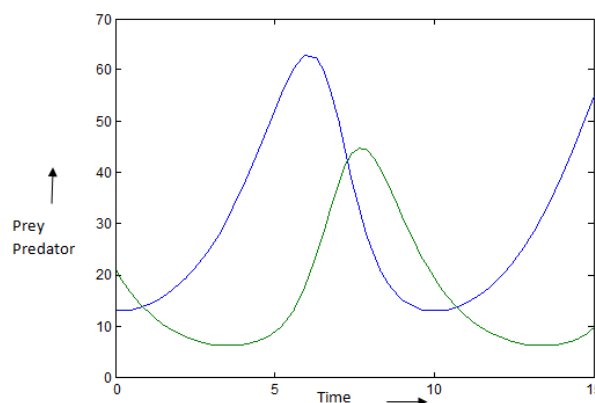


**Figure 13.** Diagram of the prey with increasing predation strength  $\alpha$ .



**Figure 14.** Diagram of the predator with increasing predation strength  $\hat{\alpha}$ .

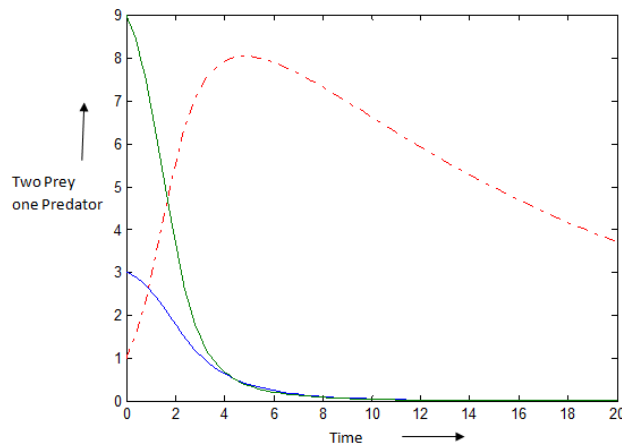
In model 2, the parameter values are taken as,  $a=0.5471$ ;  $b=0.0281$ ;  $c=0.0266$ ;  $d=0.8439$ . Here let  $t=13$  and  $P=21$  and for these values we plot the Fig.15. We can see from the Fig.15 that at time  $t=6$  the number of prey is maximum and it is becoming approximately 63. On the other hand the number of Predator at time  $t=8$  is maximum and it is becoming approximately 42. This shows that there is competition between predator and prey.



**Figure 15.** Population growth rate forms for prey and predator. Here blue color denotes the growth rate of prey(X) and green denote the growth rate of predator(Y).

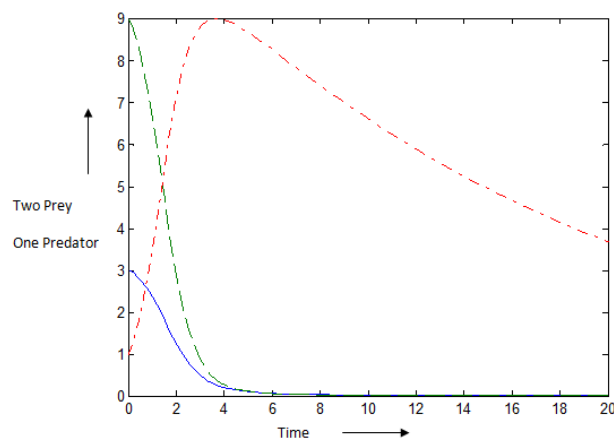
In model 3, the parameter values are taken as  $\delta_1 = 0.02$ ,  $\delta_2 = 0.03$ ,  $d=0.06$ ,  $b_1=0.131$ ,  $b_2=0.17$ ,  $x_1=3$ ,  $x_2=9$ ,  $y=1$  and  $t$  is varied from 0 to 20. Now for  $n=1$ ; we plot figure 16. and here we plot two prey and one predator. We can see from the figure 16. that at time  $t=0$  the number of preys ( $x_1$  and  $x_2$ ) are maximum and it is 3 and 9 respectively. On the other hand the number of Predator at time  $t=5.5$  is maximum and it is becoming approximately 8. This shows that initially

number of preys is higher than predator but gradually it decreases than the no. of predators. Moreover predator is highest when prey is nearly minimum and gradually predator becomes less with the decrease in number of prey.



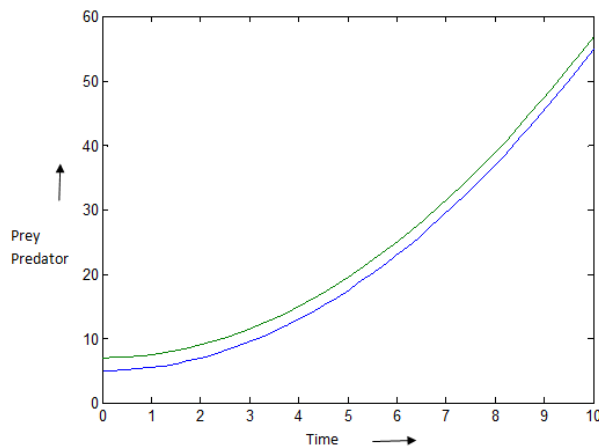
**Figure 16.** Population growth forms for two prey and one predator in case of  $n=1$ . Here blue and green color denotes the growth rates of two prey and red denote the growth rate of predator(Y).

Now for  $n>1$  we plot the figure 17 and other values are remaining same. Here figure 17 and 18 both graphs are almost similar, the only difference between the highest point of predator. In this figure the number of predator is maximum at time  $t=5$  and it is becoming 9.



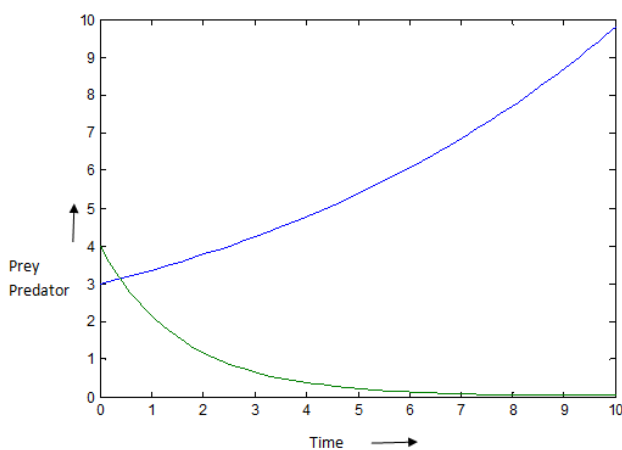
**Figure 17.** Population growth forms for two prey and one predator in case of  $n>1$ . Here blue and green color denotes the growth rates of two prey and red denote the growth rate of predator(Y).

In model 4, the parameter values are taken as,  $r=0.03$ ;  $k=0.052$ ;  $n=0.15$ ;  $m=13$ ;  $q=0.05$ ;  $s=0.09$ ;  $x=5$ ;  $y=7$  and for these values we plot figure 18. Here we can see from the figure 18. that at time  $t=10$  the number of prey is maximum and it is becoming approximately 53. On the other hand the number of Predator at time  $t=10$  is maximum and it is becoming approximately 55. This shows that there is competition between predator and prey.



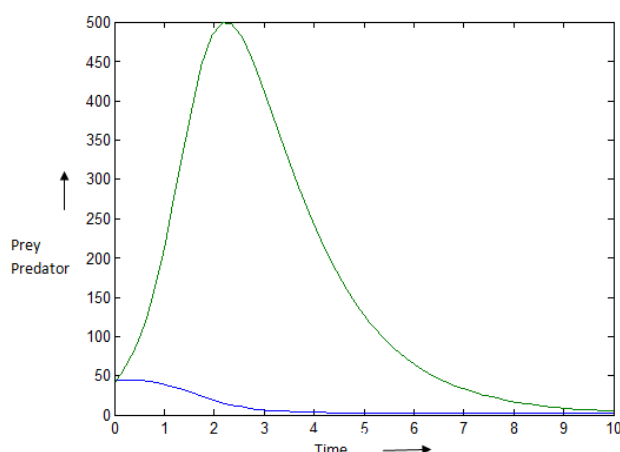
**Figure 18.** Population growth forms for prey and predator. Here blue color denotes the growth rates of prey (X) and green denote the growth rate of predator(Y).

In model 5, the parameter values are taken as  $\alpha = 0.02$ ;  $\beta = 0.07$ ;  $k=0.12$ ,  $l=0.8$ . Now for  $X=3$  and  $Y=4$  we plot the figure 19. Here we can see from the figure 19 that at time  $t=10$  the number of prey is maximum and it is becoming approximately 10. On the other hand the number of Predator at time  $t=0$  is maximum and it is 4. This shows that when predator is maximum at that moment prey is minimum and when predator decreases comparatively prey increases. That means there is competition between predator and prey.



**Figure 19.** This figure presents a predator-prey situation. Here blue color denotes the growth rates of prey (X) and green denote the growth rate of predator(Y).

Now for  $X=45$ ,  $Y=42$  we plot the figure 20 and other values are remaining same. We can see from the figure 20 that at time  $t=0$  the number of prey is maximum and it is becoming approximately 45. On the other hand the number of Predator at time  $t=2.5$  is maximum and it is 500.



**Figure 20.** This figure presents a predator-prey situation. Here blue color denotes the growth rates of prey (X) and green denote the growth rate of predator (Y).

## 9 Conclusions

In this paper we formulated five basic mathematical models which is based on the existence of different Prey - Predator. Most of the recent research on food web theory in ecology centres around the local dynamics of a community, but the evolution of food web dynamics across different spatial scales has also received considerable attention [12, 8]. Such research on ecological network theory in the future would involve rigorous modelling approaches, both analytical and through simulations, in combination with field and laboratory experimental studies, to resolve the crucial questions in conservation and restoration ecology. In this work we have analyzed the significant impact produced by the phenomenon known as Allee effect in a simple Leslie predator-prey model also called Leslie-Gower model [11]. Ecologically speaking, the prey and predator species coexist for any parameter values. The simplest models of population dynamics reveal the delicate balance that exists in almost all ecological systems. Refined Lotka-Volterra models appear to be the appropriate level of mathematical sophistication to describe simple predator-prey models.

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