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Varieties Generated by Two Element Groupoids and the Terms Mode and Differential

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Abstract In this paper, we give an example of differential variety of non-associative groupoid generated by three element. We show which of the varieties generated by two element non-associative groupoids are mode and differential, or contain a mode or differential varieties.

1 Introduction

In algebra the concept 'variety' was introduced by Philip Hall in 1949, other terms used instead of variety are 'equationally definable class' (Tarski), and 'Primitive class' (Malcev).

The study of varieties initiated by Garret Birkhoff in 1935. He stated one of the most important theorem in universl algebra known as Birkhoff's theorem or HSP theorem. There are two definitions of variety, and they are equivalent by Birkhoff's Theorem (HSP Theorem), [3].

Definition 1. *[4] A non empty class $\mathcal{K}$ of algebras of type τ is called a* ***variety*** *if it is closed under subalgebras, homomorphic images, and direct products.*

Definition 2. *[4] A* ***variety*** *is a class of all algebraic structures of a given type satisfying a given set of identities.*

For example, the class of **semigroup** $\langle G, \cdot\rangle$ forms a variety of algebra of type (2) and a sufficient defining equation (identity) is the associative law

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z) \qquad \text{for all } x,\ y,\ z\ \in\ G$$

Definition 3. *[5] A differential groupoid is a structure with one fundamental operation satisfying the following identities:*

(1) $x.x \approx x$ *idempotent law*

(2) $(x.y).(z.t) \approx (x.z).(y.t)$ *entropic law*

(3) $x.(y.z) \approx x.y$ *reductive law*

Definition 4. *[5] A groupoid $\langle G, .\rangle$ is said to be a mode if it is both idempotent and entropic.*

So, we can define a differential groupoid as a mode groupoid satisfying the identity:

$$x(yz) \approx xy \qquad \textit{reductive law.}$$

Example 1. *[2] A rectangular band* ***R*** = $\langle\{a, b, c, d\}, .\rangle$ *with the following multiplication table R is a groupoid satisfying the identities*

(1) $x.x \approx x$

(2) $x.(y.z) \approx (x.y).z$

(3) $x.(y.z) \approx x.z.$

.	a	b	c	d
a	a	b	a	b
b	a	b	a	b
c	c	d	c	d
d	c	d	c	d

Table 1. Groupoid **R**

Every rectangular band is entropic, since

$$(x.y).(z.w) = (x.(y.z)).w = (x.z).w = x.(z.w) = x.w$$

$$x.w = x.(y.w) = (x.y).w = (x.(z.y)).w = (x.z).(y.w)$$

and it satisfies the idempotent law, so this groupoid is mode, but is not differential since it does not satisfy the reductive law

$$a.(b.c) = a.c = a \neq a.b = b \; for \; a, b, c \in R.$$

Thus, the class of all rectangular bands forms a variety of modes.

*Every rectangular band is isomorphic to a product of a left-zero semigroup (a semigroup satisfying the identity $x.y \approx x$) and a right-zero semigroup (a semigroup satisfying the identity $x.y \approx y$), and conversely. Therefore the variety $\mathcal{R}b$ of all rectangular band is generated by a four-element algebra **R** (the direct product of a two-element left zero semigroup and a two-element right-zero semigroup).*

Example 2. *Consider the groupoid **G** = $\langle\{a, b, c\}, .\rangle$ with the following multiplication table*

.	a	b	c
a	a	a	b
b	b	b	a
c	c	c	c

Table 2. Groupoid **G**

***G** is non-associative groupoid satisfies the idempotent law since*

$$a.a = a \; , \; b.b = b \; \text{ and } c.c = c,$$

*it satisfies entropic law ($(x.y).(z.t) \approx (x.z).(y.t)$), and reductive law ($x.(y.z) \approx x.y$), so **G** is differential groupoid.*
*Let $\mathcal{V}$ be a variety generated by the differential groupoid **G** = $\langle\{a, b, c\}, .\rangle$, then $\mathcal{V}$ is a variety of differential groupoids.*

On a set of two elements $\{a,b\}$ there are 16 different binary operations (groupoids), eight of them are associative. If we get the non-isomorphic groupoids we get the following(the isomorphic systems have the same properties):

$S_1 = < \{a, b\}, . >$, $aa = ab = ba = bb = a$;
$S_2 = < \{a, b\}, . >$, $aa = bb = b$, $ab = ba = a$;
$S_3 = < \{a, b\}, . >$, $aa = a$, $ab = ba = bb = b$;
$S_4 = < \{a, b\}, . >$, $aa = ba = a$, $ab = bb = b$;
$S_5 = < \{a, b\}, . >$, $ba = bb = b$, $ab = aa = a$;

S_1 and S_2 are not a mode, and therefore they are not differential groupoids.
S_3 is a mode groupoid but not differential since it does not satisfy the reductive law.

S_4 (a semigroup right zero) is a mode groupoid but not differential since it does not satisfy reductive law.
S_5 (a semigroup left zero) is a differential groupoid since it satisfies the idempotent law

$$a \cdot a = a \cdot z \text{ and } b \cdot b = b$$

entropic law since

$$(x \cdot y) \cdot (z \cdot w) \;=\; x \cdot z \;=\; (x \cdot z) \cdot y \;=\; (x \cdot z) \cdot (y \cdot w)$$

and also it satisfies the reductive law $x.(y.z) \;=\; x.y$.

The other eight groupoids are non-associative and if we take the non-isomorphic groupoids, we get the following :

A_1 = $<\{a,b\},.>$, $aa = ab = bb = a$, $ba = b$;
A_2 = $<\{a,b\},.>$, $aa = ba = bb = a$, $ab = b$;
A_3 = $<\{a,b\},.>$, $ab = ba = bb = a$, $aa = b$;
A_4 = $<\{a,b\},.>$, $aa = ba = b$, $ab = bb = a$;
A_5 = $<\{a,b\},.>$, $aa = ab = b$, $ba = bb = a$;

Let $\mathcal{V}_i$, i = 1 ,. . ., 5 be varieties of groupoids, generated by A_i, i = 1 ,. . . , 5.
A variety is said to be minimal (atom) if it is nontrivial and has no proper nontrivial subvarieties. Atoms (Minimum) varieties generated by two elements non-associative groupoids were introduced and investigated in details by N.Rabie [6].

In the following section, we will answer the following question:
Which of the varieties generated by two elements non-associative groupoids are differential or contain differential varieties ?

2 Varieties of differential groupoids

Theorem 1. Neither of the varieties $\mathcal{V}_1,\ \mathcal{V}_2,\ and\ \mathcal{V}_3$ is a variety of differential groupoids.

Proof. The groupoid $\mathbf{A}_1$ is not differential groupoid since it does not satisfy the idempotent identity, where

$$b \neq bb = a$$

also it is not entropic groupoid since

$$(b \cdot b)(a \cdot b) = a \neq (b \cdot a) \cdot (b \cdot b) = b$$

therefore $\mathcal{V}_1$ is not differential variety.

In the same way the groupoid $\mathbf{A}_2$ is not differential groupoid, since it does not satisfy the idempotent identity where

$$b \neq bb = a$$

and also it is not entropic groupoid since

$$(b \cdot b)(a \cdot b) = b \neq (b \cdot a) \cdot (b \cdot b) = a$$

therefore $\mathcal{V}_2$ is not differential variety.

Finally, the groupoid $\mathbf{A}_3$ is not differential groupoid, since it does not satisfy the idempotent identity where

$$b \neq bb = a$$

and also it is not entropic groupoid since

$$(a \cdot b)(a \cdot b) = b \neq (a \cdot a) \cdot (b \cdot b) = a$$

therefore $\mathcal{V}_3$ is not differential variety.

By Theorem in [6] Varieties $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$ are atoms (minimum), so they do not contain varieties generated by a differential groupoids. □

Theorem 2. $\mathcal{V}_4$ is not a variety of differential groupoid, but it contains a variety generated by mode groupoid.

Proof. The groupoid $\mathbf{A}_4$ is not differential groupoid, since it does not satisfy the idempotent identity where

$$b \neq b \cdot b = a$$

and also it is not entropic groupoid since

$$(a \cdot b)(a \cdot b) = a \neq (a \cdot a) \cdot (b \cdot b) = b$$

therefore $\mathcal{V}_4$ is not differential variety.

By Theorem in [6] the variety $\mathcal{V}_4$ contains a variety of semigroup right zero which is associative(characterized by identity $x \cdot y = y$), the semigroup right zero is mode groupoid since it satisfy the idempotent identity, and entropic identity, but is not differential since it does not satisfy the reductive identity(i.e., $x(yz) \approx xy$) where

$$a(ba) = a \neq ab = b$$

so, the variety of semigroup right zero is not a variety of differential groupoids, but it is a variety generated by mode groupoids then $\mathcal{V}_4$ contains a variety generated by mode groupoid. □

Theorem 3. $\mathcal{V}_5$ is not a variety of differential groupoids, but it contains a variety of differential groupoids.

Proof. The groupoid $\mathbf{A}_5$ is not differential groupoid since it is not idempotent i.e. does not satisfy the identity $x.x \approx x$, where

$$b.b = a \text{ and } a.a = b$$

but by Theorem in [6] the variety $\mathcal{V}_5$ generated by $\mathbf{A}_5$ contains a variety generated by semigroup left zero which is associative, the groupoid left zero semigroup is differential groupoid, so the variety of semigroup left zero is a variety of differential groupoids characterized by the identity $x.y \approx x$, and hence $\mathcal{V}_5$ contains a variety of differential groupoids. □

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