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THREE-DIMENSIONAL GENERALIZED DIFFERENTIAL TRANSFORM METHOD FOR SPACE-TIME FRACTIONAL DIFFUSION EQUATION IN TWO SPACE VARIABLES WITH VARIABLE COEFFICIENTS

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Abstract In recent years the generalized differential transform method is introduced and applied to solve various fractional differential equations involving two variables. In these works, we observe that a basic result giving the law of exponents for Caputo fractional derivatives has been used, which does not hold true. In the present paper, we provide corrected form of the said result as Theorem 3.1. We, next develop three-dimensional generalized differential transform method and apply this method to solve some space-fractional, time-fractional and space-time fractional diffusion equations in two space variables with variable coefficients.

1 Introduction

Fractional calculus is now considered as a practical technique in many branches of science, particularly physics [11, 22]. A growing number of works in science and engineering deal with dynamical systems described by fractional order equations that involve derivatives and integrals of non-integer order [4, 16, 27]. These new models are more adequate than the previously used integer order models, because fractional order derivatives and integrals describe the memory and hereditary properties of different substances [22]. This is the most significant advantage of the fractional order models in comparison with integer order models, in which such effects are neglected. In the context of flow in porous media, fractional space derivatives exhibit large motions through highly conductive layers or fractures, while fractional time derivatives describe particles that remain motionless for extended period of time [12].

Recent applications of fractional differential equations to a number of systems have given opportunity for physicists to study even more complicated systems. For example, the fractional diffusion equation allows describing complex systems with anomalous behavior in much the same way as simpler systems.

Various fractional partial differential equations with constant coefficients have been studied and the solutions are obtained by Laplace-Fourier transform methods [2, 9, 12, 13], Adomian decomposition method [24], matrix method [10, 23]. However, many practical problems require model with variable coefficients [3, 5, 8, 15].

In the present paper we consider the following type of space-time fractional diffusion equation in two space variables with variable coefficients

$$ {}_0D_t^{\alpha}u\left(t,x,y\right)=d\left(t,x,y\right){}_0D_x^{\beta}u\left(t,x,y\right)+e\left(t,x,y\right){}_0D_y^{\gamma}u\left(t,x,y\right)+q\left(t,x,y\right), \qquad (1.1) $$

on a finite rectangular domain $x_L < x < x_H$,$y_L < y < y_H$ and $t > 0$ with $0 < \alpha \leq 1$ and $1 < \beta \leq 2, 1 < \gamma \leq 2$ where the diffusion coefficients $d\left(t,x,y\right) > 0$ and $e\left(t,x,y\right) > 0$, the forcing function $q\left(t,x,y\right)$ can be used to represent the sources and sinks, ${}_0D_x^{\alpha}$,${}_0D_t^{\beta}$ are Caputo fractional derivatives defined by equation (2.1). It is assumed that the fractional diffusion equations under consideration have unique and sufficiently smooth solution under suitable initial conditions.

The fractional diffusion equation (1.1) has been considered earlier by many researchers, for example, Ray [25] presented analytical solutions of the space fractional diffusion equations by two-step Adomian Decomposition Method, Abrashina-Zhadaeva and Romanova [1] discussed unconditional stability of numerical vector decomposition model for the fractional diffusion equation, Zhuang and Liu [29] examined an implicit difference approximation for space-time fractional diffusion equation and discussed stability and convergence of the method, Tadjeran and Meerschaert [26] presented a numerical method combining the alternating directions implicit approach with a Crank–Nicolson discretization and a Richardson extrapolation to obtain an unconditionally stable second-order accurate finite difference method to solve a fractional super diffusive differential equation, Meerschaert *et al.* [14] discussed a practical alternating directions implicit method to solve a class of two-dimensional initial-boundary value fractional partial differential equations with variable coefficients on a finite domain.

The method which we shall develop to solve equation (1.1) is based on the generalized differential transform method [18, 19, 20], which is a generalization of the differential transform method introduced by Zhou [28] who applied this method to solve linear and nonlinear initial value problems in electric circuit analysis. This method constructs an analytical solution in the form of a series. It is different from the traditional higher order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method computationally takes long time for large orders. The differential transform is an iterative procedure for obtaining analytic Taylor series solution of ordinary or partial differential equations.

The paper is organized as follows. In Section 2, we provide definitions which shall be used in the subsequent sections. In Section 3, we establish the law of exponents for Caputo fractional derivative. In Section 4, we develop three-dimensional generalized differential transform method to solve fractional partial differential equations with three variables. In Section 5, we give three examples showing application of the method developed in Section 4, to solve space-fractional, time-fractional and space-time fractional diffusion equations in two space variables with variable coefficients.

2 Definitions

Caputo fractional derivative of order α, is defined as [7]:

$$\begin{aligned} {}_aD_x^\alpha f(x) &= \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(\xi)}{(x-\xi)^{\alpha-m+1}} d\xi, (m-1<\alpha\le m), m\in\mathbb{N} \\ &= {}_aI_x^{m-\alpha} D^m f(x), \end{aligned} \tag{2.1}$$

where $D^m = \frac{d^m}{dx^m}$, ${}_aI_x^\alpha$ stands for the **Riemann-Liouville fractional integral operator** of order $\alpha > 0$given by [17]

$${}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt. \tag{2.2}$$

Clearly

$${}_aI_x^\alpha (x-a)^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} (x-a)^{\mu+\alpha}, \mu > -1. \tag{2.3}$$

Caputo generalized Taylor's Formula [21] is as given below:
Suppose that $({}_aD_x^\alpha)^k f(x) \in C[a,b]$ for $k = 0, 1, ..., n+1$, where $0 < \alpha \le 1$, then we have

$$f(x) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\gamma(i\alpha+1)} \left(({}_aD_x^\alpha)^i f \right)(a+) + \frac{\left(({}_aD_x^\alpha)^{n+1} f \right)(\xi)}{\gamma((n+1)\alpha+1)} . (x-a)^{(n+1)\alpha}, \tag{2.4}$$

with $a \le \xi \le x, \forall x \in (a,b]$. The radius of convergence ρ, for the generalized Taylor's series

$$\sum_{i=0}^{\infty} \frac{(x-a)^{i\alpha}}{\gamma(i\alpha+1)} \left(({}_aD_x^\alpha)^i f \right)(a+), |x-a| < \rho,$$

depends on $f(x)$ and a, and is given by:

$$\rho = \lim_{n\to\infty} \left| \frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)} . \frac{\left(({}_aD_x^\alpha)^{n+1} f \right)(a+)}{(({}_aD_x^\alpha)^n f)(a+)} \right|^{1/\rho} . \tag{2.5}$$

In case $\alpha = 1$, the Caputo generalized Taylor's formula (2.4) reduces to the classical Taylor's formula.

Two-dimensional Generalized differential transform [18, 19, 20] is as given below:

Consider a function of two variables $u\,(x,y)$ and suppose that it is analytic and differentiated continuously then its two-dimensional generalized differential transform is given by

$$U_{\alpha,\beta}\,(k,h)=\frac{1}{\Gamma\,(\alpha k+1)\,\Gamma\,(\beta h+1)}\left[\left({}_aD_x^{\alpha}\right)^k\left({}_bD_y^{\beta}\right)^h u\,(x,y)\right]_{(a,b)}, \tag{2.6}$$

where $0<\alpha,\beta\leq 1, \left({}_aD_x^{\alpha}\right)^k\equiv {}_aD_x^{\alpha}\cdot{}_aD_x^{\alpha}\cdots{}_aD_x^{\alpha}(k \text{ times})$ and $U_{\alpha,\beta}\,(k,h)$ is the transformed function.

The inverse generalized differential transform of $U_{\alpha,\beta}\,(k,h)$ is given by:

$$u\,(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{\alpha,\beta}\,(k,h)\,(x-a)^{k\alpha}\,(y-b)^{h\beta}\,. \tag{2.7}$$

Three-dimensional differential transform [6] is as given below:

Consider a function of three variables $u\,(x,y,z)$ which is analytic and differentiated continuously then its three-dimensional differential transform is given by

$$U\,(k,h,m)=\frac{1}{k!h!m!}\left[\frac{\partial^{k+h+m}u\,(x,y,z)}{\partial x^k\partial y^h\partial z^m}\right]_{(a,b,c)}, \tag{2.8}$$

where $U\,(k,h,m)$ is the transformed function.

The inverse differential transform of $U\,(k,h,m)$ is given by

$$u\,(x,y,z)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}U_{\alpha,\beta,\gamma}\,(k,h,m)\,(x-a)^k\,(y-b)^h\,(z-c)^m\,. \tag{2.9}$$

3 Law of exponents for Caputo fractional derivative

Here, we obtain a theorem which gives the precise conditions under which the law of exponents holds for Caputo fractional derivatives. Similar result for law of exponents of Riemann-Liouville fractional derivatives is given in the book by Miller and Ross [17, p 105].

Theorem 3.1. *Suppose that $f\,(x)=(x-a)^{\lambda}\,g\,(x)$, where $a,\lambda>0$ and $g\,(x)$ has the generalized power series expansion $g\,(x)=\sum_{n=0}^{\infty}a_n\,(x-a)^{n\alpha}$ with radius of convergence $R>0, 0<\alpha\leq 1$. Then*

$${}_aD_x^{\gamma}\,{}_aD_x^{\beta}f\,(x)={}_aD_x^{\gamma+\beta}f\,(x)\,, \tag{3.1}$$

for all $(x-a)\in(0,R)$, the coefficients $a_n=0$ for n given by $n\alpha+\lambda-\beta=0$ and either
(a) $\lambda>\mu, \mu=\max\left(\beta+[\gamma],[\beta+\gamma]\right)$
or
(b) $\lambda\leq\mu, a_k=0$, for $k=0,1,...,\left[\frac{\mu-\lambda}{\alpha}\right]$,
here $[x]$ denotes the greatest integer less than or equal to x .

Proof. For part (a), we have by definition (2.1),

$${}_aD_x^{\beta}f\,(x)={}_aI_x^{[\beta]+1-\beta}\,{}_aD_x^{[\beta]+1}\sum_{n=0}^{\infty}a_n\,(x-a)^{n\alpha+\lambda}\;.$$

Performing term by term differentiation which is justified since the series involving derivatives upto the order $[\beta]+1$ of the term $(x-a)^{n\alpha+\lambda}$ are uniformly convergent for $(x-a)\in(0,R)$, we obtain

$${}_aD_x^{\beta}f\,(x)={}_aI_x^{[\beta]+1-\beta}\sum_{n=0}^{\infty}a_n\frac{\Gamma\,(n\alpha+\lambda+1)}{\Gamma\,(n\alpha+\lambda-[\beta])}\,(x-a)^{n\alpha+\lambda-[\beta]-1}\,.$$

Using definition (2.2) and the fact that the infinite series is uniformly convergent for $(x-a)\in(0,R)$, we change the order of integration and summation and arrive at the following with the help of known result (2.3)
${}_aD_x^{\beta}f\,(x)=\sum_{n=0}^{\infty}a_n\frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta+1)}\,(x-a)^{n\alpha+\lambda-\beta}$, for $\lambda>[\beta]$.

Applying the same argument as above, we now have

$$\begin{aligned}
{}_aD_x^{\gamma}{}_aD_x^{\beta}f\left(x\right) &= {}_aD_x^{\gamma}\sum_{n=0}^{\infty}a_n\tfrac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta+1)}\left(x-a\right)^{n\alpha+\lambda-\beta}\\
&= {}_aI_x^{[\gamma]+1-\gamma}\;{}_aD_x^{[\gamma]+1}\sum_{n=0}^{\infty}a_n\tfrac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta+1)}\left(x-a\right)^{n\alpha+\lambda-\beta}\\
&= {}_aI_x^{[\gamma]+1-\gamma}\sum_{n=0}^{\infty}a_n\tfrac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta-[\gamma])}\left(x-a\right)^{n\alpha+\lambda-\beta-[\gamma]-1}\\
&= \sum_{n=0}^{\infty}a_n\tfrac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta-\gamma+1)}\left(x-a\right)^{n\alpha+\lambda-\beta-\gamma},\ \text{for } \lambda-\beta-[\gamma]>0.
\end{aligned}\tag{3.2}$$

Next

$$\begin{aligned}
{}_aD_x^{\gamma+\beta}f\left(x\right) &= {}_aD_x^{\gamma+\beta}\sum_{n=0}^{\infty}a_n\left(x-a\right)^{n\alpha+\lambda}\\
&= {}_aI_x^{[\gamma+\beta]+1-\gamma-\beta}\;{}_aD_x^{[\gamma+\beta]+1}\sum_{n=0}^{\infty}a_n\left(x-a\right)^{n\alpha+\lambda}\\
&= {}_aI_x^{[\gamma+\beta]+1-\gamma-\beta}\sum_{n=0}^{\infty}a_n\tfrac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-[\gamma+\beta])}\left(x-a\right)^{n\alpha+\lambda-[\gamma+\beta]-1}\\
&= \sum_{n=0}^{\infty}a_n\tfrac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\gamma-\beta+1)}\left(x-a\right)^{n\alpha+\lambda-\gamma-\beta},\ \text{for } \lambda-[\gamma+\beta]>0.
\end{aligned}\tag{3.3}$$

Which is precisely ${}_aD_x^{\gamma}{}_aD_x^{\beta}f\left(x\right)$, as given by (3.2).

The conditions mentioned with (3), (3.2) and (3.3) can be combined and written as condition given in part (a).

For part (b), i.e. when $\lambda\le\mu$, we take $a_k=0$ for $k=0,1,...,l-1$, where $l-1=\left[\frac{\mu-\lambda}{\alpha}\right]$ we have due to the uniform convergence of derived series upto the order $[\beta]+1$,

$$\begin{aligned}
{}_aD_x^{\beta}f\left(x\right) &= {}_aD_x^{\beta}\sum_{n=0}^{\infty}a_n\left(x-a\right)^{n\alpha+\lambda}\\
&= \sum_{n=l}^{\infty}a_n\tfrac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta+1)}\left(x-a\right)^{n\alpha+\lambda-\beta}\\
&= \sum_{r=0}^{\infty}a_{r+l}\tfrac{\Gamma((r+l)\alpha+\lambda+1)}{\Gamma((r+l)\alpha+\lambda-\beta+1)}\left(x-a\right)^{(r+l)\alpha+\lambda-\beta}.
\end{aligned}\tag{3.4}$$

If we let $\lambda'=l\alpha+\lambda$, then (3.4) becomes same as (3) (with λ replaced by λ') and the proof proceeds as in part (a). □

The following result for the law of exponents for Caputo fractional derivatives has been obtained earlier by Odibat, Momani and Erturk in their papers [18, 19, 20].

Suppose that $f\left(x\right)=x^{\lambda}g\left(x\right)$, where $\lambda>-1$ and $g\left(x\right)$ has the generalized power series expansion $g\left(x\right)=\sum_{n=0}^{\infty}a_n\left(x-a\right)^{n\alpha}$ with radius of convergence $R>0, 0<\alpha\le1$.Then

$${}_aD_x^{\gamma}{}_aD_x^{\beta}f\left(x\right)={}_aD_x^{\gamma+\beta}f\left(x\right),$$

for all $x\in(0,R)$ if:
$\beta<\lambda+1$ and α arbitrary
or
$\beta\ge\lambda+1,\gamma$ arbitrary and $a_k=0$ for $k=0,1,...,m-1$ where $m-1<\beta\le m$.

But the above result is erroneous as evident from the following example:

For $0<\alpha\le1$, we have

$${}_0D_x^{2\alpha}x^{\alpha}={}_0I_x^{[2\alpha]+1-2\alpha}{}_0D_x^{[2\alpha]+1}x^{\alpha}={}_0I_x^{[2\alpha]+1-2\alpha}\frac{\Gamma\left(\alpha+1\right)}{\Gamma\left(\alpha-[2\alpha]\right)}x^{\alpha-[2\alpha]-1}=\frac{\Gamma\left(\alpha+1\right)}{\Gamma\left(1-\alpha\right)}x^{-\alpha}\tag{3.5}$$

and
${}_0D_x^{\alpha}{}_0D_x^{\alpha}x^{\alpha}={}_0D_x^{\alpha}\left(\Gamma\left(\alpha+1\right)\right)=0$. (Since Caputo fractional derivative of constant is zero.)

Clearly

$${}_0D_x^{2\alpha}x^{\alpha}\ne{}_0D_x^{\alpha}{}_0D_x^{\alpha}x^{\alpha}.\tag{3.6}$$

But according to the above result given by Odibat, Momani and Erturk [18, 19, 20], the condition given in (a) is satisfied so we must have ${}_0D_x^{2\alpha}x^{\alpha}={}_0D_x^{\alpha}{}_0D_x^{\alpha}x^{\alpha}$.

It may be observed that in accordance with our Theorem 3.1, here $\lambda=\beta=\gamma=\alpha, a_0=1, a_1,a_2,a_3,....=0$, $n\alpha+\lambda-\beta=0$ for $n=0$ showing that the conditions of Theorem 3.1 are not satisfied therefore ${}_0D_x^{2\alpha}x^{\alpha}\ne{}_0D_x^{\alpha}{}_0D_x^{\alpha}x^{\alpha}$ is justified.

The Theorem 3.1 given above may be treated as corrected form of the above result.

4 Three-dimensional generalized differential transform method

In this section we shall develop three-dimensional generalized differential transform method for fractional partial differential equations in three variables. Consider a function of three variables $u\left(x,y,z\right)$ then we define three-dimensional generalized differential transform of function

$u(x,y,z)$ as

$$U_{\alpha,\beta,\gamma}(k,h,m)=\frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)\Gamma(\gamma m+1)}\left[\left({}_aD_x^{\alpha}\right)^k\left({}_bD_y^{\beta}\right)^h\left({}_cD_z^{\gamma}\right)^m u(x,y,z)\right]_{(a,b,c)} \tag{4.1}$$

where $0<\alpha,\beta,\gamma\le 1$, $\left({}_aD_x^{\alpha}\right)^k={}_aD_x^{\alpha}\cdot{}_aD_x^{\alpha}\cdots{}_aD_x^{\alpha}$ (k times), $U_{\alpha,\beta,\gamma}(k,h,m)$ stands for the transformed function. The inverse differential transform of $U_{\alpha,\beta,\gamma}(k,h,m)$ is given by:

$$u(x,y,z)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}U_{\alpha,\beta,\gamma}(k,h,m)(x-a)^{k\alpha}(y-b)^{h\beta}(z-c)^{m\gamma}. \tag{4.2}$$

In case $\alpha=\beta=\gamma=1$, the three-dimensional generalized differential transform (4.1) and its inverse (4.2) reduces to the three-dimensional differential transform and its inverse given by (2.8) and (2.9) respectively.

Some fundamental properties for three-dimensional generalized differential transform are given below.

Theorem 4.1. *Let $U_{\alpha,\beta,\gamma}(k,h,m), V_{\alpha,\beta,\gamma}(k,h,m)$ and $W_{\alpha,\beta,\gamma}(k,h,m)$ be three-dimensional generalized differential transforms of the functions $u(x,y,z), v(x,y,z)$ and $w(x,y,z)$ respectively, then we can easily arrive at the following results on the lines of properties of three-dimensional differential transform [6] and generalized differential transform method [18]*

(i) If $u(x,y,z)=v(x,y,z)\pm w(x,y,z)$, then $U_{\alpha,\beta,\gamma}(k,h,m)=V_{\alpha,\beta,\gamma}(k,h,m)\pm W_{\alpha,\beta,\gamma}(k,h,m)$.

(ii) If $u(x,y,z)=av(x,y,z)$, then $U_{\alpha,\beta,\gamma}(k,h,m)=aV_{\alpha,\beta,\gamma}(k,h,m)$.

(iii) If $u(x,y,z)=v(x,y,z)\,w(x,y,z)$, then $U_{\alpha,\beta,\gamma}(k,h,m)=\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}V_{\alpha,\beta,\gamma}(r,h-s,m-p)\,W_{\alpha,\beta,\gamma}(k-r,s,p)$.

(iv) If $u(x,y,z)={}_aD_x^{\alpha}v(x,y,z)$ and $0<\alpha\le 1$, then $U_{\alpha,\beta,\gamma}(k,h,m)=\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}V_{\alpha,\beta,\gamma}(k+1,h,m)$.

(v) If $u(x,y,z)={}_aD_x^{\alpha}{}_bD_y^{\beta}v(x,y,z)$ where $0<\alpha,\beta\le 1$, then $U_{\alpha,\beta,\gamma}(k,h,m)=\frac{\Gamma(\alpha(k+1)+1)\Gamma(\beta(h+1)+1)}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}V_{\alpha,\beta,\gamma}(k+1,h+1,m)$.

(vi) If $u(x,y,z)={}_aD_x^{\alpha}{}_bD_y^{\beta}{}_cD_z^{\gamma}v(x,y,z)$ where $0<\alpha,\beta,\gamma\le 1$, then $U_{\alpha,\beta,\gamma}(k,h,m)=\frac{\Gamma(\alpha(k+1)+1)\Gamma(\beta(h+1)+1)\Gamma(\gamma(m+1)+1)}{\Gamma(\alpha k+1)\Gamma(\beta h+1)\Gamma(\gamma m+1)}V_{\alpha,\beta,\gamma}(k+1,h+1,m+1)$.

(vii) If $u(x,y,z)=(x-a)^{n_1\alpha}(y-b)^{n_2\beta}(z-c)^{n_3\gamma}, n_1,n_2,n_3\in\mathbb{N}$, then $U_{\alpha,\beta,\gamma}(k,h,m)=\delta(k-n_1)\,\delta(h-n_2)\,\delta(m-n_3)$

where δ is defined as

$$\delta(k)=\begin{cases}1, & \text{when } k=0\\ 0, & \text{otherwise}\end{cases}.$$

5 Examples

In this section, we solve space-fractional, time-fractional and space-time fractional diffusion equations in two space variables with variable coefficients using three-dimensional generalized differential transform method developed in Section 4.

Example 5.1. Consider the space fractional diffusion equation

$$\frac{\partial u(t,x,y)}{\partial t}=d(t,x,y)\left({}_0D_x^{1/5}\right)^9u(t,x,y)+e(t,x,y)\left({}_0D_y^{1/5}\right)^8u(t,x,y)+q(t,x,y) \tag{5.1}$$

on a finite rectangular domain $0<x<1, 0<y<1$, for $0\le t\le T_{end}$ with the diffusion coefficients $d(t,x,y)=\Gamma(2.2)\,x^{2.8}y/6$, $e(t,x,y)=2xy^{2.6}/\Gamma(4.6)$ and the forcing function $q(t,x,y)=-(1+2xy)\,e^{-t}x^3y^{3.6}$,
with the initial condition

$$u(0,x,y)=x^3y^{3.6}. \tag{5.2}$$

We apply the three-dimensional generalized differential transform (4.1) with $a = 0 = b = c$and $\alpha = 1, \beta = \frac{1}{5}, \gamma = \frac{1}{5}$ to both sides of equation (5.1) and making use of Theorem 2(a), (c), (d), obtain the transformed equation as

$$\begin{aligned}
&U_{1,\frac{1}{5},\frac{1}{5}}(k+1,h,m) \\
&= \tfrac{1}{k+1}\left[\tfrac{\Gamma(2.2)}{6}\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}\delta(k-r)\,\delta(s-14)\,\delta(p-5)\,\tfrac{\Gamma\left(\frac{(9+h-s)}{5}+1\right)}{\Gamma\left(\frac{h-s}{5}+1\right)}U_{1,\frac{1}{5},\frac{1}{5}}(r,9+h-s,m-p)\right. \\
&+\tfrac{2}{\Gamma(4.6)}\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}\delta(k-r)\,\delta(s-5)\,\delta(p-13)\,\tfrac{\Gamma\left(\frac{(8+m-p)}{5}+1\right)}{\Gamma\left(\frac{m-p}{5}+1\right)}U_{1,\frac{1}{5},\frac{1}{5}}(r,h-s,8+m-p) \\
&\left.-\tfrac{(-1)^k}{k!}\delta(h-15)\,\delta(m-18)-2\tfrac{(-1)^k}{k!}\delta(h-20)\,\delta(m-23)\right].
\end{aligned} \tag{5.3}$$

The three-dimensional generalized differential transform of initial condition (5.2) is given by

$$U_{1,\frac{1}{5},\frac{1}{5}}(0,h,m) = \begin{cases} 1 & \text{when } h=15, m=18 \\ 0 & \text{otherwise} \end{cases}. \tag{5.4}$$

Utilizing the recurrence relation (5.3) and the transformed initial condition (5.4), we obtain

$$U_{1,\frac{1}{5},\frac{1}{5}}(k,h,m) = \begin{cases} \frac{(-1)^k}{k!} & \text{when } h=15, m=18 \\ 0 & \text{otherwise} \end{cases}. \tag{5.5}$$

From the inverse transform given by equation (4.2) , we have

$$u(t,x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}U_{1,\frac{1}{5},\frac{1}{5}}(k,h,m)\,t^k x^{h/5} y^{m/5}. \tag{5.6}$$

Using the values of $U_{1,\frac{1}{5},\frac{1}{5}}(k,h,m)$ from equation (5.5) in equation (5.6), the exact solution of space fractional diffusion equation (5.1) is obtained as

$$u(t,x,y) = x^3 y^{3.6} e^{-t}. \tag{5.7}$$

This may be verified by direct substitution in the equation (5.8).

Further in view of Theorem 3.1, we find that $\left({}_0D_x^{1/5}\right)^9 u(t,x,y) = {}_0D_x^{9/5}u(t,x,y)$ and $\left({}_0D_y^{1/5}\right)^8 u(t,x,y) = {}_0D_y^{8/5}u(t,x,y)$. Thus the space fractional diffusion equation (5.1) can be written as

$$\frac{\partial u(t,x,y)}{\partial t} = d(t,x,y)\,{}_0D_x^{9/5}u(t,x,y) + e(t,x,y)\,{}_0D_y^{8/5}u(t,x,y) + q(t,x,y). \tag{5.8}$$

The above problem has been solved earlier by Ray [25] using two step Adomian decomposition method.

Example 5.2. Consider the time fractional diffusion equation [29]

$${}_0D_t^{0.4}u(t,x,y) = \frac{2t^{1.6}}{\pi^2\Gamma(0.6)}\frac{\partial^2 u(t,x,y)}{\partial x^2} + \frac{12t^{1.6}}{\pi^2\Gamma(0.6)}\frac{\partial^2 u(t,x,y)}{\partial y^2} + \frac{25t^{1.6}}{12\Gamma(0.6)}\left(t^2+2\right)\sin\pi x\sin\pi y \tag{5.9}$$

on a finite rectangular domain $0 < x < 1, 0 < y < 1$, for $0 \le t \le T_{end}$ with the initial condition

$$u(0,x,y) = \sin\pi x\sin\pi y. \tag{5.10}$$

The above problem has been examined numerically by Zhuang and Liu [29].

We apply the three-dimensional generalized differential transform (4.1) with $a = 0 = b = c$and $\alpha = \frac{2}{5}, \beta = 1, \gamma = 1$ to both sides of equation (5.9) and making use of Theorem 2(a), (c), (d) , obtain the transformed equation as

$$\begin{aligned}
&U_{\frac{2}{5},1,1}(k+1,h,m) \\
&= \tfrac{\Gamma\left(\frac{2}{5}k+1\right)}{\Gamma\left(\frac{2}{5}(k+1)+1\right)}\left[\tfrac{2}{\pi^2\Gamma(0.6)}\sum_{r=0}^{k}\delta(k-r-4)\,(h+2)\,(h+1)\,U_{\frac{2}{5},1,1}(r,h+2,m)\right. \\
&+\tfrac{1}{12\pi^2\Gamma(0.6)}\sum_{r=0}^{k}\delta(k-r-4)\,(m+2)\,(m+1)\,U_{\frac{2}{5},1,1}(r,h,m+2) \\
&\left.+\left(\tfrac{(-1)^{hm+1}+1}{2}\right)\tfrac{25}{12\Gamma(0.6)}\tfrac{\pi^{h+m}}{h!m!}(-1)^{\frac{h+m}{2}-1}\left\{\delta(k-9)+2\delta(k-4)\right\}\right].
\end{aligned} \tag{5.11}$$

The three-dimensional generalized differential transform of initial condition (5.10) is given by

$$U_{\frac{2}{5},1,1}\left(0,h,m\right)=\begin{cases}\frac{\pi^{h+m}}{h!m!}\left(-1\right)^{\frac{h+m}{2}-1} & \text{when } h,m \text{ are odd}\\ 0 & \text{otherwise}\end{cases}. \tag{5.12}$$

Utilizing the recurrence relation (5.11) and the transformed initial condition (5.12), we obtain

$$U_{\frac{2}{5},1,1}\left(k,h,m\right)=\begin{cases}\frac{\pi^{h+m}}{h!m!}\left(-1\right)^{\frac{h+m}{2}-1} & \text{when } h,m \text{ are odd and } k=0 \text{ or } 5\\ 0 & \text{otherwise}\end{cases}. \tag{5.13}$$

From the inverse transform given by (4.2) , we have

$$u\left(t,x,y\right)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}U_{\frac{2}{5},1,1}\left(k,h,m\right)t^{2k/5}x^{h}y^{m}. \tag{5.14}$$

Using the values of $U_{\frac{2}{5},1,1}\left(k,h,m\right)$from (5.13) in (5.14), the exact solution of time-fractional diffusion equation (5.9) is obtained as

$$u\left(t,x,y\right)=\left(1+t^{2}\right)\sin\pi x\sin\pi y. \tag{5.15}$$

This may be verified by direct substitution in the equation (5.9).

Example 5.3. Consider the space-time fractional diffusion equation

$$\begin{aligned}{}_0D_t^{1/2}u\left(t,x,y\right)=\tfrac{\Gamma(2.8)}{\Gamma(4)}x^{1.2}\left({}_0D_x^{1/5}\right)^{6}u\left(t,x,y\right)+\tfrac{\Gamma(2.2)}{\Gamma(4)}y^{1.8}\left({}_0D_y^{1/5}\right)^{9}u\left(t,x,y\right)\\ +x^{3}y^{3}\left(\tfrac{8}{3\Gamma(0.5)}t^{1.5}-2t^{2}-2\right)\end{aligned} \tag{5.16}$$

on a finite rectangular domain $0<x<1, 0<y<1$, for $0\leq t\leq T_{end}$ with the initial condition

$$u\left(0,x,y\right)=x^{3}y^{3}. \tag{5.17}$$

We apply the three-dimensional generalized differential transform (4.1) with $a=0=b=c$and $\alpha=\frac{1}{2},\beta=\frac{1}{5},\gamma=\frac{1}{5}$ to both sides of equation (5.16) and making use of Theorem 4.1(a), (c), (d), obtain the transformed equation as

$$\begin{aligned}&U_{\frac{1}{2},\frac{1}{5},\frac{1}{5}}\left(k+1,h,m\right)\\&=\tfrac{\Gamma\left(\frac{k}{2}+1\right)}{\Gamma\left(\frac{k+1}{2}+1\right)}\left[\tfrac{\Gamma(2.8)}{\Gamma(4)}\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}\delta\left(k-r\right)\delta\left(s-6\right)\delta\left(p\right)\tfrac{\Gamma\left(\frac{6+h-s}{5}+1\right)}{\Gamma\left(\frac{h-s}{5}+1\right)}U_{\frac{1}{2},\frac{1}{5},\frac{1}{5}}\left(r,6+h-s,m-p\right)\right.\\&\quad+\tfrac{\Gamma(2.2)}{\Gamma(4)}\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}\delta\left(k-r\right)\delta\left(p-9\right)\delta\left(s\right)\tfrac{\Gamma\left(\frac{9+m-p}{5}+1\right)}{\Gamma\left(\frac{m-p}{5}+1\right)}U_{\frac{1}{2},\frac{1}{5},\frac{1}{5}}\left(r,h-s,9+m-p\right)\\&\quad\left.+\delta\left(h-15\right)\delta\left(m-15\right)\left\{\tfrac{8}{3\Gamma(1/2)}\delta\left(k-3\right)-2\delta\left(k-4\right)-2\delta\left(k\right)\right\}\right].\end{aligned} \tag{5.18}$$

The three-dimensional generalized differential transform of initial condition (5.17) is given by

$$U_{\frac{1}{2},\frac{1}{5},\frac{1}{5}}\left(0,h,m\right)=\begin{cases}1 & \text{when } h=15, m=15\\ 0 & \text{otherwise}\end{cases}. \tag{5.19}$$

Utilizing the recurrence relation (5.18) and the transformed initial condition (5.19), we obtain

$$U_{\frac{1}{2},\frac{1}{5},\frac{1}{5}}\left(k,h,m\right)=\begin{cases}1 & \text{when } h=m=15 \text{ and } k=0 \text{ or } 4\\ 0 & \text{otherwise}\end{cases}. \tag{5.20}$$

From the inverse transform given by equation (4.2), we have

$$u\left(t,x,y\right)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}U_{\frac{1}{2},\frac{1}{5},\frac{1}{5}}\left(k,h,m\right)t^{k/2}x^{h/5}y^{m/5}. \tag{5.21}$$

Using the values of $U_{\frac{1}{2},\frac{1}{5},\frac{1}{5}}\left(k,h,m\right)$ from (5.20) in (5.21), the exact solution of space-time fractional diffusion equation (5.16) is obtained as

$$u\left(t,x,y\right)=\left(t^{2}+1\right)x^{3}y^{3}. \tag{5.22}$$

This may be verified by direct substitution in the equation (5.16).

Further in view of Theorem 3.1, we find that $\left({}_0D_x^{1/5}\right)^6 u\left(t,x,y\right) = {}_0D_x^{6/5}u\left(t,x,y\right)$ and $\left({}_0D_y^{1/5}\right)^9 u\left(t,x,y\right) = {}_0D_y^{9/5}u\left(t,x,y\right)$. Thus the space-time fractional diffusion equation (5.16) can be written as

$$\begin{aligned}{}_0D_t^{1/2}u\left(t,x,y\right) = \tfrac{\Gamma(2.8)}{\Gamma(4)}x^{1.2}{}_0D_x^{6/5}u\left(t,x,y\right) + \tfrac{\Gamma(2.2)}{\Gamma(4)}y^{1.8}{}_0D_y^{9/5}u\left(t,x,y\right)\\ +x^3y^3\left(\tfrac{8}{3\Gamma(0.5)}t^{1.5} - 2t^2 - 2\right).\end{aligned} \tag{5.23}$$

The above problem has been examined numerically by Zhuang and Liu [29].

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