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A FAMILY OF MITTAG-LEFFLER TYPE FUNCTIONS AND THEIR PROPERTIES

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Abstract Contemporary research has proved that Mittag-Leffler function is the solution of fractional differential and integral equations. Fractional Calculus is rapidly gaining recognition as an important branch of Mathematical Sciences.

In this paper, we study a newly defined Mittag-Leffler type E-function that unifies many special functions including some newly defined **generalized trigonometric functions**. We derive its validity conditions and integral representation. Finally we obtain various integral transforms of the E-function.

1 Introduction

The Mittag-Leffler (M-L) function [6] introduced in 1903 due to Gösta Mittag-Leffler is a generalization of the exponential function e^z. The first application of this function was noticed in 1930, when Hille and Tamarkin [2] provided a solution of the Abel-Volterra type integral equation of the 2nd kind in terms of the M-L function.

Many generalizations of M-L function were developed and studied by Wiman [12], Kiryakova [5], Saxena-Nishimoto [9], and many other authors, they have proved its importance in many physical phenomena, it motivates the researcher to study a newly defined M-L type E-function [1].

Definition 1.1. Mittag-Leffler Type Functions

- In 1903, Gösta Mittag-Leffler [6], the Swedish mathematician introduced the function E_α(z), defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + 1)}\, z^n \tag{1.1}$$

  where $z, \alpha \in \mathbb{C}; \Re(\alpha) \geq 0 \, and \, |z| < \infty$.

- In 1905, Wiman [12], extended (1.1) in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n + \beta)}\, z^n \tag{1.2}$$

  where $z, \alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0 \, and \, \Re(\beta) > 0$.

- In 2000, Kiryakova [5], has studied “multiindex M-L functions” defined by

$$E_{({1}/{\rho_i}),(\mu_i)}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\mu_1 + {n}/{\rho_1}) \ldots \Gamma(\mu_m + {n}/{\rho_m})}\, z^n \tag{1.3}$$

  where $m > 1$, is an integer, $\rho_1, \ldots, \rho_m > 0$ and $\mu_1, \ldots, \mu_m$ are arbitrary real numbers.

- In 2010, Saxena and Nishimoto [9], studied an extension of M-L type function as

$$E_{\gamma,\kappa}[(\alpha_1,\beta_1),...,(\alpha_m,\beta_m);z]=\sum_{n=0}^{\infty}\frac{(\gamma)_{n\kappa}}{\prod_{j=1}^{m}\Gamma\left(\alpha_j n+\beta_j\right)}\frac{z^n}{n!} \tag{1.4}$$

where $z,\alpha_j,\beta_j,\gamma\in\mathbb{C},\sum_{j=1}^{m}\Re(\alpha_j)>\Re(\kappa)-1, j=1,\ldots,m\, and\, \Re(\kappa)>0.$

- In 2012, Kalla, Haidey and Virchenko [4], introduced multiparameter M-L type function in the following form

$$HE_{\mu_1,\mu_2,\ldots,\mu_r}^{\lambda_1,\lambda_2,\ldots,\lambda_r}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\prod_{i=1}^{r}\Gamma\left(1+\mu_i+\lambda_i n\right)}\left(\frac{z}{\Lambda}\right)^{\Lambda n+M} \tag{1.5}$$

where $\mu_i\in\mathbb{C},\lambda_i>0,i=1,2,\ldots,r;\sum_{i=1}^{r}\mu_i=M\, and \sum_{i=1}^{r}\lambda_i=\Lambda.$

2 Definition, Convergence Conditions and Special Cases of the E-Function

Definition 2.1. M-L type E-function is defined as follows

$$
{}_{\tau}E_k^h\left[z\Big|\begin{array}{c}(\rho,a);(\gamma_i,q_i,s_i)_{1,h}\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k}\end{array}\right]={}_{\tau}E_k^h\left[z\Big|\begin{array}{c}(\rho,a);(\gamma_1,q_1,s_1),...,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),...,(\delta_k,p_k,r_k)\end{array}\right]
$$

$$
=\sum_{n=0}^{\infty}\frac{\left[(\gamma_1)_{q_1 n}\right]^{s_1}\left[(\gamma_2)_{q_2 n}\right]^{s_2}...\left[(\gamma_h)_{q_h n}\right]^{s_h}(-1)^{\rho n}z^{an+\tau}}{\left[(\delta_1)_{p_1 n}\right]^{r_1}\left[(\delta_2)_{p_2 n}\right]^{r_2}...\left[(\delta_k)_{p_k n}\right]^{r_k}\Gamma(\alpha n+\beta)} \tag{2.1}
$$

where

$$
z,\alpha,\beta,\gamma_i,\delta_j\in\mathbb{C};\Re(\alpha)\geq 0,\Re(\beta)>0,\Re(\gamma_i)>0,\Re(\delta_j)>0,\Re(q_i)\geq 0,
$$

$$
\Re(p_j)\geq 0; s_i,r_j,a,\tau\in\mathbb{R};\rho\in\{0,1\},\left(\sum_{i=1}^{h}q_i s_i<\sum_{j=1}^{k}p_j r_j+\Re(\alpha)\right) or
$$

$$
\left(\sum_{i=1}^{h}q_i s_i=\sum_{j=1}^{k}p_j r_j+\Re(\alpha)\, when \prod_{i=1}^{h}(q_i)^{q_i s_i}\left[\alpha^{\alpha}\prod_{j=1}^{k}(p_j)^{p_j r_j}\right]^{-1}|z^a|<1\right)
$$

$$
for\ i=1,2,...,h; j=1,2,...,k. \tag{2.2}
$$

2.1 Domain of Convergence

Equation (2.1) can be denoted as

$$
{}_{\tau}E_k^h\left[z\Big|\begin{array}{c}(\rho,a);(\gamma_i,q_i,s_i)_{1,h}\\(\alpha,\beta);(\delta_j,p_j,r_j)_{1,k}\end{array}\right]=\sum_{n=0}^{\infty}c_n \tag{2.3}
$$

where

$$
c_n=\frac{\left[(\gamma_1)_{q_1 n}\right]^{s_1}\left[(\gamma_2)_{q_2 n}\right]^{s_2}...\left[(\gamma_h)_{q_h n}\right]^{s_h}(-1)^{\rho n}z^{an+\tau}}{\left[(\delta_1)_{p_1 n}\right]^{r_1}\left[(\delta_2)_{p_2 n}\right]^{r_2}...\left[(\delta_k)_{p_k n}\right]^{r_k}\Gamma(\alpha n+\beta)} \tag{2.4}
$$

Now applying results due to Olver [7, pp.118-119], Tricomi and Erdélyi [11, pp.133 (1)] in the ratio $\left|\frac{c_{n+1}}{c_n}\right|$ then after simplification, we get

$$\left|\frac{c_{n+1}}{c_n}\right| = \prod_{i=1}^{h}(q_i n)^{q_i s_i}\left[1+\frac{q_i(2\gamma_i+q_i-1)}{2q_i n}+O\left\{\frac{1}{\left|(q_i n)^2\right|}\right\}\right]^{s_i}\times$$

$$\prod_{j=1}^{k}(p_j n)^{-p_j r_j}\left[1+\frac{-p_j(2\delta_j+p_j-1)}{2p_j n}+O\left\{\frac{1}{\left|(p_j n)^2\right|}\right\}\right]^{r_j}\times$$

$$(\alpha n)^{-\alpha}\left[1+\frac{-\alpha(2\beta+\alpha-1)}{2\alpha n}+O\left\{\frac{1}{\left|(\alpha n)^2\right|}\right\}\right]\left|(-1)^{\rho} z^{a}\right| \tag{2.5}$$

Now taking the limit $n\to\infty$, we have

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \prod_{i=1}^{h}(q_i)^{q_i s_i}\prod_{j=1}^{k}(p_j)^{-p_j r_j}(\alpha)^{-\alpha}|z^a|\lim_{n\to\infty} n^{\left(\sum_{i=1}^{h} q_i s_i-\sum_{j=1}^{k} p_j r_j-\alpha\right)} \tag{2.6}$$

Now applying D'Alembert's simple ratio test, we get

(i)

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = 0 < 1 \quad provided \quad \sum_{i=1}^{h} q_i s_i < \sum_{j=1}^{k} p_j r_j + \Re(\alpha), then$$

the given series is convergent for all finite values of $\prod_{i=1}^{h}(q_i)^{q_i s_i}\left[\alpha^{\alpha}\prod_{j=1}^{k}(p_j)^{p_j r_j}\right]^{-1}|z^a|$.

(ii)

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| < 1 \quad provided \quad \sum_{i=1}^{h} q_i s_i = \sum_{j=1}^{k} p_j r_j + \Re(\alpha), then$$

the given series is convergent for $\prod_{i=1}^{h}(q_i)^{q_i s_i}\left[\alpha^{\alpha}\prod_{j=1}^{k}(p_j)^{p_j r_j}\right]^{-1}|z^a| < 1$.

2.2 Special Cases

(i) Put $h=1, s_1=0; k=1, r_1=0$, in (2.1) we get **generalized Sine function** as

$${}_{\tau}E_1^1\left[z\left|\begin{matrix}(\rho,a);(\gamma_1,q_1,0)\\(\alpha,\beta);(\delta_1,p_1,0)\end{matrix}\right.\right] = \sum_{n=0}^{\infty}(-1)^{\rho n}\frac{z^{an+\tau}}{\Gamma(\alpha n+\beta)} = sin^*(z) \tag{2.7}$$

(ii) Put $h=1, s_1=0; k=1, r_1=0$; τ=0, in (2.1) we get **generalized Cosine function** as

$${}_{0}E_1^1\left[z\left|\begin{matrix}(\rho,a);(\gamma_1,q_1,0)\\(\alpha,\beta);(\delta_1,p_1,0)\end{matrix}\right.\right] = \sum_{n=0}^{\infty}(-1)^{\rho n}\frac{z^{an}}{\Gamma(\alpha n+\beta)} = cos^*(z) \tag{2.8}$$

where $sin^*(z)$ and $cos^*(z)$ are defined as generalization of sine and cosine functions respectively.

3 Mellin-Barnes Type Contour Integral Representation of E-Function

Theorem 3.1. *If convergence conditions (2.2) are satisfied then the E-function ${}_{\tau}E_k^h[z]$ can be represented as the Mellin-Barnes type integral as follows*

$${}_{\tau}E_k^h\left[z\left|\begin{matrix}(\rho,a);(\gamma_1,q_1,s_1),...,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),...,(\delta_k,p_k,r_k)\end{matrix}\right.\right] = \frac{\prod_{v=1}^{k}[\Gamma(\delta_v)]^{r_v}}{\prod_{u=1}^{h}[\Gamma(\gamma_u)]^{s_u}}\times$$

$$\times \frac{z^{\tau}}{2\pi i}\int_{\mathcal{L}} \frac{\Gamma(\zeta)\Gamma(1-\zeta)\prod_{i=1}^{h}\left[\Gamma(\gamma_i - q_i\zeta)\right]^{s_i}}{\Gamma(\beta-\alpha\zeta)\prod_{j=1}^{k}\left[\Gamma(\delta_j - p_j\zeta)\right]^{r_j}}\left[(-1)^{\rho}(-z^{a})\right]^{-\zeta} d\zeta \tag{3.1}$$

where $\mathcal{L}$ is a suitable contour of integration that runs from $c-i\infty$ to $c+i\infty, c\in\mathbb{R}$ and intended to separate the poles of the integrand at $\zeta = -n$ for all $n \in \mathbb{N}_0\,(to\,the\,left)$ from those at $\zeta = n+1$ and at $\zeta = \frac{\gamma_i+n}{q_i}, i = 1,2,...,h;$ for all $n \in \mathbb{N}_0\,(to\,the\,right)$.

Proof. Rewriting the definition (2.1) in the form

$${}_{\tau}E_k^h\left[z\,\middle|\,\begin{array}{l}(\rho,a);(\gamma_1,q_1,s_1),...,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),...,(\delta_k,p_k,r_k)\end{array}\right]$$

$$= \frac{z^{\tau}\prod_{v=1}^{k}\left[\Gamma(\delta_v)\right]^{r_v}}{\prod_{u=1}^{h}\left[\Gamma(\gamma_u)\right]^{s_u}}\sum_{n=0}^{\infty}\frac{(-1)^n\prod_{i=1}^{h}\left[\Gamma(\gamma_i+q_i n)\right]^{s_i}}{\Gamma(\alpha n+\beta)\prod_{j=1}^{k}\left[\Gamma(\delta_j+p_j n)\right]^{r_j}}\left[(-1)^{\rho}(-z^{a})\right]^{n} \tag{3.2}$$

$$= \frac{z^{\tau}\prod_{v=1}^{k}\left[\Gamma(\delta_v)\right]^{r_v}}{\prod_{u=1}^{h}\left[\Gamma(\gamma_u)\right]^{s_u}}\sum_{n=0}^{\infty}\lim_{\zeta\to -n}\Gamma(\zeta)\Gamma(1-\zeta)(\zeta+n)\,g(\zeta)\left[(-1)^{\rho}(-z^{a})\right]^{-\zeta} \tag{3.3}$$

where

$$g(\zeta) = \frac{\prod_{i=1}^{h}\left[\Gamma(\gamma_i-q_i\zeta)\right]^{s_i}}{\Gamma(\beta-\alpha\zeta)\prod_{j=1}^{k}\left[\Gamma(\delta_j-p_j\zeta)\right]^{r_j}} \tag{3.4}$$

Then

$${}_{\tau}E_k^h\left[z\,\middle|\,\begin{array}{l}(\rho,a);(\gamma_1,q_1,s_1),...,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),...,(\delta_k,p_k,r_k)\end{array}\right] = \frac{\prod_{v=1}^{k}\left[\Gamma(\delta_v)\right]^{r_v}}{\prod_{u=1}^{h}\left[\Gamma(\gamma_u)\right]^{s_u}}\times$$

$$\times \frac{z^{\tau}}{2\pi i}\int_{\mathcal{L}} \frac{\Gamma(\zeta)\Gamma(1-\zeta)\prod_{i=1}^{h}\left[\Gamma(\gamma_i - q_i\zeta)\right]^{s_i}}{\Gamma(\beta-\alpha\zeta)\prod_{j=1}^{k}\left[\Gamma(\delta_j - p_j\zeta)\right]^{r_j}}\left[(-1)^{\rho}(-z^{a})\right]^{-\zeta} d\zeta \tag{3.5}$$

This completes the proof. □

4 Some Integral Transforms

Theorem 4.1. (Mellin transform) *Let conditions associated with Mellin-Barnes type contour integral representation (3.1) of E-function are satisfied and $\Re(\zeta) > 0$, then the Mellin transform of the E-function is*

$$M\left[\frac{1}{\left\{(-1)^{\rho}(-z)\right\}^{\frac{\tau}{a}}}\,{}_{\tau}E_k^h\left(\left\{(-1)^{\rho}(-z)\right\}^{\frac{1}{a}}\,\middle|\,\begin{array}{l}(\rho,a);(\gamma_1,q_1,s_1),...,(\gamma_h,q_h,s_h)\\(\alpha,\beta);(\delta_1,p_1,r_1),...,(\delta_k,p_k,r_k)\end{array}\right);\zeta\right]$$

$$= \frac{\Gamma(\zeta)\Gamma(1-\zeta)}{\Gamma(\beta-\alpha\zeta)} \frac{\prod_{i=1}^{h}\left[\frac{\Gamma(\gamma_i - q_i\zeta)}{\Gamma(\gamma_i)}\right]^{s_i}}{\prod_{j=1}^{k}\left[\frac{\Gamma(\delta_j - p_j\zeta)}{\Gamma(\delta_j)}\right]^{r_j}} \tag{4.1}$$

provided that the parameters are adjusted in such a way that the right-hand side is meaningful.

Proof. According to Theorem 3.1, the E-function can be written as follows

$$\frac{1}{\left\{(-1)^{\rho}(-z)\right\}^{\frac{\tau}{a}}} {}_{\tau}E_k^h \left(\left\{(-1)^{\rho}(-z)\right\}^{\frac{1}{a}} \mid \begin{array}{l} (\rho, a);(\gamma_1, q_1, s_1), ..., (\gamma_h, q_h, s_h) \\ (\alpha, \beta);(\delta_1, p_1, r_1), ..., (\delta_k, p_k, r_k) \end{array} \right)$$

$$= \frac{1}{2\pi i}\int_{\mathcal{L}} g(\zeta)(z)^{-\zeta} d\zeta \tag{4.2}$$

where

$$g(\zeta) = \frac{\Gamma(\zeta)\Gamma(1-\zeta)}{\Gamma(\beta-\alpha\zeta)} \frac{\prod_{i=1}^{h}\left[\frac{\Gamma(\gamma_i - q_i\zeta)}{\Gamma(\gamma_i)}\right]^{s_i}}{\prod_{j=1}^{k}\left[\frac{\Gamma(\delta_j - p_j\zeta)}{\Gamma(\delta_j)}\right]^{r_j}} \tag{4.3}$$

Then by using definition of the Mellin transform in (4.2), we have

$$L.H.S. = M^{-1}[g(\zeta); z] \tag{4.4}$$

or

$$M\left[\frac{1}{\left\{(-1)^{\rho}(-z)\right\}^{\frac{\tau}{a}}} {}_{\tau}E_k^h \left(\left\{(-1)^{\rho}(-z)\right\}^{\frac{1}{a}} \mid \begin{array}{l} (\rho, a);(\gamma_1, q_1, s_1), ..., (\gamma_h, q_h, s_h) \\ (\alpha, \beta);(\delta_1, p_1, r_1), ..., (\delta_k, p_k, r_k) \end{array} \right); \zeta\right]$$

$$= \frac{\Gamma[\zeta]\Gamma[1-\zeta]}{\Gamma(\beta-\alpha\zeta)} \frac{\prod_{i=1}^{h}\left[\frac{\Gamma(\gamma_i - q_i\zeta)}{\Gamma(\gamma_i)}\right]^{s_i}}{\prod_{j=1}^{k}\left[\frac{\Gamma(\delta_j - p_j\zeta)}{\Gamma(\delta_j)}\right]^{r_j}} \tag{4.5}$$

This completes the proof. □

Theorem 4.2. (Laplace transform) *If conditions associated with Mellin-Barnes type contour integral representation (3.1) of E-function are satisfied then the Laplace transform of the E-function is*

$$L\left[z^{\mu-1} {}_{\tau}E_k^h \left(xz^{\sigma} \mid \begin{array}{l} (\rho, a);(\gamma_1, q_1, s_1), ..., (\gamma_h, q_h, s_h) \\ (\alpha, \beta);(\delta_1, p_1, r_1), ..., (\delta_k, p_k, r_k) \end{array} \right); \nu\right]$$

$$= \frac{1}{\nu^{\mu}}\left(\frac{x}{\nu^{\sigma}}\right)^{\tau} \frac{\prod_{v=1}^{k}\left[\Gamma(\delta_v)\right]^{r_v}}{\prod_{u=1}^{h}\left[\Gamma(\gamma_u)\right]^{s_u}} \times$$

$$\times \overline{H}_{h+2,k+2}^{1,h+2}\left[(-1)^{\rho}\left\{-\left(\frac{x}{\nu^{\sigma}}\right)^{a}\right\} \mid \begin{array}{c} (0,1;1),(1-\mu-\sigma\tau,\sigma a;1),(1-\gamma_i, q_i; s_i)_1^h; \text{—} \\ (0,1);(1-\beta,\alpha;1),(1-\delta_j, p_j; r_j)_1^k \end{array}\right] \tag{4.6}$$

provided that the function on the right-hand side is convergent and has a meaning.

Proof. We obtain the Laplace transform of the E-function as follows

$$L\left[z^{\mu-1} {}_{\tau}E_k^h \left(xz^{\sigma} \mid \begin{array}{l} (\rho, a);(\gamma_1, q_1, s_1), ..., (\gamma_h, q_h, s_h) \\ (\alpha, \beta);(\delta_1, p_1, r_1), ..., (\delta_k, p_k, r_k) \end{array} \right); \nu\right]$$

$$= \int_0^{\infty} z^{\mu-1} e^{-\nu z} {}_{\tau}E_k^h \left(xz^{\sigma} \,\middle|\, \begin{matrix} (\rho, a); (\gamma_1, q_1, s_1), ..., (\gamma_h, q_h, s_h) \\ (\alpha, \beta); (\delta_1, p_1, r_1), ..., (\delta_k, p_k, r_k) \end{matrix} \right) dz, \Re(\nu) > 0 \tag{4.7}$$

Now using (3.1) and interchanging the order of integrations, which is permissible under suitable convergence conditions, we have

$$L.H.S. = \frac{\prod_{v=1}^{k} [\Gamma(\delta_v)]^{r_v}}{\prod_{u=1}^{h} [\Gamma(\gamma_u)]^{s_u}} \frac{x^{\tau}}{2\pi i} \int_{\mathcal{L}} g(\zeta) [(-1)^{\rho} \{-(x^a)\}]^{-\zeta} \left\{ \int_0^{\infty} e^{-\nu z} z^{\mu+\sigma\tau-\sigma a\zeta-1} dz \right\} d\zeta \tag{4.8}$$

where $g(\zeta)$ can be written as

$$g(\zeta) = \frac{\Gamma(0+\zeta)\Gamma(1-0-\zeta) \prod_{i=1}^{h} [\Gamma\{1-(1-\gamma_i)-q_i\zeta\}]^{s_i}}{\Gamma[1-(1-\beta)-\alpha\zeta] \prod_{j=1}^{k} [\Gamma\{1-(1-\delta_j)-p_j\zeta\}]^{r_j}} \tag{4.9}$$

Now applying gamma integral then comparing it with the definition of Inayat-Hussain $\overline{H}$-function [3], we get

$$L.H.S. = \frac{1}{\nu^{\mu}} \left(\frac{x}{\nu^{\sigma}} \right)^{\tau} \frac{\prod_{v=1}^{k} [\Gamma(\delta_v)]^{r_v}}{\prod_{u=1}^{h} [\Gamma(\gamma_u)]^{s_u}} \times$$

$$\times \overline{H}_{h+2,k+2}^{1,h+2} \left[(-1)^{\rho} \left\{ -\left(\frac{x}{\nu^{\sigma}} \right)^{a} \right\} \,\middle|\, \begin{matrix} (0,1;1), (1-\mu-\sigma\tau, \sigma a; 1), (1-\gamma_i, q_i; s_i)_1^h ; \text{---} \\ (0,1); (1-\beta, \alpha; 1), (1-\delta_j, p_j; r_j)_1^k \end{matrix} \right] \tag{4.10}$$

This completes the proof. □

Theorem 4.3. (Euler-Beta transform) *If conditions associated with Mellin-Barnes type contour integral representation (3.1) of E-function are satisfied then the Euler-Beta transform of the E-function is*

$$B\left[{}_{\tau}E_k^h \left(xz^{\sigma} \,\middle|\, \begin{matrix} (\rho, a); (\gamma_1, q_1, s_1), ..., (\gamma_h, q_h, s_h) \\ (\alpha, \beta); (\delta_1, p_1, r_1), ..., (\delta_k, p_k, r_k) \end{matrix} \right); \mu, \nu : 0, 1 \right] = \Gamma(\nu)(x)^{\tau} \frac{\prod_{v=1}^{k} [\Gamma(\delta_v)]^{r_v}}{\prod_{u=1}^{h} [\Gamma(\gamma_u)]^{s_u}} \times$$

$$\times \overline{H}_{h+2,k+3}^{1,h+2} \left[(-1)^{\rho} (-x^a) \,\middle|\, \begin{matrix} (0,1;1), (1-\mu-\sigma\tau, \sigma a; 1), (1-\gamma_i, q_i; s_i)_1^h ; \text{---} \\ (0,1); (1-\beta, \alpha; 1), (1-\mu-\nu-\sigma\tau, \sigma a; 1), (1-\delta_j, p_j; r_j)_1^k \end{matrix} \right] \tag{4.11}$$

provided that the function on the right-hand side is convergent and has a meaning.

Proof. The proof can be developed on the lines similar to the proof of Theorem 4.2. □

Theorem 4.4. (Whittaker transform) *If conditions associated with Mellin-Barnes type contour integral representation (3.1) of E-function are satisfied then the Whittaker transform of the E-function is*

$$\mathcal{W}\left[{}_{\tau}E_k^h \left(xz^{\sigma} \,\middle|\, \begin{matrix} (\rho, a); (\gamma_1, q_1, s_1), ..., (\gamma_h, q_h, s_h) \\ (\alpha, \beta); (\delta_1, p_1, r_1), ..., (\delta_k, p_k, r_k) \end{matrix} \right); \lambda, \mu, \nu \right] = x^{\tau} \frac{\prod_{v=1}^{k} [\Gamma(\delta_v)]^{r_v}}{\prod_{u=1}^{h} [\Gamma(\gamma_u)]^{s_u}} \times$$

$$\times \overline{H}^{1,h+3}_{h+3,k+3}\left[(-1)^{\rho}\left(-x^{a}\right) \;\middle|\; \begin{matrix} (0,1;1),\left(\frac{1}{2}\pm\mu-\nu-\sigma\tau,\sigma a;1\right),\left(1-\gamma_{i},q_{i};s_{i}\right)_{1}^{h};\text{---} \\ (0,1);(1-\beta,\alpha;1),(\lambda-\nu-\sigma\tau,\sigma a;1),\left(1-\delta_{j},p_{j};r_{j}\right)_{1}^{k} \end{matrix}\right] \tag{4.12}$$

provided that the function on the right-hand side is convergent and has a meaning.

Proof. The proof can be developed on the lines similar to the proof of Theorem 4.2. □

Concluding Remarks

When physical phenomena deviate from exponential behaviour then the M-L type functions play the major role being generalization of exponential function. When experiments are performed at micro and nano levels then such kind of phenomena are naturally obtained. It is believed that nonequilibrium statistical mechanics is governed by fractional calculus. The generalization of the M-L type function discussed in this paper will be useful in solving various problems of Mathematical Sciences.

Also generalizations of sine (2.7) and cosine (2.8) functions, introduced first time, are apparent paradox from which one day, useful consequences will be drawn.

The present paper provides a scope of defining M-L function of many parameters as a MATLAB function that will enable the researchers to solve more complex problems using MATLAB. Currently MATLAB provides MLFFIT1.M [8] and MLFFIT2.M [10], in which the M-L function in one and two parameters respectively, are used.

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