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A nonlinear hyperbolic problem for viscoelastic equations

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Abstract. In this paper, we consider a nonlinear boundary value problem for viscoelastic equations with a source term. By basing on *Faedo-Galerkin* approximations and compactness argument, this work is devoted to prove the existence, uniqueness, and also continuous dependence with respect to the initial data of solutions.

1 Introduction

In this paper, we consider a semilinear hyperbolic boundary value problem governed by partial differential equations, which describe the evolution of nonlinear viscoelastic materials with *Dirichlet* and *Neumann* boundary conditions as follows:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - div\sigma(u) + |u|^{\nu} u = f, \text{ in } \Omega \times (0,T), \\ \sigma(u) = \lambda F(\varepsilon(u)) + \mu G(\varepsilon(u')), \text{ in } \Omega \times (0,T), \\ u = 0 \text{ on } \Gamma_1 \times (0,T), \\ u = 0 \text{ on } \Gamma_1 \times (0,T), \ \sigma(u)\eta = 0 \text{ on } \Gamma_2 \times (0,T), \\ u(x,0) = u_0(x), \ u'(x,0) = u_1(x), \ x \in \Omega, \end{cases}$$

where F, G are nonlinear functions and ν, λ, μ are positive reel numbers. In the case where $\mu = 0$ and $\lambda = 1$, if $\sigma(u) = \nabla u$ or $\sigma(u) = |\nabla u|^{p-2} \nabla u$, $p \geq 2$, the correspondent problems were considered by *J.L. Lions* [3]. Precisely, under assumption on ν, he showed the existence, uniqueness and the regularity of a solution. For $\lambda = 1, \mu = 0$, where F is a linear function, *Rahmoune* and *Benabderrahmane* in [5, 6], without supposing any assumption on ν, have showed the local existence, uniqueness and the regularity of solutions, by using *Faedo Galerkin* techniques and compactness method.

In this paper assume that the data functions F, G, f, u_0, u_1 satisfy certain hypotheses, then by using *Faedo Galerkin* approximations and compactness argument, we will prove the local existence of solutions. Our main goal in this work is, without any assumption on ν like in [5, 6], to show the uniqueness and the continuous dependence with respect to the initial data of solutions.

2 Problem statement

Let Ω be an open and bounded domain in $\mathbb{R}^n$, recall that the boundary Γ of Ω is assumed to be regular and is composed of two relatively closed parts : Γ_1, Γ_2, with mutually disjoint relatively open interiors. We assume that $meas\,(\Gamma_1) > 0$. We pose $\Sigma_i = \Gamma_i \times (0,T), i = 1, 2$, where T is a finite real. Let η be the unit outward normal vector on Γ. Here and throughout this paper, the summation convention over repeated indices is used.

The classical formulation of the problem is as follows. Find a displacement field $u : \Omega \times (0,T) \to \mathbb{R}^n$, a stress field $\sigma : \Omega \times (0,T) \to S_n$, such that

$$u'' - div\sigma(u) + |u|^{\nu} u = f \text{ in } Q, \ \nu > -1, \tag{2.1}$$

$$\sigma(u) = \lambda F(x, \varepsilon(u)) + \mu G(x, \varepsilon(u')) \text{ in } Q, \tag{2.2}$$

$$\begin{cases} u = 0 \text{ on } \Sigma_1, \\ \sigma(u)\eta = 0 \text{ on } \Sigma_2, \end{cases} \tag{2.3}$$

$$\left\{ \begin{array}{l} u(x,0) = u_0(x) \text{ in } \Omega, \\ u'(x,0) = u_1(x) \text{ in } \Omega, \end{array} \right. \tag{2.4}$$

where $\mathcal{S}_n$ is the space of symmetric second-order tensors in $\mathbb{R}^n$. u and f represent the displacement field and the density of volume forces, respectively. (2.1), without the source term $|u|^{\nu} u$, is the equilibrium equations for the stress, in which "div" denotes the divergence operator for stress tensor $\sigma = (\sigma_{ij}), i,j = 1,2,...,n$. (2.2) represents the viscoelastic constitutive law, $\varepsilon(u) = \frac{1}{2}\left(\nabla u + \nabla^T u\right)$ denotes the linearized strain tensor. Next, (2.3) and (2.4) are the displacement and traction boundary conditions and the initial conditions, respectively. To simplify the notations, we do not indicate explicitly the dependence of u and σ with respect to $x \in \Omega$ and $t \in (0,T)$.

In order to proceed with the variational formulation, we need the following space:

$$\mathcal{H}= L^2(\Omega)_s^{n\times n} = \left\{\sigma = (\sigma_{ij}) \in \mathcal{S}_n : \sigma_{ij} = \sigma_{ji} \in \ L^2(\Omega)\right\},$$

which is a real *Hilbert* spaces endowed with the scalar product defined by

$$\langle \sigma,\tau \rangle = \int_\Omega \sigma_{ij}\tau_{ij}dx.$$

The associated norm is denoted by $\|.\|_{\mathcal{H}}$. Also we consider the notation $\|v\|_{L^2(\Omega)} = |v| = \left(\int_\Omega |v|^2 dx\right)^{\frac{1}{2}}$.

For studying the problem (2.1)–(2.4), we will need the following hypotheses:

We assume that the nonlinear elasticity operator $F : \Omega \times S_n \to S_n$ satisfies :

$$\left\{ \begin{array}{l} \text{For all } \varepsilon_1, \varepsilon_2 \in S_n, \text{ then there exist } m_1 > 0 \text{ and } L > 0 \text{ such that} \\ (a)\ \ (F(x,\varepsilon_1) - F(x,\varepsilon_2)) \,.\, (\varepsilon_1 - \varepsilon_2) \geq m_1 \left|\varepsilon_1 - \varepsilon_2\right|^2, \text{ a.e. } x \in \Omega; \\ (b)\ \ |F(x,\varepsilon_1) - F(x,\varepsilon_2)| \leq L\left|\varepsilon_1 - \varepsilon_2\right|, \text{ a.e. } x \in \Omega; \\ (c) \text{ The mapping } x \to F(x,\varepsilon) \text{ is } \textit{Lebesgue} \text{ measurable on } \Omega; \\ (d)\ F(x;0) = 0. \end{array} \right. \tag{2.5}$$

The nonlinear viscosity operator $G : \Omega \times S_n \to S_n$ verifies :

$$\left\{ \begin{array}{l} \text{For all } \varepsilon_1, \varepsilon_2 \in S_n, \text{ then there exists } m_2 > 0 \text{ such that} \\ (a)\ \ \ (G(\varepsilon_1) - G(\varepsilon_2)) \,.\, (\varepsilon_1 - \varepsilon_2) \geq m_2 \left|\varepsilon_1 - \varepsilon_2\right|^2, \text{ a.e. } x \in \Omega; \\ (b) \text{ The mapping } x \to G(x,\varepsilon) \text{ is } \textit{Lebesgue} \text{ measurable on } \Omega. \end{array} \right. \tag{2.6}$$

Also we assume that the given data f, u_0 and u_1 verify

$$f \in L^2(Q), \tag{2.7}$$

$$u_0 \in V \cap L^p(\Omega),\ p = \nu + 2, \tag{2.8}$$

$$u_1 \in L^2(\Omega). \tag{2.9}$$

It is easy, as in [5, 6] to verify the following result.

Lemma 2.1. *Assume that hypotheses (2.5) and (2.6) hold. Then the functions, still denoted by F, $G : \mathcal{H} \longrightarrow \mathcal{H}$, defined by*

$$F\left(\varepsilon(.)\right) = F\left(.,\varepsilon(.)\right),\ \ G\left(\varepsilon(.)\right) = \ G\left(.,\varepsilon(.)\right),\ \textit{a.e. in } \Omega$$

are continuous on $\mathcal{H}$.

By a standard procedure based on *Green*'s formula, we derive the following variational formulation of the mechanical problem (2.1)–(2.4):

Find a displacement field $u \in V \cap L^p(\Omega),\ p = \nu + 2$ such that, for all $v \in V \cap L^p(\Omega)$, we have

$$\left\{ \begin{array}{c} (u'', v) + \lambda a\left(u, v\right) + \mu\left(G\left(\varepsilon(u')\right), \varepsilon(v)\right) + \left(|u|^{\nu} u, v\right) = (f, v), \\ u(x,0) = u_0(x),\ u'(x,0) = u_1(x),\ x \in \Omega, \end{array} \right.$$

where

$$V = \left\{v \in H^1(\Omega), v = 0 \text{ on } \Gamma_1\right\} \text{ and } a(u,v) = \int_\Omega F\left(\varepsilon(u)\right)\varepsilon(v)dx.$$

3 Existence and Uniqueness

Our main goal in this section is, by basing on *Faedo-Galerkin* approximations and compactness argument, to show the local existence and uniqueness of a weak solution.

3.1 Existence

Theorem 3.1. *Assume that (2.5)–(2.9) hold. Then there exists at least one solution to problem (2.1)–(2.4) and it satisfies*

$$u \in L^{\infty}(0,T;V\cap L^{p}(\Omega)),\ \ p=\nu+2, \tag{3.1}$$

$$u' \in L^{\infty}\left(0,T;L^{2}(\Omega)\right)\cap L^{2}\left(0,T;V\right). \tag{3.2}$$

Assume that the result of the Theorem 3.1 is verified, then we show the following result, which guarantees that the initial conditions are well-defined.

Lemma 3.2. *Assume that (2.5)–(2.9) hold. Then the initial conditions in (2.4) are well-defined.*

Proof. Using hypotheses (2.5)–(2.9) and the results of 3.1, we get

$$u\in L^{2}\left(0,T;V\right) \text{ and } u'\in L^{2}\left(0,T;V\right).$$

Using a known result like in [3], then the application $u:[0,T]\to V$, is continuous, possibly after a modification on a subset of $[0,T]$ with zero measure, then $u\left(0\right)$ is well-defined, therefore the first condition in (2.4) has a sense.

Using, (3.1) and (3.2), then we have $\varepsilon(u)$, $\varepsilon(u')\in L^{\infty}\left(0,T;L^{2}(\Omega)\right)$. Thus, using the fact that F and G are continuous and $L^{2}(\Omega)\subset V'$, we get $F\left(\varepsilon(u)\right),G\left(\varepsilon(u')\right)\in L^{\infty}\left(0,T;L^{2}(\Omega)\right)$. Consequently, we deduce

$$div\sigma\left(u\right)=div\left(\lambda F\left(\varepsilon(u)\right)+\mu G\left(\varepsilon(u')\right)\right)\in L^{\infty}\left(0,T;V'\right).$$

On the other hand, we have

$$\int_{\Omega}\left|\left(\left|u\right|^{\nu}u\right)\right|^{p'}dx\le\int_{\Omega}\left|u\right|^{(\nu+1)p'}dx=\int_{\Omega}\left|u\right|^{(p-1)\frac{p}{p-1}}dx=\left\|u\right\|_{L^{p}(\Omega)}^{p},\ \frac{1}{p}+\frac{1}{p'}=1.$$

Thus, for all $u\in L^{p}(\Omega)$, we arrive at

$$\left|u\right|^{\nu}u\in L^{\infty}\left(0,T;L^{p'}(\Omega)\right).$$

Then, from (2.1) it follows

$$u''=f+div\sigma\left(u\right)-\left|u\right|^{\nu}u\in L^{2}\left(0,T;L^{2}(\Omega)\right)+L^{\infty}\left(0,T;V'+L^{p'}(\Omega)\right),$$

where V' is the dual space of V and

$$V'+L^{p'}(\Omega)=\left\{u+v;\ u\in V'\text{and } v\in L^{p'}(\Omega)\right\}.$$

Using the fact that $L^{2}(\Omega)\subset V'+L^{p'}(\Omega)$, it is easy to deduce that

$$u''\in L^{2}\left(0,T;V'+L^{p'}(\Omega)\right). \tag{3.3}$$

As in [3], using (3.2) we get

$$u':[0,T]\longrightarrow V'+L^{p'}(\Omega)$$

is continuous, possibly after a modification on a subset of $[0,T]$ with zero measure, then $u'\left(0\right)$ is well-defined, therefore the second initial condition in (2.4) has a sense. □

Proof of Theorem 3.1 . It consists of four steps:

Step 1 : Approximate Solutions.
It introduces a sequence (w_n) of functions having the following properties:
$*$ $\forall j=1,...,m; w_j\in V\cap L^{p}(\Omega)$;
$*$ The family $\{w_1,w_2,...,w_m\}$ is linearly independent;
$*$ The $V_m=span\left\{w_1,w_2,...,w_m\right\}$ generated by $\{w_1,w_2,...,w_m\}$ is dense in $V\cap L^{p}(\Omega)$.

Let $u_m = u_m(t)$ be an approximate solution such that

$$u_m(t) = \sum_{i=1}^{m} K_{jm}(t) w_i. \tag{3.4}$$

The K_{jm} being to be determined by the following system:

$$\begin{gathered}(u_m''(t), w_j) + \lambda a(u_m(t), w_j) + \mu \left(G\left(\varepsilon(u_m'(t))\right), \varepsilon(w_j)\right) + \\ +(|u_m|^{\nu} u_m(t), w_j) = (f(t), w_j),\ 1 \leq j \leq m,\end{gathered} \tag{3.5}$$

$$u_m(0) = u_{0m} = \sum_{i=1}^{m} \alpha_{im} w_i \overset{m \longrightarrow \infty}{\longrightarrow} u_0 \text{ in } V \cap L^p(\Omega), \tag{3.6}$$

$$u_m'(0) = u_{1m} = \sum_{i=1}^{m} \beta_{im} w_i \overset{m \longrightarrow \infty}{\longrightarrow} u_1 \text{ in } L^2(\Omega). \tag{3.7}$$

Since (3.5)–(3.7) is a normal system of ordinary differential equations, then there exist at least u_m, solutions to the problem (2.1)–(2.4) having the following regularities:

$$u_m(t) \in L^2(0, t_m; V_m),\ u_m'(t) \in L^2(0, t_m; V_m).$$

In order to verify that $\forall k \in \mathbb{N}$, then $t_m = T$, we will show some a priori estimates uniform with respect to m.

Step 2: Estimates on (u_m). We set

$$\|u\|_1 = \left(\int_{\Omega} F\left(\varepsilon(u)\right) \varepsilon(u) dx\right)^{\frac{1}{2}}. \tag{3.8}$$

Then, using (2.5), it can be shown that $\|u\|_1$ is a norm on V equivalent to the norm $\|u\|$ in $H^1(\Omega)$.

Multiplying the equation (3.5) by $K_{jm}'(t)$ and summing over $j = 1$ to m, we get

$$\begin{cases} (u_m''(t), u_m'(t)) + \lambda a\left(u_m(t), u_m'(t)\right) + \mu\left(G\left(\varepsilon(u_m'(t))\right), \varepsilon(u_m'(t))\right) + \\ \qquad + \left(|u_m|^{\nu} u_m(t), u_m'(t)\right) = \left(f(t), u_m'(t)\right). \end{cases} \tag{3.9}$$

Since $u_m \in L^2(0, t_m; V_m),\ u_m' \in L^2(0, t_m; V_m)$, we then deduce that $\varepsilon(u_m), \varepsilon(u_m') \in L^2\left(0, T; L^2(\Omega)\right)$.

From Lemma 2.1 it follows

$$F\left(\varepsilon(u_m)\right),\ F\left(\varepsilon(u_m')\right) \in L^2\left(0, T; L^2(\Omega)\right).$$

Observing that

$$\frac{d}{dt} a\left(u_m(t), u_m(t)\right) = \left(F\varepsilon(u_m(t)), \varepsilon(u_m'(t))\right) + \left(\frac{d}{dt}\left(F\varepsilon(u_m(t))\right), \varepsilon(u_m(t))\right).$$

From (3.8)we have

$$\left(F\left(\varepsilon(u_m(t))\right), \varepsilon(u_m'(t))\right) = \frac{d}{dt}\|u_m(t)\|_1^2 - \left(\frac{d}{dt}F\left(\varepsilon(u_m(t))\right), \varepsilon(u_m(t))\right).$$

Using *Cauchy-Shwarz*'s, *Hollder*'s inequalities and hypotheses (2.5), then we have

$$\left|\left(\frac{d}{dt}F\left(\varepsilon(u_m(t))\right), \varepsilon(u_m(t))\right)\right| \leq L\left|\varepsilon(u_m'(t))\right|\left|\varepsilon(u_m(t))\right| \leq C_3\|u_m'(t)\|\|u_m(t)\|. \tag{3.10}$$

Also, we have

$$\begin{gathered}\tfrac{1}{2}\frac{d}{dt}|u_m'(t)|^2 = (u_m''(t), u_m'(t)), \\ \tfrac{1}{p}\frac{d}{dt}\|u_m(x,t)\|_{L^p(\Omega)}^p = \left(|u_m|^{\nu} u_m(t), u_m'(t)\right),\ p = \nu + 2.\end{gathered}$$

Using hypotheses (2.6), then there is a constant $C_2 > 0$ such that

$$\left(G(\varepsilon(u_m'(t))), \varepsilon(u_m'(t))\right) \geq C_2\|u_m'(t)\|^2.$$

Then, using *Cauchy-Schwarz's* and *Young's* inequalities and (3.10) from (3.9) it finds

$$\begin{gathered}\frac{d}{dt}\left[\tfrac{1}{2}\left|u_m'(t)\right|^2+\lambda C_1\left\|u_m\left(t\right)\right\|^2\right]+\mu C_2\left\|u_m'\left(t\right)\right\|^2+\tfrac{1}{p}\frac{d}{dt}\left\|u_m(x,t)\right)\|_{L^p(\Omega)}^p\leq\\ \leq\left|(f(t)\right|\left|u_m'(t)\right|+C_3\left\|u_m'\left(t\right)\right\|\left\|u_m\left(t\right)\right\|\leq\\ \leq\tfrac{1}{2}\left|f(t)\right|^2+\tfrac{1}{2}\left|u_m'(t)\right|^2+\tfrac{1}{2}\mu C_2\left\|u_m'\left(t\right)\right\|^2+\tfrac{1}{2}\tfrac{C_3^2}{\mu C_2}\left\|u_m\left(t\right)\right\|^2.\end{gathered} \tag{3.11}$$

Integrating the last inequality over $(0,t)$, we deduce that

$$\begin{gathered}\tfrac{1}{2}\left|u_m'(t)\right|^2+\lambda C_1\left\|u_m(t)\right\|^2+\tfrac{1}{2}\mu C_2\int_0^t\left\|u_m'\left(s\right)\right\|^2ds+\tfrac{1}{p}\left\|u_m(t)\right\|_{L^p(\Omega)}^p\leq\\ \leq\tfrac{1}{2}\left|u_{1m}\right|^2+\lambda C_1\left\|u_{0m}\right\|^2+\tfrac{1}{p}\left\|u_{0m}\right\|_{L^p(\Omega)}^p+\tfrac{1}{2}\int_0^t\left|f(s)\right|^2ds+\\ +\tfrac{1}{2}\int_0^t\left|u_m'(s)\right|^2ds+C_4\int_0^t\left\|u_m\left(s\right)\right\|^2ds,\end{gathered} \tag{3.12}$$

where $C_4=\frac{1}{2}\frac{C_3^2}{\mu C_2}$.

By using hypotheses (3.7), (3.6) and (2.7), then there exists a constant $C_5>0$ such that

$$\frac{1}{2}\left|u_{1m}\right|+\lambda C_1\left\|u_{0m}\right\|^2+\frac{1}{p}\left\|u_{0m}\right\|_{L^p(\Omega)}^p+\frac{1}{2}\int_0^t\left|f(s)\right|^2ds\leq C_5.$$

Then from (3.12) it follows that

$$\begin{gathered}\left|u_m'(t)\right|^2+\left\|u_m(t)\right\|^2+\int_0^t\left\|u_m'\left(s\right)\right\|^2ds+\left\|u_m(t)\right\|_{L^p(\Omega)}^p\leq\\ \leq C_6+C_7\int_0^t\left(\left|u_m'(s)\right|^2+\left\|u_m(s)\right\|^2\right)ds,\ \forall m\in\mathbb{N}^*,\end{gathered} \tag{3.13}$$

where $C_6=\frac{C_5}{\min\left(\frac{1}{2},\lambda C_1,\frac{1}{2}\mu C_2,\frac{1}{p}\right)}$ and $C_7=\frac{\max\left(\frac{1}{2},C_4\right)}{\min\left(\frac{1}{2},\lambda C_1,\frac{1}{2}\mu C_2,\frac{1}{p}\right)}$.

Consequently in particular, for all $t\in(0,T)$

$$\left|u_m'(t)\right|^2+\left\|u_m(t)\right\|^2\leq C_6+C_7\int_0^t\left(\left|u_m'(s)\right|^2+\left\|u_m\left(s\right)\right\|^2\right)ds$$

Therefore, employing *Gronwall*'s inequality, we deduce that there exists a constant $C>0$, independent of m, such that

$$\left|u_m'(t)\right|+\left\|u_m(t)\right\|\leq C. \tag{3.14}$$

From (3.13), we have

$$\left\|u_m(t)\right\|_{L^p(\Omega)}+\int_0^t\left\|u_m'\left(s\right)\right\|^2ds\leq C\text{ (independent of }m). \tag{3.15}$$

From where, we deduce that t_m is independent of m.

Passing to the limit when $m\longrightarrow\infty$, then from (3.15) it follows

$$\begin{cases}\left(u_m\right)\text{ is bounded in }L^\infty\left(0,T;V\cap L^p(\Omega)\right),\\ \left(u_m'\right)\text{ is bounded in }L^\infty\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;V\right).\end{cases} \tag{3.16}$$

Step 3 : Passage to the limit.

It follows from (3.16) that there exists a subsequence (u_μ) of (u_m) such that

$$u_\mu\longrightarrow u\text{ weakly star in }L^\infty\left(0,T;V\cap L^p(\Omega)\right), \tag{3.17}$$

$$u_\mu'\longrightarrow u'\text{ weakly star in }L^\infty\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;V\right). \tag{3.18}$$

From (3.16), it is obtained that the sequences (u_m) and (u_m') are bounded in $L^2(0,T;V)\subset L^2(0,T;L^2(\Omega))=L^2(Q)$ and $L^2\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;V\right)\subset L^2(0,T;L^2(\Omega))=L^2(Q)$, respectively.

Thanks to the *Aubin–Lions* theorem (see [4]), we deduce

$$u_\mu \longrightarrow u \text{ strongly in } L^2(Q). \tag{3.19}$$

From (3.15) it results that $(|u_m|^\nu u_m)$ is bounded in $L^\infty\left(0,T;L^{p'}(\Omega)\right)$, consequently

$$|u_\mu|^\nu u_\mu \to |u|^\nu u \text{ weakly in } L^\infty\left(0,T;L^{p'}(\Omega)\right), \tag{3.20}$$

where $\frac{1}{p}+\frac{1}{p'}=1$ and $p=\nu+2$.

Let j be fixed and $\mu > j$. Then by (3.5) we have

$$\begin{aligned}\left(u''_\mu(t),w_j\right)+\lambda a\left(u_\mu(t),w_j\right)+\mu\left(-divG\left(\varepsilon(u'_\mu(t))\right),w_j\right)+\\ +\left(|u_\mu|^\nu u_\mu(t),w_j\right)=\left(f(t),w_j\right).\end{aligned} \tag{3.21}$$

Then, (3.17) and (3.18) imply that

$$\begin{cases} a(u_\mu,w_j)\longrightarrow a(u,w_j) \text{ weakly star in } L^\infty(0,T),\\ (u'_\mu,w_j)\longrightarrow (u',w_j) \text{ weakly star in } L^\infty(0,T),\\ \left(-divG\left(\varepsilon(u'_\mu(t))\right),w_j\right)\to(\chi,w_j) \text{ weakly star in } L^\infty(0,T).\end{cases}$$

And

$$\left(u''_\mu(t),w_j\right)\longrightarrow\left(u''(t),w_j\right) \text{ in } \mathcal{D}'(0,T).$$

Using (3.20) to obtain

$$\left(|u_\mu|^\nu u_\mu,w_j\right)\longrightarrow\left(|u|^\nu u,w_j\right) \text{ weakly star in } L^\infty(0,T).$$

Then (3.21) becomes

$$\left(u''(t),w_j\right)+\lambda a(u(t),w_j)+\mu\left(\chi,w_j\right)+\left(|u|^\nu u(t),w_j\right)=\left(f(t),w_j\right).$$

Finally, since the space V_m is dense in $V\cap L^p(\Omega)$, for all $v\in V\cap L^p(\Omega)$ we obtain

$$\left(u''(t),v\right)+\lambda a\left(u(t),v\right)+\mu\left(\chi,v\right)+\left(|u|^\nu u(t),v\right)=\left(f(t),v\right).$$

Then, u satisfies

$$u''(t)-\lambda divF\left(\varepsilon(u)\right)+\mu\chi+|u|^\nu u=f. \tag{3.22}$$

Step 4 : Verification of the initial conditions.

From (3.17) and (3.18), it follows that

$$u_\mu(0)\to u(0) \text{ weakly in } L^2(\Omega).$$

Then, using (3.6) we deduce that

$$u_\mu(0)=u_{0\mu}\to u_0 \text{ in } V\cap L^p(\Omega).$$

Thus, the first condition in (2.4) is obtained.

Again, by using (3.3), we get

$$\begin{cases}\frac{d}{dt}\left(u'_\mu(t),w_j\right)\longrightarrow\left(f(t),w_j\right)-\lambda a\left(u(t),w_j\right)-\left(\chi,w_j\right)=\\ \quad=\frac{d}{dt}\left(u'(t),w_j\right) \text{ weakly in } L^2\left(0,T\right)+L^{p'}(0,T),\end{cases}$$

then, for $j=1,...,m$, we have

$$\left(u'_\mu(0),w_j\right)=\left(u_{1\mu},w_j\right)\longrightarrow\left(u'(0),w_j\right).$$

Consequently

$$\left(u_1,w_j\right)=\left(u'(0),w_j\right), \text{ for } j=1,...,m,$$

Then the second condition in (2.4) is satisfied.

In order to complete the proof of 3.1, we need to prove that

$$\chi=-divG\left(\varepsilon(u'(t))\right)$$

For any $\varphi \in L^2(0,T;V)$, let us consider $\mathcal{A}(\varphi(t)) = -divG(\varepsilon(\varphi(t)))$. Our goal now is to show that $\chi = \mathcal{A}(u'(t))$. Indeed, using *Green*'s formula and the boundary conditions (2.3), then for all $v \in L^2(0,T;V)$, from (2.6) we get

$$X_\mu = \int_0^t \left(\mathcal{A}\left(u'_\mu(s)\right) - \mathcal{A}(v'(s)), u'_\mu(s) - v'(s)\right) ds \geq 0 \tag{3.23}$$

and also, from (3.5) it results

$$\int_0^t \left(\mathcal{A}\left(u'_\mu(s)\right), u'_\mu(s)\right) ds = -\frac{1}{2}\left|u'_\mu(t)\right|^2 - \lambda \int_0^t a\left(u_\mu(s), u'_\mu(s)\right) ds - \\ -\frac{1}{p}\|u_\mu(x,t)\|^p_{L^p(\Omega)} + \frac{1}{p}\|u_{0\mu}\|^p_{L^p(\Omega)} + \frac{1}{2}|u_{1\mu}|^2 + \int_0^t \left(f(s), u'_\mu(s)\right) ds.$$

Hence, using (3.23), we arrive at

$$X_\mu = -\frac{1}{2}\left|u'_\mu(t)\right|^2 - \lambda \int_0^t a\left(u_\mu(s), u'_\mu(t)\right) ds - \frac{1}{p}\|u_\mu(x,t)\|^p_{L^p(\Omega)} + \\ +\frac{1}{p}\|u_{0\mu}\|^p_{L^p(\Omega)} + \frac{1}{2}|u_{1\mu}|^2 + \int_0^t \left(f(s), u'_\mu(s)\right) ds - \int_0^t \left(\mathcal{A}\left(u'_\mu(s)\right), v'(s)\right) ds - \\ - \int_0^t \left(\mathcal{A}(v'(s)), u'_\mu(s) - v'(s)\right) ds.$$

Consequently, using the following inequalities

$$\begin{cases} \liminf_\mu \left|u'_\mu(t)\right|^2 \geq |u'(t)|^2, \\ \liminf_\mu \|u_\mu(x,t)\|^p_{L^p(\Omega)} \geq \|u(x,t)\|^p_{L^p(\Omega)}, \\ \liminf_\mu \int_0^t a\left(u_\mu(s), u'_\mu(s)\right) ds \geq \int_0^t a(u(s), u'(s)) ds, \end{cases}$$

to obtain that

$$\limsup_\mu X_\mu \leq -\frac{1}{2}|u'(t)|^2 - \lambda \int_0^t a(u(s), u'(s)) ds - \frac{1}{p}\|u(x,t)\|^p_{L^p(\Omega)} + \\ +\frac{1}{p}\|u_0\|^p_{L^p(\Omega)} + \frac{1}{2}|u_1|^2 + \int_0^t (f(s), u'(s)) ds - \\ -\mu \int_0^t (\chi, v'(s)) ds - \int_0^t (\mathcal{A}(v'(s)), u'(s) - v'(s)) ds. \tag{3.24}$$

We multiply the (3.22) by u' and we use integration by parts to arrive at

$$\mu \int_0^t (\chi, u'(s)) ds = \int_0^t (f(s), u'(s)) ds - \frac{1}{2}|u'(t)|^2 - \lambda \int_0^t a(u(s), u'(s)) ds - \\ -\frac{1}{p}\|u(x,t)\|^p_{L^p(\Omega)} + \frac{1}{p}\|u_0\|^p_{L^p(\Omega)} + \frac{1}{2}|u_1|^2.$$

This last equality with (3.23) and (3.24) yields

$$\int_0^t (\mu\chi - \mathcal{A}(v'(s)), u'(s) - v'(s)) ds \geq 0. \tag{3.25}$$

Now, for all $w \in L^2(0,T;V)$, we put $v' = u' - \alpha w$, $\alpha > 0$, then (3.25) becomes

$$\int_0^t (\chi - \mathcal{A}(u'(s) - \alpha w'(s)), w(s)) ds \geq 0. \tag{3.26}$$

Finally, when $\alpha \longrightarrow 0$ from (3.26) it results

$$\int_0^t (\chi - \mathcal{A}(u'(s)), w(s)) ds \geq 0, \ \forall w \in L^2(0,T;V),$$

which implies that $\chi = \mathcal{A}\left(u'\left(t\right)\right)$.

Then, for all $w_j \in V_m$ and $j = 1, ..., m$, from (3.21) it results

$$\frac{d^2}{dt^2}\left(u\left(t\right), w_j\right) + \lambda a(u\left(t\right), w_j) + \mu\left(-divG\left(\varepsilon(u'(t))\right), w_j\right) = \left(f\left(t\right), w_j\right).$$

Finally, using the fact that V_m is dense in V , then for all $v \in V$ we gate

$$\left(u''\left(t\right), v\right) + \lambda a\left(u\left(t\right), v\right) + \mu\left(-divG\left(\varepsilon(u'(t))\right), v\right) = \left(f\left(t\right), v\right). \tag{3.27}$$

Thus, the (2.1) is satisfied. □

3.2 Uniqueness

Many authors have showed the uniqueness of the solution by supposing that $\nu \leq \dfrac{2}{n-2}$, of particular problems. Our goal in this section is, for all $\nu > -1$, to show the uniqueness of the solution of our problem.

Theorem 3.3. *Under the hypotheses in 3.1. Then for all $\nu > -1$, the solution u, given by 3.1, is unique.*

Proof. Let u, v be two solutions to problem (2.1)–(2.4). Define $w = u - v$ and using 3.1, then w satisfy the following system

$$\begin{gathered} w'' - \lambda div\left(F(\varepsilon(u)) - F(\varepsilon(v))\right) - \mu div\left(G\left(\varepsilon(u')\right) - G\left(\varepsilon(v')\right)\right) + \\ +(|u|^{\nu} u - |v|^{\nu} v) = 0 \text{ in } Q, \end{gathered} \tag{3.28}$$

$$w(0) = w'(0) = 0 \text{ in } \Omega, \tag{3.29}$$

$$w = 0 \text{ on } \Sigma_1,\ \sigma(w)\eta = 0 \text{ on } \Sigma_2, \tag{3.30}$$

$$w \in L^{\infty}(0, T; V \cap L^p(\Omega)),\ p = \nu + 2, \tag{3.31}$$

$$w' \in L^{\infty}\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; V\right). \tag{3.32}$$

Multiplying the equation (3.28) by w' and integrating over Ω. Then, using *Green*'s formula and (3.30) we obtain

$$\begin{gathered} \tfrac{1}{2}\tfrac{d}{dt}\left|w'(t)\right|^2 + \lambda\left(F(\varepsilon(u)) - F(\varepsilon(v)), \varepsilon(u') - \varepsilon(v')\right) + \\ +\mu\left(G\left(\varepsilon(u')\right) - G\left(\varepsilon(v')\right), \varepsilon(u') - \varepsilon(v')\right) = \int_{\Omega}\left(|v|^{\nu} v - |u|^{\nu} u\right) w' dx. \end{gathered} \tag{3.33}$$

Now, we use (2.5), (3.29) and (3.8) to obtain

$$\begin{aligned} \left(F(\varepsilon(u)) - F(\varepsilon(v)), \varepsilon(u') - \varepsilon(v')\right) &= \frac{d}{dt}\left(F(\varepsilon(u)) - F(\varepsilon(v)), \varepsilon(u) - \varepsilon(v)\right) - \\ &- \left(\frac{d}{dt}\left(F(\varepsilon(u)) - F(\varepsilon(v))\right), \varepsilon(u) - \varepsilon(v)\right) \geq \\ &\geq C_1\frac{d}{dt}\left\|w\right\|^2 - \left(\frac{d}{dt}\left(F(\varepsilon(u)) - F(\varepsilon(v))\right), \varepsilon(u) - \varepsilon(v)\right). \end{aligned}$$

Then, from (2.6), there exists a constant $C_2 > 0$ such that

$$\left(G\left(\varepsilon(u')\right) - G\varepsilon(v'), \varepsilon(u') - \varepsilon(v')\right) \geq C_2\left\|w'\right\|^2.$$

Therefore (3.33) takes the form

$$\begin{gathered} \tfrac{d}{dt}\left(\tfrac{1}{2}\left|w'(t)\right|^2 + \lambda C_1\left\|w(t)\right\|^2\right) + \mu C_2\left\|w'(t)\right\|^2 \leq \int_{\Omega}(|v|^{\nu} v - |u|^{\nu} u) w' dx + \\ + \int_{\Omega}\frac{d}{dt}\left(F(\varepsilon(u)) - F(\varepsilon(v))\right)\left(\varepsilon(u) - \varepsilon(v)\right) dx. \end{gathered} \tag{3.34}$$

From hypotheses (2.5) we have the following estimates:

$$\begin{gathered} \left|\int_{\Omega}\frac{d}{dt}\left(F(\varepsilon(u)) - F(\varepsilon(v))\right)\left(\varepsilon(u)\right) - \left(\varepsilon(v)\right) dx\right| \leq \\ \leq L\int_{\Omega}\left|\varepsilon(u') - \varepsilon(v')\right|\left|\varepsilon(u) - \varepsilon(v)\right| dx \leq C_3\left\|w'\right\|\left\|w\right\|. \end{gathered} \tag{3.35}$$

Now, as in [4], we estimate the first term on the right hand side of (3.34) as follows

$$\left|\int_{\Omega}(|v|^{\nu} v-|u|^{\nu} u)w'dx\right| \leq (\nu+1)\int_{\Omega}\sup\left(|u|^{\nu},|v|^{\nu}\right)|w|\,|w'|\,dx.$$

Using *Holder*'s inequality with $\frac{1}{n}+\frac{1}{q}+\frac{1}{2}=1$, we get

$$\left|\int_{\Omega}(|v|^{\nu} v-|u|^{\nu} u)w'dx\right| \leq C(\nu+1)\left(\||u|^{\nu}\|_{L^n(\Omega)}+\||v|^{\nu}\|_{L^n(\Omega)}\right)||w(t)||_{L^q(\Omega)}|w'(t)|,$$

where C is a positive constant.

Also, as in [1], for all $k,\ q\in\mathbb{N}^*$, we have

$$\|v\|_{L^{kq}(\Omega)}=\left\||v|^k\right\|_{L^q(\Omega)}^{\frac{1}{k}}. \tag{3.36}$$

Finally, For all $n>2$ and $\nu>-1$, putting $k=E\left(\frac{\nu(n-2)}{2}\right)+1$, where $E(x)$ is the integer part of x, then k satisfies

$$\nu\leq\frac{2k}{n-2},\ k\in\mathbb{N}^*,\ n\neq 2. \tag{3.37}$$

Thus, $\nu n\leq kq$.

Then, using (3.36), (3.37) and the *Sobolev* embedding $H^1(\Omega)\hookrightarrow L^q(\Omega)$, we arrive at

$$\begin{aligned}\||v|^{\nu}\|_{L^n(\Omega)}=\|v\|_{L^{\nu n}(\Omega)}^{\nu}\leq\|v\|_{L^{kq}(\Omega)}^{\nu}=\left\||v|^k\right\|_{L^q(\Omega)}^{\frac{\nu}{k}}\leq\\ \leq C\|v\|_{L^q(\Omega)}^{\nu}\leq C\|v\|^{\nu}.\end{aligned}$$

Thus we have

$$\||v|^{\nu}\|_{L^n(\Omega)}\leq C\|v\|^{\nu}, \tag{3.38}$$

which implies that

$$\left|\int_{\Omega}\left(|v|^{\nu} v-|u|^{\nu} u\right)w'dx\right|\leq C(\nu+1)\left(||u||^{\nu}+\|v\|^{\nu}\right)||w||\,|w'|.$$

Since $u,\,v\in L^{\infty}(0,T;V\cap L^p(\Omega))$, then there exists $C_4>0$ such that

$$\left|\int_{\Omega}(|v|^{\nu} v-|u|^{\nu} u)w'dx\right|\leq C_4||w||\,|w'|. \tag{3.39}$$

Combining (3.34), (3.35) and (3.39) and using *Young*'s, *Holder*'s inequalities to obtain

$$\begin{aligned}\tfrac{d}{dt}\left(\tfrac{1}{2}|w'(t)|^2+\lambda C_1\|w(t)\|^2\right)+\mu C_2\|w'(t)\|^2\leq\\ \leq C_3\|w'(t)\|\,\|w(t)\|+C_4||w(t)||\,|w'(t)|\leq\tfrac{1}{2}\mu C_2\|w'(t)\|^2+\\ +C_5\|w(t)\|^2+\tfrac{1}{2}C_6\left(|w'(t)|^2+\|w(t)\|^2\right).\end{aligned}$$

Integrating the above inequality over $(0,t)$ and using (3.29), we get

$$\begin{aligned}\tfrac{1}{2}|w'(t)|^2+\lambda C_1\|w(t)\|^2+\tfrac{1}{2}\mu C_2\int_0^t\|w'(s)\|^2ds\leq\\ \leq 0+C_7\int_0^t\left(|w'(s)|^2+\|w(s)\|^2\right)ds.\end{aligned}$$

Using *Gronwall*'s inequality to deduce that $w=0$. □

4 Continuous dependence with respect to the initial data

Our goal in this section is to show the continuous dependence with respect to the initial data of solutions.

Let $W(Q)=\left\{\varphi\in L^{\infty}(0,T;V)\,/\,\varphi'\in L^{\infty}\left(0,T;L^2(\Omega)\right)\right\}$ be the *Banach* space equipped with the norm

$$\|\varphi\|_{W(Q)}=\|\varphi\|_{L^{\infty}(0,T;V)}+\|\varphi'\|_{L^{\infty}(0,T;L^2(\Omega))}.$$

Let $u \in W(Q)$ be the solution of (2.1)–(2.4) associated to the initial data $\{f, u_0, u_1\}$. Then, using 3.1 and 3.3, we can define the application π as follows:

$$\begin{cases} \pi : L^2(Q) \times V \times L^2(\Omega) \to W(Q) \\ \{f, u_0, u_1\} \mapsto u. \end{cases} \tag{4.1}$$

Our main result announces as follows.

Theorem 4.1. *Let $u, v \in W(Q)$ be solutions of (2.1)–(2.4) associated to the initial data $\{f, u_0, u_1\}, \{g, v_0, v_1\} \in L^2(Q) \times V \times L^2(\Omega)$, respectively. Under the assumptions in 3.1 and 3.3. Then, the application π defined by (4.1) is continuous, i.e. there exists a function $C(u, v)$ such that*

$$|u'(t) - v'(t)|^2 + \|u(t) - v(t)\|^2 \le C(u, v) \left(|u_1 - v_1|^2 + \|u_0 - v_0\|^2 + \int_0^t |(f - g)(s)|^2 ds \right).$$

Proof. Let $u, v \in W(Q)$ solutions of (2.1)–(2.4) associated to the initial data $\{f, u_0, u_1\}, \{g, v_0, v_1\} \in L^2(Q) \times V \times L^2(\Omega)$.

Then, putting $w = u - v$ we get

$$\begin{aligned} w'' - \lambda div\left(F(\varepsilon(u)) - F(\varepsilon(v))\right) - \mu div\left(G(\varepsilon(u')) - G(\varepsilon(v'))\right) = \\ = (f - g) + \left(|v|^\nu v - |u|^\nu u\right). \end{aligned}$$

Multiplying the last equation by w', using *Green*'s formula and (3.30) we arrive at

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|w'(t)|^2 + \lambda \int_\Omega \left(F(\varepsilon(u)) - F(\varepsilon(v))\right)\varepsilon(w')dx + \mu \int_\Omega \left(G(\varepsilon(u')) - G(\varepsilon(v'))\right)\varepsilon(w')dx = \\ = (f - g, w') + \int_\Omega \left(|v|^\nu v - |u|^\nu u\right) w' dx. \end{aligned} \tag{4.2}$$

Thus, from (2.5) we conclude that

$$\begin{aligned} \int_\Omega \left(F(\varepsilon(u)) - F(\varepsilon(v))\right)\varepsilon(w')dx \ge \\ \ge \frac{d}{dt} C_1 \|w\|^2 - \int_\Omega \left(\frac{d}{dt}\left(F(\varepsilon(u)) - F(\varepsilon(v))\right)\right)\varepsilon(w)dx \end{aligned}$$

and

$$\left| \int_\Omega \left(\frac{d}{dt}\left(F(\varepsilon(u)) - F(\varepsilon(v))\right)\right)\varepsilon(w)dx \right| \le C_3 \|w'\| \|w\|.$$

On the other hand, by hypotheses (2.6) we have

$$\int_\Omega \left(G(\varepsilon(u')) - G(\varepsilon(v'))\right)\varepsilon(w')dx \ge C_2 \|w'\|^2.$$

Therefore, using *Holder*'s, *Young*'s inequalities and integrating the result over $(0, t)$ from (4.2), we get

$$\begin{aligned} \tfrac{1}{2}|w'(t)|^2 + \lambda C_1 \|w(t)\|^2 + \mu C_2 \int_0^t \|w'(s)\|^2 ds \le \tfrac{1}{2}|u_1 - v_1|^2 + \lambda C_1 \|u_0 - v_0\|^2 + \\ + \tfrac{1}{2}\int_0^t |(f - g)(s)|^2 ds + \tfrac{1}{2}\int_0^t |w'(s)|^2 ds + \tfrac{1}{2}C_4 \int_0^t \|w(s)\|^2 ds + \\ + \tfrac{1}{2}\mu C_2 \int_0^t \|w'(s)\|^2 ds + \int_0^t \int_\Omega \left(|v(s)|^\nu v(s) - |u(s)|^\nu u(s)\right) w'(s)\, dx ds. \end{aligned} \tag{4.3}$$

Combining these estimates (3.39) and (4.3) to deduce

$$\begin{aligned} |w'(t)|^2 + 2\lambda C_1 \|w(t)\|^2 + \mu C_2 \int_0^t \|w'(s)\|^2 ds \le |u_1 - v_1|^2 + 2C_1\lambda \|u_0 - v_0\|^2 + \\ + \int_0^t |(f - g)(s)|^2 ds + \int_0^t |w'(s)|^2 ds + C_4 \int_0^t \|w(s)\|^2 ds + C_5 \int_0^t \left(||w(s)||^2 + |w'(s)|^2\right) ds. \end{aligned}$$

Consequently, the last inequality implies

$$|w'(t)|^2 + 2\lambda C_1 \|w(t)\|^2 \leq |u_1 - v_1|^2 + 2\lambda C_1 \|u_0 - v_0\|^2 + \\ + \int_0^t |(f-g)(s)|^2 ds + 2C_6 \int_0^t \left(|w'(s)|^2 + \|w(s)\|^2\right) ds.$$

Finally, using *Gronwall*'s inequality to arrive at

$$|w'(t)|^2 + \|w(t)\|^2 \leq C(u,v) \left(|u_1 - v_1|^2 + \|u_0 - v_0\|^2 + \int_0^t |(f-g)(s)|^2 ds\right),$$

where $C(u,v)$ is a bounded function in Q. Thus, the proof of 3.1 is completed. □

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