ON SEQUENCES OF NUMBERS IN GENERALIZED ARITHMETIC AND GEOMETRIC PROGRESSIONS

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Abstract The paper provides a further generalization of the sequences of numbers in generalized arithmetic and geometric progressions [1].

1 Introduction

The usual arithmetic sequence of numbers takes the form:

$$a, a + d, a + 2d, a + 3d, \ldots, a + (n-1)d, a + nd, \ldots$$

while the geometric sequence of numbers has the form

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}, ar^n, \ldots$$

Formally speaking, an arithmetic sequence is a number sequence in which every term except the first is obtained by adding a fixed number, called the common difference, to the preceding term and a geometric sequence is a number sequence in which every term except the first is obtained by multiplying the previous term by a constant, called the common ratio. The sequence $1, 3, 5, 7, 9, 11, \ldots$ is an example of arithmetic sequence with common difference 2 and the sequence $2, 4, 8, 16, \ldots$ is a geometric sequence with common ratio 2. Certain generalizations of arithmetic and geometric sequence were presented in [1], [3], [4]. Particularly, in [3], Zhang and Zhang introduced the concept of sequences of numbers in arithmetic progression with alternate common differences and in [4], Zhang, et.al. provided a generalization of the sequence. It was then extended by Majumdar [1] to sequences of numbers in geometric progression with alternate common ratios and the periodic sequence with two common ratios. The author [1] also provided a simpler and shorter forms and proofs of some cases of the results presented by Zhang and Zhang in [3]. Recently, Rabago [2] further generalized these concepts by introducing additional common differences and common ratios. Here we will provide another generalization of the sequences of numbers defined in [1] and [3] by providing a definition to what we call sequences of numbers with $m$ alternate common differences (Section 2) and sequence of numbers with $m$ alternate common ratios (Section 3).

Throughout in the paper we denote the greatest integer contained in $x$ as $\lfloor x \rfloor$.

2 Sequence of numbers with $m$ alternate common differences

We start-off this section with the definition of what we call sequence of numbers with $m$ alternate common differences.

Definition 2.1. A sequence of numbers $\{a_n\}$ is called a sequence of numbers with $m$ alternate common differences if for a fixed natural number $m$ and for all $j = 1, 2, \ldots, m$,

$$a_{m(k-1)+j+1} - a_{m(k-1)+j} = d_j,$$

for all $k \in \mathbb{N}$. Here $d_j$ is the $j$-th common difference of $\{a_n\}$.

With the above definition, a sequence of numbers with $m$ alternate common differences takes the following form:

$$a, a + d_1, a + d_1 + d_2, \ldots, a + d_1 + d_2 + \cdots + d_m, a + 2d_1 + d_2 + \cdots + d_m,$$

$$a + 2d_1 + 2d_2 + \cdots + d_m, a + 2d_1 + 2d_2 + \cdots + 2d_m, \ldots$$

(2.1)
The sequence
\[2, 3, 5, 8, 9, 11, 14, 15, 17, 20, \ldots\]
is an example of a sequence of numbers with 3 alternate common differences. The common differences are \(d_1 = 1, d_2 = 2,\) and \(d_3 = 3.\)

**Theorem 2.2.** Let \(\{a_n\}\) be a sequence of number that takes the form (2.1). Then, the formula for the \(n^{th}\) term of the sequence \(\{a_n\}\) is given by

\[a_n = a_1 + \sum_{i=1}^{m} \left[ \frac{n + (m - 1) - i}{m} \right] d_i. \tag{2.2}\]

**Proof.** Obviously, (2.2) holds for \(n \leq m.\) We only need to show that (2.2) is true for \(n > m\) to prove the validity of the formula. Suppose (2.2) holds for some natural number \(k.\) Hence,

\[a_k = a_1 + \sum_{i=1}^{m} \left[ \frac{k + (m - 1) - i}{m} \right] d_i.\]

Let \(k = m(p - 1) + j\) and \(p \in \mathbb{N}.\) Now, for every \(j = 1, 2, \ldots, m \in \mathbb{N},\) we have \(a_{k+1} = a_k + d_j.\)

Thus,

\[a_{k+1} = a_1 + \sum_{i=1}^{m} \left[ \frac{k + (m - 1) - i}{m} \right] d_i + d_j = a_1 + \sum_{i=1}^{m} \left[ \frac{m(p - 1) + j + (m - 1) - i}{m} \right] d_i + d_j = a_1 + \sum_{i=1}^{m} \left[ \frac{p + j - 1 - i}{m} \right] d_i + \sum_{i=j}^{m} \left[ \frac{p + j - 1 - i}{m} \right] d_i + d_j = a_1 + \sum_{i=1}^{j} pd_i + \sum_{i=j+1}^{m} (p-1)d_i + d_j = a_1 + \sum_{i=1}^{j} pd_i + \sum_{i=j+1}^{m} (p-1)d_i = a_1 + \sum_{i=1}^{j} \left[ \frac{p + j - 1 - i}{m} \right] d_i + \sum_{i=j+1}^{m} \left[ \frac{p + j - 1 - i}{m} \right] d_i + \sum_{i=j+1}^{m} \left[ \frac{m(p - 1) + j + (m - 1) - i}{m} \right] d_i = a_1 + \sum_{i=1}^{m} \left[ \frac{(k + 1) + (m - 1) - i}{m} \right] d_i.
\]

\[\square\]

Below is a table of formulas for the \(n^{th}\) term \(a_n\) of the given sequence for specific values of \(m.\)

<table>
<thead>
<tr>
<th>(m)</th>
<th>(n^{th}) term (a_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(a_1 + (n - 1)d)</td>
</tr>
<tr>
<td>2</td>
<td>(a_1 + \frac{n}{2}d_1 + \frac{n-1}{2}d_2)</td>
</tr>
<tr>
<td>3</td>
<td>(a_1 + \frac{n+1}{3}d_1 + \frac{n}{3}d_2 + \frac{n-1}{3}d_3)</td>
</tr>
<tr>
<td>4</td>
<td>(a_1 + \frac{n+2}{4}d_1 + \frac{n+1}{4}d_2 + \frac{n}{4}d_3 + \frac{n-1}{4}d_4)</td>
</tr>
<tr>
<td>5</td>
<td>(a_1 + \frac{n+3}{5}d_1 + \frac{n+2}{5}d_2 + \frac{n+1}{5}d_3 + \frac{n}{5}d_4 + \frac{n-1}{5}d_5)</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
</tr>
</tbody>
</table>
Corollary 2.3. Let $m$ and $n$ be natural numbers. If $m|\(n - 1\)$ then we have

$$a_n = a_1 + \left(\frac{n - 1}{m}\right) \sum_{i=1}^{m} d_i.$$

Proof. Suppose $m|\(n - 1\)$ then $n - 1 = mk$ for some $k \in \mathbb{N}$. Then,

$$a_n = a_1 + \sum_{i=1}^{m} \left[ k + \frac{m - i}{m} \right] d_i = a_1 + k \sum_{i=1}^{m} d_i = a_1 + \left(\frac{n - 1}{m}\right) \sum_{i=1}^{m} d_i.$$ 

□

Corollary 2.4. If $m|n$, we have

$$a_n = a_1 + \left(\frac{n}{m}\right) \sum_{i=1}^{m} d_i - d_m.$$ 

Proof. Suppose $m|n$ then $n = mk$ for some $k \in \mathbb{N}$. So,

$$a_n = a_1 + \sum_{i=1}^{m} \left[ k + \frac{1 - i}{m} \right] d_i = a_1 + k \sum_{i=1}^{m} d_i - d_m = a_1 + \left(\frac{n}{m}\right) \sum_{i=1}^{m} d_i - d_m.$$ 

□

Lemma 2.5. For any natural numbers $m$ and $n$, we have

$$\sum_{i=1}^{m} \left\lfloor \frac{n + (m - 1) - i}{m} \right\rfloor = n - 1 = \sum_{i=1}^{m} \left\lfloor \frac{i + n - 2}{m} \right\rfloor.$$

Proof. Note that

$$\left\lfloor \frac{n}{m} \right\rfloor = k \quad \Rightarrow \quad k \leq \frac{n}{m} < k + 1.$$

Hence,

$$\sum_{i=1}^{m} \left\lfloor \frac{n + (m - 1) - i}{m} \right\rfloor = \left\lfloor \frac{n + (m - 2)}{m} \right\rfloor + \ldots + \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{n - 1}{m} \right\rfloor$$

$$= k + k + \ldots + k + (k - 1) = mk - 1 = n - 1.$$ 

□

Theorem 2.6. If $d_i = d_1$ for $i \leq m - 1$, we have

$$a_n = a_1 + \left( n - 1 - \left\lfloor \frac{n - 1}{m} \right\rfloor \right) d_1 + \left\lfloor \frac{n - 1}{m} \right\rfloor d_m. \quad (2.3)$$

Proof. Let $d_i = d_1$, for $i \leq m - 1$, in (2.2). Hence,

$$a_n = a_1 + \sum_{i=1}^{m} \left\lfloor \frac{n + (m - 1) - i}{m} \right\rfloor d_i$$

$$= a_1 + d_1 \sum_{i=1}^{m-1} \left\lfloor \frac{n + (m - 1) - i}{m} \right\rfloor + \left\lfloor \frac{n - 1}{m} \right\rfloor d_m$$

$$= a_1 + d_1 \left( \left\lfloor \frac{n + (m - 2)}{m} \right\rfloor + \left\lfloor \frac{n + (m - 3)}{m} \right\rfloor + \ldots + \left\lfloor \frac{n}{m} \right\rfloor \right) + \left\lfloor \frac{n - 1}{m} \right\rfloor d_m$$

$$= a_1 + d_1 \left\{ \sum_{i=1}^{m} \left\lfloor \frac{n + (m - 1) - i}{m} \right\rfloor - \left\lfloor \frac{n - 1}{m} \right\rfloor \right\} + \left\lfloor \frac{n - 1}{m} \right\rfloor d_m$$

$$= a_1 + \left( n - 1 - \left\lfloor \frac{n - 1}{m} \right\rfloor \right) d_1 + \left\lfloor \frac{n - 1}{m} \right\rfloor d_m.$$ 

□
Theorem 2.7. Let \( \{a_n\} \) be a sequence of number that takes the form (2.1). Then, the formula for the sum of the first \( n \) terms of the sequence \( \{a_n\} \) is given by

\[
S_n = na_1 + \sum_{i=1}^{m} \left( N - i \right) \left( N - m \left[ \frac{N - i}{m} + 1 \right] \right) d_i,
\]

(2.4)

where \( N = n + m - 1 \).

Proof. Consider a sequence \( \{a_n\} \) that takes the form (2.1). Then,

\[
\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} \left( a_1 + \sum_{i=1}^{m} \left( \frac{j + (m - 1) - i}{m} \right) d_i \right) = na_1 + \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \frac{j + (m - 1) - i}{m} \right) d_i
\]

\[
= na_1 + \sum_{i=1}^{m} \left( \frac{n + (m - 1) - i}{m} \right) \left( n + m - i \right) \left( \frac{n + 2m - (1 + i)}{m} \right) d_i
\]

Letting \( N = n + m - 1 \), conclusion follows. \( \square \)

Theorem 2.8. The sum of the first \( n \) terms of the sequence \( \{a_n\} \) that takes the form (2.1) with \( d_i = d_1 \), for \( i \leq m - 1 \), is given by

\[
S_n = na_1 + \frac{n(n - 1)}{2} d_1 + \left( d_m - d_1 \right) \left( \frac{n - 1}{m} \right) \left( \frac{n - m - 1}{m} \right).
\]

(2.5)

Proof. Consider a sequence \( \{a_n\} \) that takes the form (2.1). Then,

\[
\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} \left( a_1 + \left( j - 1 - \left[ \frac{j - 1}{m} \right] \right) d_1 + \left[ \frac{j - 1}{m} \right] d_m \right)
\]

\[
= na_1 + \frac{n(n - 1)}{2} d_1 - \sum_{j=1}^{n} \left[ \frac{j - 1}{m} \right] d_1 + \sum_{j=1}^{n} \left[ \frac{j - 1}{m} \right] d_m
\]

\[
= na_1 + \frac{n(n - 1)}{2} d_1 + \left( d_m - d_1 \right) \sum_{j=1}^{n} \left[ \frac{j - 1}{m} \right]
\]

\[
= na_1 + \frac{n(n - 1)}{2} d_1 + \left( d_m - d_1 \right) \left( \frac{n - 1}{m} \right) \left( \frac{n - m - 1}{m} \right).
\]

\( \square \)

3 Sequence of numbers with \( m \) alternate common ratios

We define the sequence of numbers with \( m \) alternate common ratios \( \{a_n\} \) as follows:

Definition 3.1. A sequence of numbers \( \{a_n\} \) is called a sequence of numbers with \( m \) alternate common ratios if for a fixed natural number \( m \) and for all \( j = 1, 2, \ldots, m \),

\[
\frac{a_m(k-1)+j+1}{a_m(k-1)+j} = r_j,
\]

for all \( k \in \mathbb{N} \). Here \( r_j \) is the \( j \)-th common ratio of \( \{a_n\} \).

With the above definition, we can see immediately that a sequence of numbers \( \{a_n\} \) with \( m \) alternate common ratios has the following form:

\[
a, ar_1, ar_1r_2, \ldots, ar_1r_2 \cdots r_m, ar_1r_2^2 \cdots r_m, ar_1r_2^2 \cdots r_m, \ldots
\]

(3.1)

The sequence

\[1, 2, 6, 24, 48, 144, 576, 1152, \ldots\]

is an example of a sequence of numbers \( \{a_n\} \) with 3 alternate common ratios. The common ratios are \( r_1 = 2, r_2 = 3, \) and \( r_3 = 4 \).
Theorem 3.2. Let \( \{a_n\} \) be a sequence of numbers that takes the form (3.1). Then, the formula for the \( n \)th term of the sequence \( \{a_n\} \) is given by

\[
a_n = a_1 \prod_{i=1}^{m} r_i^{e_i},
\]

where \( e_i = \left\lfloor \frac{n+(m-1)-i}{m} \right\rfloor \).

In particular, if \( m|(n-1) \), we have

\[
a_n = a_1 \left( \prod_{i=1}^{m} r_i^{(\frac{n-1}{m})} \right),
\]

and if \( m|(n-1) \),

\[
a_n = a_1 r_m \left( \prod_{i=1}^{m} r_i^{(\frac{n}{m})} \right).
\]

Proof. The proof is by induction on \( n \). Obviously, (3.2) holds for \( n \leq m \). We will show that (3.2) is true for \( n > m \). Suppose (3.2) holds when for some natural number \( k \). That is,

\[
a_k = a_1 \prod_{i=1}^{m} r_i^{e_i},
\]

where \( e_i = \left\lfloor \frac{m(p-1)+j+(m-1)-i}{m} \right\rfloor \).

Let \( k = m(p-1)+j \) and \( p \in \mathbb{N} \). Now, for every \( j = 1, 2, \ldots, m \in \mathbb{N} \), we have \( a_{k+1} = a_k \cdot r_j \). Thus,

\[
a_{k+1} = a_1 \prod_{i=1}^{m} r_i^{e_i} \cdot r_j,
\]

where \( e_i = \left\lfloor \frac{m(p-1)+j+(m-1)-i}{m} \right\rfloor \)

\[
= a_1 \prod_{i=1}^{j-1} r_i^{e_i} \cdot \prod_{i=j}^{m} r_i^{e_i} \cdot r_j,
\]

where \( e_i = \left\lfloor p + \frac{j-1-i}{m} \right\rfloor \)

\[
= a_1 \prod_{i=1}^{j-1} r_i^{f_i} \cdot \prod_{i=j}^{m} r_i^{g_i} \cdot r_j,
\]

where \( f_i = \left\lfloor \frac{p + \frac{j-i}{m}}{m} \right\rfloor \), \( g_i = \left\lfloor \frac{p-1 + \frac{m+j-i}{m}}{m} \right\rfloor \)

\[
= a_1 \prod_{i=1}^{j-1} r_i^{h_i},
\]

where \( h_i = \left\lfloor \frac{(k+1)+(m-1)-i}{m} \right\rfloor \).

If \( m|n \) (resp. \( m|(n-1) \)) then (3.3) (resp. (3.4)) follows immediately.

Theorem 3.3. Consider a sequence \( \{a_n\} \) that takes the form (3.1) and suppose \( r_i = r_1 \), for \( i \leq m-1 \). Then,

\[
a_n = a_1 r_1^{e} \cdot r_m^{s},
\]

where \( e = n-1 - \left\lfloor \frac{n-1}{m} \right\rfloor \) and \( s = \left\lfloor \frac{n-1}{m} \right\rfloor \).

Proof. Let \( r_1 = r_s \), for \( i \leq m-1 \), in (3.2). Hence,

\[
a_n = a_1 \prod_{i=1}^{m} r_i^{e_i},
\]

where \( e_i = \left\lfloor \frac{n + (m-1) - i}{m} \right\rfloor \)

\[
= a_1 r_1^{e} \cdot r_m^{s},
\]

where \( e = \sum_{i=1}^{m-1} \left\lfloor \frac{n + (m-1) - i}{m} \right\rfloor \) and \( s = \left\lfloor \frac{n-1}{m} \right\rfloor \).

But, by Lemma (2.5), \( e = n-1 - \left\lfloor \frac{n-1}{m} \right\rfloor \). Thus, \( a_n = a_1 r_1^{s} \cdot r_m^{s} \).
Theorem 3.4. Let \( \{a_n\} \) be a sequence of number that takes the form (3.1). Then, the formula for the sum of the first \( n \) terms of the sequence \( \{a_n\} \) is given by

\[
S_n = a_1 \left( 1 + R \left( \frac{1 - r^{n+1}}{1 - r} \right) \right) + a_1 r^{n-1} \sum_{i=1}^{m-1} \left( \left\lfloor \frac{N - i}{m} \right\rfloor - \left\lfloor \frac{N - 1 - i}{m} \right\rfloor \right) \prod_{k=1}^{i} r_k,
\]

where \( R = \sum_{i=1}^{m} \prod_{j=1}^{i} r_j, \quad r = \prod_{i=1}^{m} r_i, \quad e_{n-1} = \left\lfloor \frac{n-1}{m} \right\rfloor \) and \( N = n + m - 1 \).

Proof. Consider a sequence \( \{a_n\} \) that takes the form (3.1) and let \( R = \sum_{i=1}^{m} \prod_{j=1}^{i} r_j, \quad r = \prod_{i=1}^{m} r_i, \quad p = e_{n-1} = \left\lfloor \frac{n-1}{m} \right\rfloor \) then

\[
\sum_{j=1}^{n} a_j = a_1 \sum_{j=1}^{n} \prod_{i=1}^{j} r_i^{e_j} \quad \text{where } e_j = \left\lfloor \frac{j+(m-1)-i}{m} \right\rfloor.
\]

Expanding the expression, we obtain

\[
\sum_{j=1}^{n} a_j = a_1 + a_1 R \sum_{j=0}^{p-1} r_j^{e_n(m-3)} r_j^{e_n(m-4)} \cdots r_j^{e_{n-2}} + \cdots + a_1 r_1^{e_n(m-2)} r_1^{e_n(m-1)} \cdots r_1^{e_{n-1}}.
\]

Simplifying and rewriting the expression in compact form, we obtain

\[
\sum_{j=1}^{n} a_j = a_1 \left( 1 + R \left( \frac{1 - r^{p}}{1 - r} \right) \right) + a_1 r^{p-1} \sum_{i=1}^{m} M_i \prod_{k=1}^{i} r_k
\]

where \( M_i = \left( \left\lfloor \frac{i-1}{m} \right\rfloor - \left\lfloor \frac{i-1-1}{m} \right\rfloor \right) \) which is the desired result. \( \square \)

Theorem 3.5. Let \( \{a_n\} \) be a sequence of number that takes the form (3.1) with \( r_i = r_1, \) for all \( i \leq m - 1 \). Then, the formula for the sum of the first \( n \) terms of the sequence \( \{a_n\} \) is given by

\[
S_n = a_1 \left( \frac{1 - r^{n+1}}{1 - r} \right) \left( \frac{1 - r^{n-1} r_m}{1 - r_1 r_m} \right) + a_1 r^{n-1} r_m \left( \frac{1 - r^{n-mp}}{1 - r_1 r_m} \right),
\]

where \( p = \left\lfloor \frac{n-1}{m} \right\rfloor \).

Proof. Consider a sequence \( \{a_n\} \) that takes the form (3.1) with \( r_i = r_1, \) for all \( i \leq m - 1 \) and let \( p = \left\lfloor \frac{n-1}{m} \right\rfloor \),

\[
\sum_{j=1}^{n} a_j = a_1 \sum_{j=1}^{n} r_1^{j-1} \left( \frac{r_m}{r_1} \right)^{\left\lfloor \frac{j-1}{m} \right\rfloor}
\]

\[
= a_1 \left\{ \sum_{j=1}^{m} r_1^{j-1} + \left( \frac{r_m}{r_1} \right) \sum_{j=m+1}^{2m} r_1^{j-1} + \left( \frac{r_m}{r_1} \right)^2 \sum_{j=2m+1}^{3m} r_1^{j-1} + \cdots \right\} + a_1 \left( \frac{r_m}{r_1} \right)^{n-1} \sum_{j=m+1}^{n} r_1^{j-1}
\]

\[
= a_1 \left\{ \sum_{j=1}^{m} r_1^{j-1} + \left( \frac{r_m}{r_1} \right) \sum_{j=1}^{m} r_1^{j-1} + \left( \frac{r_m}{r_1} \right)^2 \sum_{j=1}^{m} r_1^{j-1} + \cdots \right\} + a_1 \left( \frac{r_m}{r_1} \right)^{n-mp} \sum_{j=1}^{n} r_1^{j-1}
\]

\[
= a_1 \left( \frac{1 - r_m}{1 - r_1} \right) \left\{ 1 + \left( \frac{r_m}{r_1} \right) + \left( \frac{r_m}{r_1} \right)^2 + \cdots + \left( \frac{r_m}{r_1} \right)^{p-1} \right\}
\]

\[
+ a_1 \left( \frac{r_m}{r_1} \right)^{n-mp} \sum_{j=1}^{n} r_1^{j-1}
\]

which is desired. \( \square \)
4 Some Remarks

If we replace \(m\) by \(t\) in (2.2) and define \(t\) as the period of the sequence \(\{a_n\}\) and by considering \(d_i = d_i^1\) for \(i \leq m - 1\) as the first common difference of the sequence and \(d_m = d_m^2\) as the second difference then we obtain,

\[
a_n = a_1 + \left( n - 1 - \left \lfloor \frac{n - 1}{t} \right \rfloor \right) d_1^1 + \left \lfloor \frac{n - 1}{t} \right \rfloor d_2^1.
\] (4.1)

Equation (4.1) is exactly the formula for the \(n^{th}\) term of a periodic number sequence with two common differences obtained by Zhang and Zhang in [4]. Furthermore, it can be observed from (4.1) that \(a_n \to a_1 + (n-1)d_1\) as \(m \to \infty\). Similarly, if \(d_i = d_1\) for all \(i \leq m\), \(a_n = a_1 + (n-1)d_1\).

In (2.5), on the other hand, would have \(S_n = na_1 + \frac{n(n-1)}{2}d_1\) if \(m \to \infty\) and a similar result will be obtained if \(d_i = d_1\) for all \(i \leq m\).

Also, note that in (3.2), \(a_n \to a_1 r_1^{n-1}\) if we apply the same argument letting either \(m \to \infty\) or \(r_i = r_1\) for all \(i \leq m\). Furthermore, the limit of the sum given by

\[
\sum_{j=1}^{n} a_1 r_1^{j-1} \left( \frac{r_m}{r_1} \right)^{\left \lfloor \frac{n-j}{m} \right \rfloor} \to a_1 \left( \frac{1 - r_1^n}{1 - r_1} \right) \text{ as } m \to \infty.
\]

References


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