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ON SEQUENCES OF NUMBERS IN GENERALIZED ARITHMETIC AND GEOMETRIC PROGRESSIONS

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Abstract The paper provides a further generalization of the sequences of numbers in generalized arithmetic and geometric progressions [1].

1 Introduction

The usual *arithmetic sequence* of numbers takes the form:

$$a, a+d, a+2d, a+3d, \dots, a+(n-1)d, a+nd, \dots$$

while the *geometric sequence* of numbers has the form

$$a, ar, ar^2, ar^3, \dots, ar^{n-1}, ar^n, \dots.$$

Formally speaking, an arithmetic sequence is a number sequence in which every term except the first is obtained by adding a fixed number, called the *common difference*, to the preceeding term and a geometric sequence is a number sequence in which every term except the first is obtained by multiplying the previous term by a constant, called the *common ratio.* The sequence $1, 3, 5, 7, 9, 11, \dots$ is an example of arithmetic sequence with common difference 2 and the sequence $2, 4, 8, 16, \dots$ is a geometric sequence with common ratio 2. Certain generalizations of arithmetic and geometric sequence were presented in [1], [3], [4]. Particularly, in [3], Zhang and Zhang introduced the concept of sequences of numbers in arithmetic progression with alternate common differences and in [4], Zhang, et.al. provided a generalization of the sequence. It was then extended by Majumdar [1] to sequences of numbers in geometric progression with alternate common ratios and the periodic sequence with two common ratios. The author [1] also provided a simpler and shorter forms and proofs of some cases of the results presented by Zhang and Zhang in [3]. Recently, Rabago [2] further generalized these concepts by introducing additional common differences and common ratios. Here we will provide another generalization of the sequences of numbers defined in [1] and [3] by providing a definition to what we call sequences of numbers with m alternate common differences (Section 2) and sequence of numbers with m alternate common ratios (Section 3).

Throughout in the paper we denote the greatest integer contained in x as $\lfloor x \rfloor$.

2 Sequence of numbers with m alternate common differences

We start-off this section with the definition of what we call sequence of numbers with m alternate common differences.

Definition 2.1. A sequence of numbers $\{a_n\}$ is called a sequence of numbers with m alternate common differences if for a fixed natural number m and for all $j = 1, 2, \dots, m$,

$$a_{m(k-1)+j+1} - a_{m(k-1)+j} = d_j,$$

for all $k \in \mathbb{N}$. Here d_j is the j-th common difference of $\{a_n\}$.

With the above definition, a sequence of numbers with m alternate common differences takes the following form:

$$a, a+d_1, a+d_1+d_2, \dots, a+d_1+d_2+\dots+d_m, a+2d_1+d_2+\dots+d_m,$$
$$a+2d_1+2d_2+\dots+d_m, a+2d_1+2d_2+\dots+2d_m, \dots \qquad (2.1)$$

The sequence

$$2, 3, 5, 8, 9, 11, 14, 15, 17, 20, \ldots$$

is an example of a sequence of numbers with 3 alternate common differences. The common differences are $d_1 = 1, d_2 = 2$, and $d_3 = 3$.

Theorem 2.2. *Let $\{a_n\}$ be a sequence of number that takes the form* (2.1)*. Then, the formula for the n^{th} term of the sequence $\{a_n\}$ is given by*

$$a_n = a_1 + \sum_{i=1}^{m} \left\lfloor \frac{n + (m-1) - i}{m} \right\rfloor d_i. \tag{2.2}$$

Proof. Obviously, (2.2) holds for $n \le m$. We only need to show that (2.2) is true for $n > m$ to prove the validity of the fomula. Suppose (2.2) holds for some natural number k. Hence,

$$a_k = a_1 + \sum_{i=1}^{m} \left\lfloor \frac{k + (m-1) - i}{m} \right\rfloor d_i.$$

Let $k = m(p-1) + j$ and $p \in \mathbb{N}$. Now, for every $j = 1, 2, \ldots, m \in \mathbb{N}$, we have $a_{k+1} = a_k + d_j$. Thus,

$$\begin{aligned}
a_{k+1} &= a_1 + \sum_{i=1}^{m} \left\lfloor \frac{k + (m-1) - i}{m} \right\rfloor d_i + d_j \\
&= a_1 + \sum_{i=1}^{m} \left\lfloor \frac{m(p-1) + j + (m-1) - i}{m} \right\rfloor d_i + d_j \\
&= a_1 + \sum_{i=1}^{j-1} \left\lfloor p + \frac{j-1-i}{m} \right\rfloor d_i + \sum_{i=j}^{m} \left\lfloor p + \frac{j-1-i}{m} \right\rfloor d_i + d_j \\
&= a_1 + \sum_{i=1}^{j-1} p d_i + \sum_{i=j}^{m} (p-1) d_i + d_j \\
&= a_1 + \sum_{i=1}^{j} p d_i + \sum_{i=j+1}^{m} (p-1) d_i \\
&= a_1 + \sum_{i=1}^{j} \left\lfloor p + \frac{j-i}{m} \right\rfloor d_i + \sum_{i=j+1}^{m} \left\lfloor p - 1 + \frac{m+j-i}{m} \right\rfloor d_i \\
&= a_1 + \sum_{i=1}^{j} \left\lfloor \frac{(m(p-1)+j) + 1 + (m-1) - i}{m} \right\rfloor d_i \\
&\quad + \sum_{i=j+1}^{m} \left\lfloor \frac{(m(p-1)+j) + 1 + (m-1) - i}{m} \right\rfloor d_i \\
&= a_1 + \sum_{i=1}^{m} \left\lfloor \frac{(k+1) + (m-1) - i}{m} \right\rfloor d_i.
\end{aligned}$$

□

Below is a table of formulas for the n^{th} term a_n of the given sequence for specific values of m.

m	n^{th} term a_n
1	$a_1 + (n-1)d$
2	$a_1 + \left\lfloor \frac{n}{2} \right\rfloor d_1 + \left\lfloor \frac{n-1}{2} \right\rfloor d_2$
3	$a_1 + \left\lfloor \frac{n+1}{3} \right\rfloor d_1 + \left\lfloor \frac{n}{3} \right\rfloor d_2 + \left\lfloor \frac{n-1}{3} \right\rfloor d_3$
4	$a_1 + \left\lfloor \frac{n+2}{4} \right\rfloor d_1 + \left\lfloor \frac{n+1}{4} \right\rfloor d_2 + \left\lfloor \frac{n}{4} \right\rfloor d_3 + \left\lfloor \frac{n-1}{4} \right\rfloor d_4$
5	$a_1 + \left\lfloor \frac{n+3}{5} \right\rfloor d_1 + \left\lfloor \frac{n+2}{5} \right\rfloor d_2 + \left\lfloor \frac{n+1}{5} \right\rfloor d_3 + \left\lfloor \frac{n}{5} \right\rfloor d_4 + \left\lfloor \frac{n-1}{5} \right\rfloor d_5$
$\vdots$	$\vdots$

Corollary 2.3. *Let m and n be natural numbers. If $m|(n-1)$ then we have*

$$a_n = a_1 + \left(\frac{n-1}{m}\right)\sum_{i=1}^{m} d_i.$$

Proof. Suppose $m|(n-1)$ then $n-1 = mk$ for some $k \in \mathbb{N}$. Then,

$$a_n = a_1 + \sum_{i=1}^{m}\left\lfloor k + \frac{m-i}{m}\right\rfloor d_i = a_1 + k\sum_{i=1}^{m} d_i = a_1 + \left(\frac{n-1}{m}\right)\sum_{i=1}^{m} d_i.$$

□

Corollary 2.4. *If $m|n$, we have*

$$a_n = a_1 + \left(\frac{n}{m}\right)\sum_{i=1}^{m} d_i - d_m.$$

Proof. Suppose $m|n$ then $n = mk$ for some $k \in \mathbb{N}$. So,

$$a_n = a_1 + \sum_{i=1}^{m}\left\lfloor k + 1 - \frac{i+1}{m}\right\rfloor d_i = a_1 + k\sum_{i=1}^{m} d_i - d_m = a_1 + \left(\frac{n}{m}\right)\sum_{i=1}^{m} d_i - d_m.$$

□

Lemma 2.5. *For any natural numbers m and n, we have*

$$\sum_{i=1}^{m}\left\lfloor \frac{n+(m-1)-i}{m}\right\rfloor = n-1 = \sum_{i=1}^{m}\left\lfloor \frac{i+n-2}{m}\right\rfloor.$$

Proof. Note that

$$\left\lfloor \frac{n}{m}\right\rfloor = k \quad \Rightarrow \quad k \le \frac{n}{m} < k+1.$$

Hence,

$$\begin{aligned}\sum_{i=1}^{m}\left\lfloor \frac{n+(m-1)-i}{m}\right\rfloor &= \left\lfloor \frac{n+(m-2)}{m}\right\rfloor + \ldots + \left\lfloor \frac{n}{m}\right\rfloor + \left\lfloor \frac{n-1}{m}\right\rfloor \\ &= k + k + \ldots + k + (k-1) = mk - 1 = n-1.\end{aligned}$$

□

Theorem 2.6. *If $d_i = d_1$ for $i \le m-1$, we have*

$$a_n = a_1 + \left(n - 1 - \left\lfloor \frac{n-1}{m}\right\rfloor\right) d_1 + \left\lfloor \frac{n-1}{m}\right\rfloor d_m. \tag{2.3}$$

Proof. Let $d_i = d_1$, for $i \le m-1$, in (2.2). Hence,

$$\begin{aligned}a_n &= a_1 + \sum_{i=1}^{m}\left\lfloor \frac{n+(m-1)-i}{m}\right\rfloor d_i \\ &= a_1 + d_1\sum_{i=1}^{m-1}\left\lfloor \frac{n+(m-1)-i}{m}\right\rfloor + \left\lfloor \frac{n-1}{m}\right\rfloor d_m \\ &= a_1 + d_1\left(\left\lfloor \frac{n+(m-2)}{m}\right\rfloor + \left\lfloor \frac{n+(m-3)}{m}\right\rfloor + \ldots + \left\lfloor \frac{n}{m}\right\rfloor\right) + \left\lfloor \frac{n-1}{m}\right\rfloor d_m \\ &= a_1 + d_1\left\{\left(\sum_{i=1}^{m}\left\lfloor \frac{n+(m-1)-i}{m}\right\rfloor\right) - \left\lfloor \frac{n-1}{m}\right\rfloor\right\} + \left\lfloor \frac{n-1}{m}\right\rfloor d_m \\ &= a_1 + \left(n-1-\left\lfloor \frac{n-1}{m}\right\rfloor\right) d_1 + \left\lfloor \frac{n-1}{m}\right\rfloor d_m.\end{aligned}$$

□

Theorem 2.7. *Let $\{a_n\}$ be a sequence of number that takes the form* (2.1)*. Then, the formula for the sum of the first n terms of the sequence $\{a_n\}$ is given by*

$$S_n = na_1 + \sum_{i=1}^{m} \left\lfloor \frac{N-i}{m} \right\rfloor \left(N - i - \frac{m}{2} \left\lfloor \frac{N-i}{m} + 1 \right\rfloor \right) d_i, \tag{2.4}$$

where $N = n + m - 1$.

Proof. Consider a sequence $\{a_n\}$ that takes the form (2.1). Then,

$$\begin{aligned} \sum_{j=1}^{n} a_j &= \sum_{j=1}^{n} \left(a_1 + \sum_{i=1}^{m} \left\lfloor \frac{j+(m-1)-i}{m} \right\rfloor d_i \right) = na_1 + \sum_{i=1}^{m} \sum_{j=1}^{n} \left\lfloor \frac{j+(m-1)-i}{m} \right\rfloor d_i \\ &= na_1 + \sum_{i=1}^{m} \left\lfloor \frac{n+(m-1)-i}{m} \right\rfloor \left(n + m - i - \frac{m}{2} \left\lfloor \frac{n+2m-(1+i)}{m} \right\rfloor \right) d_i \end{aligned}$$

Letting $N = n + m - 1$, conclusion follows. □

Theorem 2.8. *The sum of the first n terms of the sequence $\{a_n\}$ that takes the form* (2.1) *with $d_i = d_1$, for $i \leq m-1$, is given by*

$$S_n = na_1 + \frac{n(n-1)}{2} d_1 + (d_m - d_1) \left\lfloor \frac{n-1}{m} \right\rfloor \left(n - \frac{m}{2} \left\lfloor \frac{n+m-1}{m} \right\rfloor \right). \tag{2.5}$$

Proof. Consider a sequence $\{a_n\}$ that takes the form (2.1). Then,

$$\begin{aligned} \sum_{j=1}^{n} a_j &= \sum_{j=1}^{n} \left(a_1 + \left(j - 1 - \left\lfloor \frac{j-1}{m} \right\rfloor \right) d_1 + \left\lfloor \frac{j-1}{m} \right\rfloor d_m \right) \\ &= na_1 + \frac{n(n-1)}{2} d_1 - \sum_{j=1}^{n} \left\lfloor \frac{j-1}{m} \right\rfloor d_1 + \sum_{j=1}^{n} \left\lfloor \frac{j-1}{m} \right\rfloor d_m \\ &= na_1 + \frac{n(n-1)}{2} d_1 + (d_m - d_1) \sum_{j=1}^{n} \left\lfloor \frac{j-1}{m} \right\rfloor \\ &= na_1 + \frac{n(n-1)}{2} d_1 + (d_m - d_1) \left\lfloor \frac{n-1}{m} \right\rfloor \left(n - \frac{m}{2} \left\lfloor \frac{n+m-1}{m} \right\rfloor \right). \end{aligned}$$

□

3 Sequence of numbers with m alternate common ratios

We define the sequence of numbers with m alternate common ratios $\{a_n\}$ as follows:

Definition 3.1. A sequence of numbers $\{a_n\}$ is called a sequence of numbers with m alternate common ratios if for a fixed natural number m and for all $j = 1, 2, \ldots, m$,

$$\frac{a_{m(k-1)+j+1}}{a_{m(k-1)+j}} = r_j,$$

for all $k \in \mathbb{N}$. Here r_j is the j-th common ratio of $\{a_n\}$.

With the above definition, we can see immediately that a sequence of numbers $\{a_n\}$ with m alternate common ratios has the following form:

$$a,\ ar_1,\ ar_1r_2,\ \ldots,\ ar_1r_2\cdots r_m,\ ar_1^2r_2\cdots r_m,\ ar_1^2r_2^2\cdots r_m,\ ar_1^2r_2^2\cdots r_m^2, \ldots. \tag{3.1}$$

The sequence

$$1, 2, 6, 24, 48, 144, 576, 1152, \ldots$$

is an example of a sequence of numbers $\{a_n\}$ with 3 alternate common ratios. The common ratios are $r_1 = 2, r_2 = 3$, and $r_3 = 4$.

Theorem 3.2. *Let $\{a_n\}$ be a sequence of number that takes the form* (3.1)*. Then, the formula for the n^{th} term of the sequence $\{a_n\}$ is given by*

$$a_n = a_1 \prod_{i=1}^{m} r_i^{e_i}, \tag{3.2}$$

where $e_i = \left\lfloor \frac{n+(m-1)-i}{m} \right\rfloor$.

In particular, if $m|(n-1)$, we have

$$a_n = a_1 \left(\prod_{i=1}^{m} r_i \right)^{\left(\frac{n-1}{m}\right)}, \tag{3.3}$$

and if $m|(n-1)$,

$$a_n = \frac{a_1}{r_m} \left(\prod_{i=1}^{m} r_i \right)^{\left(\frac{n}{m}\right)}. \tag{3.4}$$

Proof. The proof is by induction on n. Obviously, (3.2) holds for $n \leq m$. We will show that (3.2) is true for $n > m$. Suppose (3.2) holds when for some natural number k. That is,

$$a_k = a_1 \prod_{i=1}^{m} r_i^{e_i},$$

where $e_i = \left\lfloor \frac{m(p-1)+j+(m-1)-i}{m} \right\rfloor$.

Let $k = m(p-1)+j$ and $p \in \mathbb{N}$. Now, for every $j = 1, 2, \ldots, m \in \mathbb{N}$, we have $a_{k+1} = a_k \cdot r_j$. Thus,

$$\begin{aligned}
a_{k+1} &= a_1 \prod_{i=1}^{m} r_i^{e_i} \cdot r_j, \quad \text{where } e_i = \left\lfloor \frac{m(p-1)+j+(m-1)-i}{m} \right\rfloor \\
&= a_1 \prod_{i=1}^{j-1} r_i^{e_i} \cdot \prod_{i=j}^{m} r_i^{e_i} \cdot r_j, \quad \text{where } e_i = \left\lfloor p + \frac{j-1-i}{m} \right\rfloor \\
&= a_1 \prod_{i=1}^{j-1} r_i^{p} \cdot \prod_{i=j}^{m} r_i^{p-1} \cdot r_j \\
&= a_1 \prod_{i=1}^{j} r_i^{f_i} \cdot \prod_{i=j+1}^{m} r_i^{g_i}, \quad \text{where } f_i = \left\lfloor p + \frac{j-i}{m} \right\rfloor,\ g_i = \left\lfloor p - 1 + \frac{m+j-i}{m} \right\rfloor \\
&= a_1 \prod_{i=1}^{m} r_i^{h_i}, \quad \text{where } h_i = \left\lfloor \frac{(k+1)+(m-1)-i}{m} \right\rfloor.
\end{aligned}$$

If $m|n$ (resp. $m|(n-1)$) then (3.3) (resp. (3.4)) follows immediately. □

Theorem 3.3. *Consider a sequence $\{a_n\}$ that takes the form* (3.1) *and suppose $r_i = r_1$, for $i \leq m-1$. Then,*

$$a_n = a_1 r_1^{e} \cdot r_m^{s}, \tag{3.5}$$

where $e = n - 1 - \left\lfloor \frac{n-1}{m} \right\rfloor$ and $s = \left\lfloor \frac{n-1}{m} \right\rfloor$.

Proof. Let $r_1 = r_i$, for $i \leq m-1$, in (3.2). Hence,

$$\begin{aligned}
a_n &= a_1 \prod_{i=1}^{m} r_i^{e_i}, \quad \text{where } e_i = \left\lfloor \frac{n+(m-1)-i}{m} \right\rfloor \\
&= a_1 r_1^{e} \cdot r_m^{s}, \quad \text{where } e = \sum_{i=1}^{m-1} \left\lfloor \frac{n+(m-1)-i}{m} \right\rfloor \text{ and } s = \left\lfloor \frac{n-1}{m} \right\rfloor
\end{aligned}$$

But, by Lemma (2.5), $e = n - 1 - \left\lfloor \frac{n-1}{m} \right\rfloor$. Thus, $a_n = a_1 r_1^{e} \cdot r_m^{s}$. □

Theorem 3.4. *Let $\{a_n\}$ be a sequence of number that takes the form* (3.1). *Then, the formula for the sum of the first n terms of the sequence $\{a_n\}$ is given by*

$$S_n = a_1\left(1 + R\left(\frac{1-r^{e_{n-1}}}{1-r}\right)\right) + a_1 r^{e_{n-1}} \sum_{i=1}^{m-1}\left(\left\lfloor\frac{N-i}{m}\right\rfloor - \left\lfloor\frac{N-1-i}{m}\right\rfloor\right)\sum_{j=1}^{i}\prod_{k=1}^{j} r_k,$$

where $R = \sum_{i=1}^m \prod_{j=1}^i r_j$, $r = \prod_{i=1}^m r_i$, $e_{n-1} = \left\lfloor \frac{n-1}{m}\right\rfloor$ and $N = n + m - 1$.

Proof. Consider a sequence $\{a_n\}$ that takes the form (3.1) and let $R = \sum_{i=1}^m \prod_{j=1}^i r_j$, $r = \prod_{i=1}^m r_i$, $p = e_{n-1} = \left\lfloor \frac{n-1}{m}\right\rfloor$ then

$$\sum_{j=1}^n a_j = a_1 \sum_{j=1}^n \prod_{i=1}^m r_i^{e_j} \quad \text{where } e_j = \left\lfloor \tfrac{j+(m-1)-i}{m}\right\rfloor.$$

Expanding the expression, we obtain

$$\sum_{j=1}^n a_j = a_1 + a_1 R \sum_{j=0}^{p-1} r^j + a_1 r_1^{e_{n+(m-3)}} r_2^{e_{n+(n-4)}} \cdots r_m^{e_{n-2}} + \ldots + a_1 r_1^{e_{n+(m-2)}} r_2^{e_{n+(m-1)}} \cdots r_m^{e_{n-1}}.$$

Simplifying and rewriting the expression in compact form, we obtain

$$\sum_{j=1}^n a_j = a_1\left(1 + R\left(\frac{1-r^p}{1-r}\right)\right) + a_1 r^p \sum_{i=1}^{m-1} M_i \sum_{j=1}^{i}\prod_{k=1}^{j} r_k$$

where $M_i = \left(\left\lfloor\frac{N-i}{m}\right\rfloor - \left\lfloor\frac{N-1-i}{m}\right\rfloor\right)$ which is the desired result. □

Theorem 3.5. *Let $\{a_n\}$ be a sequence of number that takes the form* (3.1) *with $r_i = r_1$, for all $i \le m-1$. Then, the formula for the sum of the first n terms of the sequence $\{a_n\}$ is given by*

$$S_n = a_1\left(\frac{1-r_1^m}{1-r_1}\right)\left(\frac{1-(r_1^{m-1}r_m)^p}{1-r_1^{m-1}r_m}\right) + a_1\left(r_1^{m-1}r_m\right)^p\left(\frac{1-r_1^{n-mp}}{1-r_1}\right),$$

where $p = \left\lfloor\frac{n-1}{m}\right\rfloor$.

Proof. Consider a sequence $\{a_n\}$ that takes the form (3.1) with $r_i = r_1$, for all $i \le m-1$ and let $p = \left\lfloor\frac{n-1}{m}\right\rfloor$,

$$\begin{aligned}
\sum_{j=1}^n a_j &= a_1 \sum_{j=1}^n r_1^{j-1}\left(\frac{r_m}{r_1}\right)^{\left\lfloor\frac{j-1}{m}\right\rfloor} \\
&= a_1\left\{\sum_{j=1}^m r_1^{j-1} + \left(\frac{r_m}{r_1}\right)\sum_{j=m+1}^{2m} r_1^{j-1} + \left(\frac{r_m}{r_1}\right)^2 \sum_{j=2m+1}^{3m} r_1^{j-1} + \ldots \right. \\
&\quad \left. + \left(\frac{r_m}{r_1}\right)^{p-1}\sum_{j=(p-1)m+1}^{mp} r_1^{j-1}\right\} + a_1\left(\frac{r_m}{r_1}\right)^p \sum_{j=mp+1}^{n} r_1^{j-1} \\
&= a_1\left\{\sum_{j=1}^m r_1^{j-1} + \left(\frac{r_1^m r_m}{r_1}\right)\sum_{j=1}^{m} r_1^{j-1} + \left(\frac{r_1^m r_m}{r_1}\right)^2 \sum_{j=1}^{m} r_1^{j-1} + \ldots \right. \\
&\quad \left. + \left(\frac{r_1^m r_m}{r_1}\right)^{p-1}\sum_{j=1}^{m} r_1^{j-1}\right\} + a_1\left(\frac{r_1^m r_m}{r_1}\right)^p \sum_{j=1}^{n-mp} r_1^{j-1} \\
&= a_1\left(\frac{1-r_1^m}{1-r_1}\right)\left\{1 + \left(\frac{r_1^m r_m}{r_1}\right) + \left(\frac{r_1^m r_m}{r_1}\right)^2 + \ldots + \left(\frac{r_1^m r_m}{r_1}\right)^{p-1}\right\} \\
&\quad + a_1\left(\frac{r_1^m r_m}{r_1}\right)^p \sum_{j=1}^{n-mp} r_1^{j-1} \\
&= a_1\left(\frac{1-r_1^m}{1-r_1}\right)\left(\frac{1-(r_1^{m-1}r_m)^p}{1-r_1^{m-1}r_m}\right) + a_1\left(r_1^{m-1}r_m\right)^p\left(\frac{1-r_1^{n-mp}}{1-r_1}\right),
\end{aligned}$$

which is desired. □

4 Some Remarks

If we replace m by t in (2.2) and define t as the period of the sequence $\{a_n\}$ and by considering $d_i = d_1^*$ for $i \leq m-1$ as the first common difference of the sequence and $d_m = d_2^*$ as the second difference then we obtain,

$$a_n = a_1 + \left(n - 1 - \left\lfloor \frac{n-1}{t} \right\rfloor\right) d_1^* + \left\lfloor \frac{n-1}{t} \right\rfloor d_2^*. \tag{4.1}$$

Equation (4.1) is exactly the formula for the n^{th} term of a periodic number sequence with two common differences obtained by Zhang and Zhang in [4]. Furthermore, it can be observed from (4.1) that $a_n \to a_1+(n-1)d_1$ as $m \to \infty$. Similarly, if $d_i = d_1$ for all $i \leq m$, $a_n = a_1+(n-1)d_1$.

In (2.5), on the other hand, would have $S_n = na_1 + \frac{n(n-1)}{2}d_1$ if $m \to \infty$ and a similar result will be obtained if $d_i = d_1$ for all $i \leq m$.

Also, note that in (3.2), $a_n \to a_1 r_1^{n-1}$ if we apply the same argument letting either $m \to \infty$ or $r_i = r_1$ for all $i \leq m$. Furthermore, the limit of the sum given by

$$\sum_{j=1}^{n} a_1 r_1^{n-1} \cdot \left(\frac{r_m}{r_1}\right)^{\left\lfloor \frac{n-1}{m} \right\rfloor} \longrightarrow a_1 \left(\frac{1-r_1^n}{1-r_1}\right) \text{ as } m \to \infty.$$

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