On Legendre curves in three-dimensional contact metric manifolds

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Abstract
Legendre curves, biharmonic Legendre curves and mean curvature vector of a Legendre curve satisfying some recurrent conditions on a three dimensional contact metric manifold with Tanaka-Webster connection have been studied.

1 Introduction

A contact manifold is a $(2n+1)$-dimensional differentiable manifold which satisfies $\eta \wedge (d\eta)^n \neq 0$ for a 1-form η defined on the manifold [2]. Recently the study of contact manifolds has become a subject of growing interest due to its application in different field of science. In [1], the authors introduced a class of contact metric manifolds for which the characteristic vector field ξ belongs to the $(k, \mu)-$nullity distribution for some real numbers k and μ. Such manifolds are known as $(k, \mu)-$contact metric manifolds. The class of $(k, \mu)-$contact metric manifolds encloses both Sasakian and non-Sasakian manifolds. Before Boeckx [4], two classes of non-Sasakian $(k, \mu)-$contact metric manifolds were known. The first class consists of the unit tangent sphere bundles of spaces of constant curvature, equipped with their natural contact metric structure, and the second class contains all the three-dimensional unimodular Lie groups, except the commutative one, admitting the structure of a left invariant $(k, \mu)-$contact metric manifold [1], [4], [18]. A full classification of $(k, \mu)-$contact metric manifolds was given by E. Boeckx [1].

In the study of contact manifolds, Legendre curves play an important role, e.g., a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves. Legendre curves on contact manifolds have been studied by C. Baikoussis and D. E. Blair in the paper [3]. Belkhelfa et al [5] have investigated Legendre curves in Riemannian and Lorentzian manifolds. In [9] slant curves, as a generalization of Legendre curves, have been studied on three-dimensional Sasakian space forms. Legendre curves on almost contact and contact manifolds have also been studied in the papers [13], [16], [27]. The first author of the present paper has also studied Legendre curves in the papers [21], [22].

The study of mean curvature vector field in Euclidean space was initiated by Chen [6]. Again mean curvature of curves and submanifolds have been studied in the papers [11], [14], [16]. Motivated by these works, in the present paper we study mean curvature vector field, satisfying some recurrent conditions, of Legendre curves in three-dimensional contact metric manifolds with Tanaka-Webster connection.

From the papers [8], [12], it is known that there exists no biharmonic Legendre curve on $\mathbb{S}^3$ with respect to Levi-Civita connection ∇. The study of Legendre curves and slant curves as a generalization of Legendre curves on three-dimensional contact manifolds with Tanaka-Webster connection $\tilde{\nabla}$ [23], [28] was initiated by J. T. Cho and collaborators [7] [9]. Slant curves on contact metric manifolds with Tanaka-Webster connection have also been studied in the papers [10] and [17]. In [9], corresponding to biharmonicity of the Levi-Civita connection ∇, the authors investigated $\tilde{\nabla}$ Jacobi equations for $\tilde{\nabla}$ geodesic vector fields with Tanaka-Webster connections $\tilde{\nabla}$ on contact three manifolds.

In [9], Cho and Lee proved that with respect to Tanaka Webster connection a geodesic on a Sasakian manifold(Sasakian space form) is a slant curve. Now the natural question arises that, is the result also true for non-Sasakian contact metric manifold? To get the answer of the question,

we have shown that with respect to Tanaka Webster connection a geodesic on a non-Sasakian contact metric manifold is not a slant curve and hence not a Legendre curve. It is also established that a Legendre curve on a three dimensional contact metric manifold is a geodesic when the curve is biharmonic.

The present paper is organized as follows:

After the introduction in Section 1, we give some required preliminaries in Section 2. In Section 3, we study Legendre curves whose mean curvature vector field satisfies some recurrent conditions with respect to Tanaka-Webster connection $\tilde{\nabla}$ on a three-dimensional contact metric manifold. Here we obtain some interesting equivalent relations regarding the recurrence of the mean curvature vector field with respect to Tanaka-Webster connection. In the last section, we prove that a geodesic on a non-Sasakian contact metric manifold is not a slant curve and hence not a Legendre curve. This section also shows that a Legendre curve on a three-dimensional contact metric manifold is a geodesic when it is biharmonic.

2 Preliminaries

Let M be a $(2n + 1)-$dimensional $C^\infty-$differentiable manifold. The manifold is said to admit an almost contact metric structure (ϕ, ξ, η, g) if it satisfies the following relations [2]:

$$\phi^2 X = -X + \eta(X)\xi, \ \ \eta(\xi) = 1, \ \ g(X, \xi) = \eta(X), \tag{2.1}$$

where ϕ is a tensor field of type $(1, 1)$, ξ is a vector field, η is an 1-form and g is a Riemannian metric on M. Further from above the following can be obtained:

$$\phi\xi = 0, \ \ \eta\phi = 0, \ \ g(X, \phi Y) = -g(\phi X, Y), \ \ g(X, \phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.3}$$

A manifold equipped with an almost contact metric structure is called an almost contact metric manifold. An almost contact metric manifold is called a contact metric manifold if it satisfies $g(X, \phi Y) = d\eta(X, Y)$ for a 1-form η defined on the manifold [2].

Given a three-dimensional contact metric manifold $M(\phi, \xi, \eta, g)$, we consider a $(1, 1)$ tensor field h defined by $h = \frac{1}{2}L_\xi\phi$, where L denotes the Lie differentiation. h is a symmetric operator and satisfies $h\phi = -\phi h$. If λ is an eigenvalue of h with eigenvector X, then $-\lambda$ is also an eigenvalue of h with eigenvector ϕX. Again, we have tr$h$ = tr$\phi h = 0$, and $h\xi = 0$. Moreover, if ∇ denotes the Levi-Civita connection of g, then the following relation holds [1]:

$$\nabla_X \xi = -\phi X - \phi h X, \ \ (\nabla_X \eta)Y = g(X + hX, \phi Y). \tag{2.4}$$

The vector field ξ is a Killing vector field with respect to g if and only if $h = 0$. A contact metric manifold $M(\phi, \xi, \eta, g)$ for which ξ is a Killing vector is said to be a $K-$contact manifold. An almost contact structure on M gives rise to an almost complex structure on the product $M \times \mathbb{R}$. If this almost complex structure is integrable, the contact metric manifold is said to be Sasakian. Equivalently, a contact metric manifold is said to be Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

holds for all X, Y, where R denotes the Riemannian curvature tensor of the manifold M with respect to Levi-Civita connection. The $(k, \mu)-$nullity distribution of a contact metric manifold $M(\phi, \xi, \eta, g)$ is a distribution [1]

$$\begin{aligned} N(k, \mu) \ &: \ p \to N_p(k, \mu) \\ &= \ \{Z \in T_p(M) : R(X, Y)Z \\ &= k(g(Y, Z)X - g(X, Z)Y) + \mu(g(Y, Z)hX - g(X, Z)hY)\}, \end{aligned} \tag{2.5}$$

for any vector fields X, Y on the manifold. Hence, if the characteristic vector field ξ belongs to the $(k, \mu)-$nullity distribution, we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.6}$$

A contact metric manifold with ξ belonging to $(k, \mu)-$nullity distribution is called a $(k, \mu)-$contact metric manifold. The manifold is Sasakian if and only if $k = 1$ [1]. In particular, if $\mu = 0$, then the notion of $(k, \mu)-$nullity distribution reduces to $k-$nullity distribution introduced by S. Tanno [26]. A contact metric manifold with ξ belonging to $k-$nullity distribution is known as $N(k)-$contact metric manifold.

Given a contact structure η, we have two compatible structures. One is a Riemannian structure or metric g and then we call (M, η, g) a contact Riemannian manifold. The other is an almost CR-structure (η, L), where L is the Levi form associated with an endomorphism J on the contact distribution D such that $J^2 = -I$. In particular, if J is integrable, then we call it the integrable CR-structure. The associated almost CR-structure is said to be pseudo-Hermitian, strongly pseudo-convex if the Levi form is Hermitian and positive definite. Such a manifold is called contact strongly pseudo convex pseudo-Hermitian manifold. There is a one to one correspondence between the two associated structures by the relation $g = L + \eta \otimes \eta$, where we denote by the same letter L the natural extension of the Levi form to a $(0, 2)$ tensor field on M. From this point of view we have two geometries on a given contact structure, one is formed by the Levi-Civita connection ∇ the other is derived by the Tanaka webster connection $\tilde{\nabla}$,(gives pseudo-Hermitian geometry), which is a canonical affine connection on a strongly pseudo-convex CR-manifold [9]. Let us recall the Tanaka-Webster connection on a strongly pseudo-convex pseudo-Hermitian manifold $M(\eta, L)$ with the associated contact Riemannian structure (ϕ, ξ, η, g). The Tanaka-Webster connection $\tilde{\nabla}$ is defined by [23], [28]

$$\tilde{\nabla}_X Y = \nabla_X Y + A(X, Y), \tag{2.7}$$

where

$$A(X, Y) = \eta(X)\phi Y + \eta(Y)(\phi X + \phi h X) - g(\phi X + \phi h X, Y)\xi. \tag{2.8}$$

The torsion $\tilde{T}$ of $\tilde{\nabla}$ is given by

$$\tilde{T}(X, Y) = 2g(X, \phi Y)\xi + \eta(Y)\phi h X - \eta(X)\phi h Y. \tag{2.9}$$

Proposition 2.1. *The Tanaka-Webster connection $\tilde{\nabla}$ on a three-dimensional contact Riemannian manifold is the unique linear connection satisfying the following conditions* [25]:
(i) $\tilde{\nabla}\eta = 0, \tilde{\nabla}\xi = 0$,
(ii) $\tilde{\nabla}g = 0, \tilde{\nabla}\phi = 0$,
(iii) $\tilde{T}(X, Y) = -\eta([X, Y])\xi, \tilde{T}(\xi, \phi Y) = -\phi\tilde{T}(\xi, Y)$.
Let $\tilde{\nabla}_{\dot{\gamma}}$ denote the covariant differentiation along γ with respect to Tanaka-Webster connection on M. We shall say that γ is a Frenet curve with respect to Tanaka-Webster connection if one of the following three cases holds:
(a) γ is of osculating order 1, i.e., $\tilde{\nabla}_T T = 0$ (geodesic).
(b) γ is of osculating order 2, i.e., there exist two orthonormal vector fields $T(= \dot{\gamma})$, N and a non-negative function $\tilde{k}$ (curvature) along γ such that $\tilde{\nabla}_T T = \tilde{k}N, \tilde{\nabla}_T N = -\tilde{k}T$.
(c) γ is of osculating order 3, i.e., there exist three orthonormal vectors $T(= \dot{\gamma})$, N, B and two non-negative functions $\tilde{k}$(curvature) and $\tilde{\tau}$(torsion) along γ such that

$$\tilde{\nabla}_T T = \tilde{k}N, \tag{2.10}$$

$$\tilde{\nabla}_T N = -\tilde{k}T + \tilde{\tau}B, \tag{2.11}$$

$$\tilde{\nabla}_T B = -\tilde{\tau}N. \tag{2.12}$$

The above formulas are Serret-Frenet formulas for Tanaka-Webster connection.

With respect to pseudo-Hermitian connection, a Frenet curve of osculating order 3 for which $\tilde{k}$ is a positive constant and $\tilde{\tau} = 0$ is called a circle in M; a Frenet curve of osculating order 3 is called a helix in M if $\tilde{k}$ and $\tilde{\tau}$ both are positive constants and the curve is called a generalized helix if $\frac{\tilde{k}}{\tilde{\tau}}$ is a constant. If $\tilde{k} = 0$, the curve is geodesic.

A Frenet curve γ in a contact metric manifold is said to be a Legendre curve if it is an integral curve of the contact distribution $D = \ker\eta$, i.e., if $\eta(\dot{\gamma}) = 0$.

The curve is called a slant curve if $\eta(\dot{\gamma}) = \cos\alpha$, where α is a constant. For more details we refer [2], [3], [9], [15].

3 Legendre curves on three-dimensional contact metric manifolds with the mean curvature vector satisfying some recurrent conditions

In this section we study Legendre curves on three-dimensional contact metric manifolds with the mean curvature vector satisfying some recurrent conditions with respect to Tanaka-Webster connections. Mean curvature vector of a Legendre curve has been studied in the papers [14] and [16]. For the definition of recurrent, 2-recurrent and generalized 2-recurrent tensors we refer [19] and [20].

Definition 3.1. *With respect to Tanaka-Webster connnection* $\tilde{\nabla}$, *the mean curvature vector* $H = \tilde{\nabla}_{\dot{\gamma}}\dot{\gamma}$ *of a Legendre curve* γ *on a three-dimensional contact metric manifold will be called*

- parallel if $\tilde{\nabla}_{\dot{\gamma}}H = 0$,
- recurrent if $\tilde{\nabla}_{\dot{\gamma}}H = A(\dot{\gamma})H$,
- 2-recurrent if $\tilde{\nabla}_{\dot{\gamma}}^2 H = A(\dot{\gamma})H$,
- generalized 2-recurrent if $\tilde{\nabla}_{\dot{\gamma}}^2 H = A(\dot{\gamma})H + B(\dot{\gamma},\dot{\gamma})H$,

where A is an 1-form and B is a 2-form defined on the tangent space of γ.

In this section we consider $\dot{\gamma}, \phi\dot{\gamma}, \xi$ as orthonormal Frenet frame.

Proposition 3.1. *With respect to Tanaka-Webster connection, the mean curvature vector of a Legendre curve on a three-dimensional contact metric manifold is parallel if and only if the pseudo-Hermitian curvature $\tilde{k}$ of the curve is zero.*

Proof. By definition of H and Serret-Frenet formula, we get

$$
\begin{aligned}
\tilde{\nabla}_{\dot{\gamma}}H &= \nabla_{\dot{\gamma}}(\tilde{k}\phi\dot{\gamma}) \\
&= \tilde{k}'\phi\dot{\gamma} + \tilde{k}\tilde{\nabla}_{\dot{\gamma}}(\phi\dot{\gamma}) \\
&= \tilde{k}'\phi\dot{\gamma} + \tilde{k}(\tilde{\nabla}_{\dot{\gamma}}\phi)\dot{\gamma} + \tilde{k}\phi(\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma}).
\end{aligned} \tag{3.1}
$$

Consider H is parallel with respect to Tanaka-Webster connection. Then $\tilde{\nabla}_{\dot{\gamma}}H = 0$. Hence using Proposition 2.1, we get from the above equation

$$
-\tilde{k}^2\dot{\gamma} + \tilde{k}'\phi\dot{\gamma} = 0.
$$

Taking inner product with $\dot{\gamma}$ in both sides of the above equation we get $\tilde{k} = 0$. The converse is trivial.

Proposition 3.2. *With respect to Tanaka-Webster connection, the mean curvature vector of a Legendre curve on a three-dimensional contact metric manifold is recurrent if and only if the pseudo-Hermitian curvature $\tilde{k}$ of the curve is zero.*

Proof. Suppose H is recurrent with respect to Tanaka-Webster connection. So

$$
\tilde{\nabla}_{\dot{\gamma}}H = A(\dot{\gamma})H.
$$

After simplification and using Serret-Frenet formula we get

$$
-\tilde{k}^2\dot{\gamma} + (\tilde{k}' - A(\dot{\gamma})\tilde{k})\phi\dot{\gamma} = 0.
$$

Taking inner product with $\dot{\gamma}$ in both sides of the above equation we get $\tilde{k} = 0$. The converse is trivial.

Proposition 3.3. *With respect to Tanaka-Webster connection, the mean curvature vector of a Legendre curve on a three-dimensional contact metric manifold is 2-recurrent if and only if the pseudo-Hermitian curvature $\tilde{k}$ of the curve is zero.*

Proof. Let us consider H as 2-recurrent with respect to Tanaka-Webster connection. So $\tilde{\nabla}_{\dot{\gamma}}^2 H = A(\dot{\gamma})H$. Using Serret-Frenet formula, we get from above after straight forward calculation

$$
3\tilde{k}\tilde{k}'\dot{\gamma} + (\tilde{k}^3 - \tilde{k}'' + A(\dot{\gamma})\tilde{k})\phi\dot{\gamma} = 0.
$$

Taking inner product with $\dot{\gamma}$ in both sides of the above equation we get $\tilde{k}\tilde{k}' = 0$. Hence either $\tilde{k} = 0$ or $\tilde{k}' = 0$. Suppose $\tilde{k}' = 0$. Taking inner product with $\phi\dot{\gamma}$ in both sides of the above equation we have $\tilde{k}^3 - \tilde{k}'' + A(\dot{\gamma})\tilde{k} = 0$. Hence we must get $\tilde{k} = 0$. The converse is trivial.

Proposition 3.4. *With respect to Tanaka-Webster connection, the mean curvature vector of a Legendre curve on a three-dimensional contact metric manifold is generalized 2-recurrent if and only if the pseudo-Hermitian curvature $\tilde{k}$ of the curve is zero.*

Proof. Let us consider H as generalized 2-recurrent with respect to Tanaka-Webster connection. So

$$
\tilde{\nabla}_{\dot{\gamma}}^2 H = A(\dot{\gamma})H + B(\dot{\gamma},\dot{\gamma})H.
$$

After simplification and using Serret-Frenet formula we get as before

$$
3\tilde{k}\tilde{k}' + (\tilde{k}^3 - \tilde{k}'' + A(\dot{\gamma})\tilde{k} + B(\dot{\gamma},\dot{\gamma})\tilde{k})\phi\dot{\gamma} = 0.
$$

As before, here we also get $\tilde{k} = 0$. The converse is trivial.

Theorem 3.1. *For a Legendre curve γ on a three-dimensional contact metric manifold with Tanaka-Webster connection the following conditions are equivalent:*

- the mean curvature vector of γ is parallel,
- the mean curvature vector of γ is recurrent,

- the mean curvature vector of γ is 2-recurrent,
- the mean curvature vector of γ is generalized 2-recurrent,
- γ is a geodesic.

Proof. Proof follows from the combination of the above propositions.

Proposition 3.5. *A Legendre curve on a contact metric manifold is a geodesic with respect to Tanaka-Webster connection if and only if it is so with respect to Levi-Civita connection.*

Proof. In view of (2.4), (2.7) and (2.8), we get

$$
\begin{aligned}
\tilde{\nabla}_{\dot\gamma}\dot\gamma &= \nabla_{\dot\gamma}\dot\gamma + \eta(\dot\gamma)\phi\dot\gamma + (\nabla_{\dot\gamma}\eta)(\dot\gamma)\xi - \eta(\dot\gamma)\nabla_{\dot\gamma}\xi & (3.2) \\
&= \nabla_{\dot\gamma}\dot\gamma + 2\eta(\dot\gamma)\phi(\dot\gamma) + \eta(\dot\gamma)\phi h\dot\gamma. & (3.3)
\end{aligned}
$$

Since for a Legendre curve $\eta(\dot\gamma) = 0$, we get

$$
\tilde{\nabla}_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma \tag{3.4}
$$

The above equation proves the proposition.

By Theorem 3.1 and Proposition 3.5 we have the following

Corollary 3.1. *For a Legendre curve γ on a three-dimensional contact metric manifold with Levi-Civita connection the following conditions are equivalent:*

- the mean curvature vector of γ is parallel,
- the mean curvature vector of γ is recurrent,
- the mean curvature vector of γ is 2-recurrent,
- the mean curvature vector of γ is generalized 2-recurrent,
- γ is a geodesic.

4 Legendre curves on contact metric manifolds

In [9], Cho and Lee proved that with respect to Tanaka Webster connection a geodesic on a Sasakian manifold is a slant curve. In the following we prove that the above result is not true for a non-Sasakian contact metric manifold.

Theorem 4.1. *A geodesic on a non-Sasakian contact metric manifold is not necessarily a Slant curve and hence not a Legendre curve.*

Proof. Let us first construct the following example of non-Sasakian contact metric manifold. To construct M we have followed the paper [1].

Consider $M = R^3$, which is generated by three linearly independent vector fields e_1, e_2 and e_3 satisfying

$$
[e_2, e_3] = 2e_1, \qquad [e_3, e_1] = e_2, \qquad [e_1, e_2] = 2e_3. \tag{4.1}
$$

We take $e_1 = \xi$. Define the Riemannian metric by $g(e_i, e_j) = \delta_{ij}$ and $\eta(X) = g(X, e_1)$. Let $\phi e_3 = -e_2, \phi e_2 = e_3$. For g as an associated metric, we have $\phi^2 X = -X + \eta(X)\xi$. Hence $M(\phi, \xi, \eta, g)$ is a contact metric manifold. By Koszul formula we can calculate the following:

$$
\begin{aligned}
&\nabla_{e_1}e_1 = 0, && \nabla_{e_2}e_2 = 0, && \nabla_{e_3}e_3 = 0, \\
&\nabla_{e_1}e_2 = \tfrac{1}{2}e_3, && \nabla_{e_2}e_1 = -\tfrac{3}{2}e_3, && \nabla_{e_1}e_3 = -\tfrac{1}{2}e_2, \\
&\nabla_{e_3}e_1 = \tfrac{1}{2}e_2, && \nabla_{e_2}e_3 = \tfrac{3}{2}e_1 && \nabla_{e_3}e_2 = -\tfrac{1}{2}e_1.
\end{aligned}
$$

From above it can be shown that the manifold is a (k, μ)-contact metric manifold with $k = \tfrac{3}{4}$ and $\mu = -1$. Obviously the manifold is non-Sasakian. Using (2.7) and (2.8) we get

$$
\begin{aligned}
&\tilde{\nabla}_{e_1}e_1 = 0, && \tilde{\nabla}_{e_2}e_2 = 0, && \tilde{\nabla}_{e_3}e_3 = 0, \\
&\tilde{\nabla}_{e_1}e_2 = \tfrac{3}{2}e_3, && \tilde{\nabla}_{e_2}e_1 = \tfrac{1}{2}e_3, && \tilde{\nabla}_{e_1}e_3 = -\tfrac{3}{2}e_2, \\
&\tilde{\nabla}_{e_3}e_1 = \tfrac{1}{2}e_2, && \tilde{\nabla}_{e_2}e_3 = e_1 && \tilde{\nabla}_{e_3}e_2 = -\tfrac{1}{2}e_1.
\end{aligned}
$$

Let γ be a geodesic on M. After a straight forward calculation, using the values of $\tilde{\nabla}_{e_i}e_j$ calculated above, we get

$$
\begin{aligned}
\tilde{\nabla}_T T &= (T_1'(T_1 + T_2 + T_3) + \tfrac{1}{2}T_2 T_3)e_1 \\
&+ (T_2'(T_1 + T_2 + T_3) - T_1 T_3)e_2 \\
&+ (T_3'(T_1 + T_2 + T_3) + 2T_1 T_2)e_3. & (4.2)
\end{aligned}
$$

If the curve is a geodesic, that is, $\tilde{\nabla}_T T = 0$, we get from (4.2) by taking inner product with e_1

$$T_1' = -\frac{T_2 T_3}{2(T_1 + T_2 + T_3)}.$$

From above it follows that T_1 is not necessarily a constant. Hence $\eta(\dot{\gamma}) = T_1$ is not a constant. Therefore the geodesic is not a slant curve and hence not a Legendre curve. This completes the proof of the theorem.

In the following we show that a Legendre curve on any three-dimensional contact metric manifold is a geodesic if the curve is biharmonic.

Following [9] and [13] we give the following

Definition 4.1. *A Legendre curve on a three-dimensional contact metric manifold is called biharmonic with respect to Tanaka-Webster connections if it satisfies the equation*

$$\tilde{\nabla}_T^3 T + \tilde{\nabla}_T \tilde{T}(\tilde{\nabla}_T T, T) + \tilde{R}(\tilde{\nabla}_T T, T)T = 0. \tag{4.3}$$

If instead of pseudo-Hermitian connections $\tilde{\nabla}$, we take Levi-Civita connection ∇, then the above equation becomes

$$\nabla_T^3 T + R(\nabla_T T, T)T = 0. \tag{4.4}$$

Theorem 4.2. *With respect to Tanaka-Webster connection a biharmonic Legendre curve on a three-dimensional contact metric manifold is a geodesic.*

Proof. Let us consider a biharmonic Legendre curve on a (k, μ)-contact metric manifold M. By Proposition 2.1, we have $\tilde{\nabla}\phi = 0$. So $\nabla_X \phi Y = \phi \nabla_X Y$. Hence it can be shown that $R(\phi X, Y)Z = \phi R(X, Y)Z$ for any vector fields X, Y, Z on M. We consider $(T, \phi T, \xi)$ as Frenet frame of the Legendre curve γ, where $\dot{\gamma} = T$. Consequently, by use of Serret Frenet formula, we obtain

$$\tilde{R}(\tilde{\nabla}_T T, T)T = 0, \tag{4.5}$$

where $T = \dot{\gamma}$. By virtue of (2.9),

$$\tilde{T}(\tilde{\nabla}_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma}) = 2g(\tilde{\nabla}_{\dot{\gamma}} \dot{\gamma}, \phi \dot{\gamma})\xi.$$

By covariant differentiation along γ we get from above equation

$$\tilde{\nabla}_{\dot{\gamma}} \tilde{T}(\tilde{\nabla}_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma}) = 2g(\tilde{\nabla}_{\dot{\gamma}}^2 \dot{\gamma}, \phi \dot{\gamma}) + 2g(\tilde{\nabla}_{\dot{\gamma}} \dot{\gamma}, \tilde{\nabla}_{\dot{\gamma}} \phi \dot{\gamma})\xi + 2g(\tilde{\nabla}_{\dot{\gamma}} \dot{\gamma}, \phi \dot{\gamma})\tilde{\nabla}_{\dot{\gamma}}\xi.$$

Applying Proposition 2.1 in the above equation we get

$$\tilde{\nabla}_{\dot{\gamma}} \tilde{T}(\tilde{\nabla}_{\dot{\gamma}} \dot{\gamma}, \dot{\gamma}) = 2g(\tilde{\nabla}_{\dot{\gamma}}^2 \dot{\gamma}, \phi \dot{\gamma}) + 2g(\tilde{\nabla}_{\dot{\gamma}} \dot{\gamma}, (\tilde{\nabla}_{\dot{\gamma}}\phi)\dot{\gamma} - \phi \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma})\xi.$$

Again using Proposition 2.1 in the above equation we have

$$\tilde{\nabla}_T \tilde{T}(\tilde{\nabla}_T T, T) = 0, \tag{4.6}$$

where $T = \dot{\gamma}$.

By Serret Frenet formula for $\tilde{\nabla}$

$$\tilde{\nabla}_T^3 T = -3\tilde{k}\tilde{k}'T + (\tilde{k}'' - \tilde{k}^3 - \tilde{k}\tilde{\tau}^2)N + (2\tilde{\tau}\tilde{k}' + \tilde{k}\tilde{\tau}')B. \tag{4.7}$$

Combining (4.5), (4.6) and (4.7) we get

$$\tilde{\nabla}_T^3 T + \tilde{\nabla}_T \tilde{T}(\tilde{\nabla}_T T, T) + \tilde{R}(\tilde{\nabla}_T T, T)T = -3\tilde{k}\tilde{k}'T + (\tilde{k}'' - \tilde{k}^3 - \tilde{k}\tilde{\tau}^2)N + (2\tilde{\tau}\tilde{k}' + \tilde{k}\tilde{\tau}')B$$

From above it follows that the curve is biharmonic if and only if
- $\tilde{k}\tilde{k}' = 0$,
- $\tilde{k}'' - \tilde{k}^3 - \tilde{k}\tilde{\tau}^2 = 0$,
- $2\tilde{\tau}\tilde{k}' + \tilde{k}\tilde{\tau}' = 0$.

By the above three-conditions we get $\tilde{k} = 0$. Thus the theorem follows.

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