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VECTOR VALUED FUNCTIONS AND BOEHMIANS FOR PLANCHEREL THEOREM OF MELLIN TRANSFORM

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Abstract In this paper, the Mellin transform and Mellin-Plancherel theorem are introduced for vector-valued Boehmians and further, an isomorphism between $L^2(R)$ onto $L^2(R)$has been established. Moreover, we have investigated the results for the Mellin transform, invoking a relation between Fourier and Mellin transform.

1 Introduction

The Mellin transform of a function $f : R^n \to \mathbb{C}, f \in L_1(R^n)$, which is denoted by $\tilde{f}$, is defined by [4, p. 194]

$$\tilde{f}(s) = M[f](s) = \int_{\mathrm{R}^n} x^{s-1} f(x)dx, \tag{1.1}$$

where $s \in \mathbb{C}^n, s = \sigma + i\tau$ and $\sigma, \tau \in R^n$. The inversion formula is

$$f(x) = M^{-1}[\tilde{f}](s) = (2\pi i)^{-n} \int x^{-s} \tilde{f}(s)ds. \tag{1.2}$$

Eqs. (1.1) and (1.2), which illustrates the relation between the Mellin and the Fourier transforms, may be written in the following form

$$\tilde{f}(s) = F[e^{\sigma\, y} f(e^y)](\tau) \quad \text{[cf. [4, p. 196, Eq. (3.5)]]} \tag{1.3}$$

$$\tilde{f}(s) = F[f(e^y); is] \tag{1.4}$$

i.e.

$$\tilde{f}(is) = \int_{-\infty}^{\infty} f(e^y) e^{sy} dy \tag{1.5}$$

and

$$e^{\sigma\, y} f(e^y) = F^{-1}[\tilde{f}(\sigma + i\tau)](y) \qquad \text{[cf. [4, p.197]]} \tag{1.6}$$

i.e.

$$f(e^y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(is) e^{-sy} ds. \tag{1.7}$$

The convolution [4, p.205, Eqs. (3.24)-(3.25)], $f \vee g$, where f and g are functions from R^n (n -dimensional – Euclidean space) into $\mathbb{C}$ (the complex plane) (or $f, g \in L_1$) , is defined by

$$(f \vee g)(x) = \int_{\mathrm{R}^n} f(t) g(x/t) t^{-1} dt, \tag{1.8}$$

i.e.

$$M[f \vee g] = M[f] \cdot M[g], \tag{1.9}$$

where M stands for the Mellin transform [4, p.207].

Using Eq. (1.3), the definition of the Plancherel theorem for the Mellin transform, where $Re(s) = 1/2, x = Im(s) \in R^n$, is given as under [4, p.207] :

Let $f \in L_2(R_+^n)$and $v \in R_+^n$. If

$$g(x) = \lim_{v\to\infty} \int_{1/v}^{v} y^{ix-\frac{1}{2}} f(y)dy, \quad x \in R^n \tag{1.10}$$

then $g \in L_2(R_+^n)$and, the Parseval formula is

$$\|f\|_{L_2(R_+^n)} = (2\pi)^{-n} \|g\|_2 . \tag{1.11}$$

The inversion of which is given by

$$f(y) = (2\pi)^{-n} \lim_{\mathrm{R}\to\infty} \int_{Q[-R,R]} y^{-ix-\frac{1}{2}} g(x)dx, \tag{1.12}$$

where, $Q[-R,R] = \mathop{\times}\limits_{j=1}^{n} [-R_j, R_j], R \in R_+^n$.
The Plancherel theorem, by using (1.5) and (1.7), can be stated as under.

Let $f \in L_2(R_+^n)$

$$\tilde{f}(is) = \int_{-\infty}^{\infty} f(e^y)e^{sy}dy. \tag{1.13}$$

Then $f(e^x) \in L_2(R_+^n)$and, the Parseval formula is

$$\|\tilde{f}(is)\|_{L_2(\mathrm{R}_+^n)} = (2\pi)^{-1} \|f(e^x)\|_2 . \tag{1.14}$$

The inversion of which is given by

$$f(e^y) = (2\pi)^{-1} \int_{-\infty}^{\infty} \tilde{f}(is)e^{-sy}ds. \tag{1.15}$$

The distributional Mellin transform $f \in \mathcal{D}$ (the testing function space) is defined by

$$\tilde{f}(is) = \langle f(e^y), e^{sy} \rangle ,$$

and the distributional Parseval relations for the Mellin transform are given with respect to the relations, as under [1, p.166-168]:

$$\left\langle \overline{f(e^x)}, \varphi(e^x) \right\rangle = (2\pi)^{-1} \left\langle \overline{\tilde{f}(is)}, \tilde{\varphi}(is) \right\rangle ,$$

i.e.

$$\left\langle \overline{\tilde{f}(is)}, \tilde{\varphi}(is) \right\rangle = 2\pi \left\langle \overline{f(e^x)}, \varphi(e^x) \right\rangle . \tag{1.16}$$

$$\langle \tilde{f}(is), \tilde{\varphi}(is) \rangle = \langle f(e^x), \breve{\varphi}(e^x) \rangle \quad , \quad \breve{\varphi}(e^x) = \varphi(e^{-x}) \tag{1.17}$$

i.e.

$$\langle f(e^x), \tilde{\varphi}(is) \rangle = \langle \tilde{f}(is), \varphi(e^x) \rangle \quad . \tag{1.18}$$

Here $\varphi \in \mathcal{D}$ and $f \in \mathcal{D}'$($\mathcal{D}'$is dual of $\mathcal{D}$). The testing function space and relations (1.16) and (1.18) are also proved for the tempered distribution space $\mathcal{S}'$ of the Mellin transform [1].

Karunakaran and Thiliga [5] proved the Plancherel theorem of Fourier transform for the vector valued Boehmains. Loonker and Banerji [6] extended the results of the above citation [5] to obtain the Plancherel theorem of the wavelet transform for vector-valued Boehmians. Employing similar notations and terminologies, the Plancherel theorem of Mellin transform for vector-valued Boehmians is developed in this paper.

Consider a space $L_2(A)$ consists of A-valued Borel measurable functions on R such that $\int_R |f(x)|^2 dx < \infty$ and A is both a complex Hilbert space and a separable commuatiave Banach algebra, with an identity e such that the norm induced by the inner product and the norm in the Banach algebra are equivalent. The Plancherel theorem is, thus, developed. If A is a Hilbert space as well as a complex algebra in which the left and right multiplications are continuous, then a Banach algebraic structure is introduced such that the Banach algebra norm and the Hilbert space norm are equivalent, which allows us to use the notation A for both the complex Hilbert space and the complex Banach space [9].

We define the basic definitions of testing function space which are Banach space valued as shown in [10]. Let $\mathcal{D}_K(A)$is the linear space of all functions φ from R to A such that supp $\varphi \subseteq K$ and, for every integer k, the kth derivative of φ, namely $\varphi^{(k)}$, is continuous, where A be a complex Banach space and K a compact subset of R. The topology generated by the collection $\{\gamma_k(\varphi) : 0 \le k < \infty\}$ of seminorms, where

$$\gamma_k(\varphi) = \underset{t\in K}{supp} \ \left\|\varphi^k(t)\right\|_A .$$

Let $\{K_j\}_{j=1}^{\infty}$ be a sequence of compact subsets of R such that $K_1 \subset K_2 \subset \ldots, \bigcup_j K_j = R$ and that every compact subset of R is contained in some K_j. We define $\mathcal{D}(A) = \bigcup_j \mathcal{D}_{K_j}$ to be the inductive limit of $\mathcal{D}_{K_j}(A)$. When $A = C$, $\mathcal{D}(C) = \bigcup_j \mathcal{D}_{K_j}(A) = \mathcal{D}$ is the space of test functions. $\mathcal{E}(A)$is defined as the largest ρ-type test function space containing $\mathcal{D}(A)$.When $A = \ C,\ \mathcal{E}(A) = \mathcal{E}$ is called the space of smooth functions on R.

If B is any other complex Banach space, then $[\mathcal{D}(A) : B]$ is $[A, B]$ - valued distributions, that is the space of all continuous linear mappings from $\mathcal{D}(A)$ to B . Let τ_t denote the translation operator given by $(\tau_t\varphi)(x) = \varphi(x-t)$. Then, for $y \in [\mathcal{D}(A) : B]$, $\upsilon \in [\mathcal{E} : A]$, their convolution, denoted by $y * \upsilon$, is defined as a B-valued mapping on $\mathcal{D}$ by $(y * \upsilon)(\varphi) = y(\psi)$, where $\psi(t) = \upsilon(\tau_{-t}\varphi)$, for all $\varphi \in \mathcal{D}$. It can be shown that $\psi \in \mathcal{D}(A)$ [10, pp.99-100] . Thus, $y * \upsilon$ is well defined and the mapping $\upsilon \to y * \upsilon$ is a continuous linear mapping of $[\mathcal{E} : A]$ into $[\mathcal{D} : B]$. $\mathcal{D}(A)$ can be identified as a subspace of $[\mathcal{E} : A]$ and, in particular , if $y \in [\mathcal{D}(A) \ : B]$and $\upsilon \in \mathcal{D}(A)$, then $y * \upsilon$ is well defined and, further, it can be identified with a smooth B-valued function $u \in \mathcal{E}(B)$ in the sense that

$$(y * \upsilon)(\varphi) = \int_R u(x)\varphi(x)dx, \qquad \forall\varphi \in \mathcal{D},$$

where $u(x) = y(\tau_x\overline{\upsilon})$, $\overline{\upsilon}(t) = \upsilon(-t)$.

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Definition 1.1. *Let A be a separable and commutative complex Banach algebra and R be the measurable space, $1 \le p < \infty$, such that*

$$L_p(A) = [f] \ \left| \ f : R \to A \text{ is Borel measurable} \int_R \|f(x)\|^p \, dm(x) < \infty \right. , \tag{1.20}$$

where $dm(x) = dx/\sqrt{2\pi},\ m(x)$ being measure of x and [f] denotes the equivalence class containing f with respect to the equivalence relation $f \sim g$ if and only if $f = g$ almost everywhere on R with respect to the Lebesgue measure.

When $f : R \to A$ is Borel measurable, then f is a Bochner measurable (which is also true for the mapping $f : R \times R \to A$).

Theorem 1.1. *If $f \in L_p(A), g \in \mathcal{D}(A)$, then*

$$(f * g)(x) = \int_R f(x-y)g(y)dm(y) \tag{1.21}$$

exists, and thus, it defines a Bochner integral. □

Proof . By Definition 1.1, indeed, the mapping f is Borel measurable. If we consider A to be a separable Banach algebra over $\mathbb{C}$ and f and g be A-valued Borel measurable, then $f(x-y)g(y)$is Borel measurable as well as Bochner measurable. If $K =$ supp g and $\|g\|_0 = \sup_{x\in K} \|g(x)\|$, then

$$\int_R f(x-y)g(y)dm(y) = \int_K f(x-y)g(y)dm(y).$$

Now,

$$\int_K \|f(x-y)\| \ \|g(y)\| \, dm(y) \quad \le \quad \|g\|_0 \int_K \|f(x-y)\| \ dm(y) \le C \, \|g\|_0 < \infty,$$

where $C = \|f\|_1$, if $p = 1$ and $C = \|f\|_p \, m(K)^{1/q}$ with $\frac{1}{p} + \frac{1}{q} = 1$ if $p > 1$.
Thus, $\int_K f(x-y)g(y)dm(y)$exists, for each $x \in R$, as a Bochner integral.

Theorem 1.2. *Let* $1 \le p < \infty$. *If* $f \in L_p(A)$, $\quad g \in D(A)$, *then* $f * g \in L_p(A)$ *and* $\|f * g\|_p \le \|f\|_p \, \|g\|_1$.

Proof . Let K =supp g and $\|g\|_0 = \sup\limits_{x \in K} \|g(x)\|$.

$$\|f * g\|_p^p = \int_R \|f * g(x)\|^p \, dm(x) \le \int_R \left(\int_K \|f(x-y)\| \ \|g(y)\| \, dm(y) \right)^p dm(x). \qquad (1.22)$$

Using the Jensen's inequality and considering $\lambda = \int_K \|g(y)dm(y)\|$ and $d\mu(y) = (1/\lambda) \, \|g(y)\| \, dm(y)$, relation (1.22) yields

$$\|f * g\|_p^p \le \lambda^p \int_R \left(\int_K \|f(x-y)\|^p \, d\mu(y) \right)^p dm(x)$$

$$= \lambda^{p-1} \int_K \|g(y)\| \left(\int_R \|f(x-y)\|^p \, dm(x) \right) dm(y)$$

$$= \lambda^{p-1} \lambda \, \|f\|_p^p = \|f\|_p^p \, \|g\|_1^p \, .$$

Thus, $\|f * g\|_p \le \|f\|_p \, \|g\|_1$. $\square$

Definition 1.2. *If* $f \in L_1(A)$, *then*

$$\tilde{f}(is) = \lim_{n \to \infty} \int_{-n}^{n} f(e^x) e^{xs} dx.$$

The Banach algebra norm in A is denoted by $\|\cdot\|_A$ and the norm in A, induced by the inner product $\langle f, g \rangle$, is denoted by $\|\cdot\|_H$.

Definition 1.3. *For* $f, g \in L_2(A)$, *A- valued inner product is defined as*

$$\langle f, g \rangle = \int_R \langle f(x), g(x) \rangle \, dm(x).$$

Theorem 1.3. $L_2(A)$ *is a Hilbert space with respect to the inner product* $\langle f, g \rangle$.

Proof . By virtue of [2, Theorem 3, p.16], it is justified that the space $H^s$is a Hilbert space with respect to the inner product space for Mellin transform. Similarly, here we say that since $\|\cdot\|_A$ and $\|\cdot\|_H$ are equivalent in A , the inner product defined in Definition 1.3 proves $L_2(A)$to be a Hilbert space with respect to the inner product $\langle f, g \rangle$.

Theorem 1.4. *To each* $f \in L_2(A)$ *we can assign* $\tilde{f}(is) \in L_2(A)$ *such that*
(i) if $f \in L_1(A) \bigcap L_2(A)$, then $\|f(e^x)\|_H = \|\tilde{f}(is)\|_H$.
(ii) $f \to \tilde{f}$ is a Hilbert space isomorphism of $L_2(A)$ onto$L_2(A)$.

Proof. We have, from (1.14), $\|\tilde{f}(is)\|_{L_2(R_+^n)} = (2\pi)^{-1} \|f(e^x)\|_2$ and in Theorem 1.3 , $L_2(A)$ is proved to be a Hilbert space with respect to inner product $\langle f, g \rangle$.
Let $f \in L_2(A)$ and $f_n = \chi_{[-n,n]} f$ for all n , where $\chi_{[-n,n]}$denotes the characteristic function on $[-n, n]$, $f_n \in L_1(A) \bigcap L_2(A)$and $\|f_n - f\|_A \to 0$ as $n \to \infty$. Since the norms $\|\cdot\|_A$ and $\|\cdot\|_H$

are equivalent, $\|f_n - f\|_H \to 0$ as $n \to \infty$. From (i), $\|\tilde{f}\|_H = \|f\|_H$, because f_n is Cauchy sequence with respect to $\|\cdot\|_A$and $\tilde{f}_n$is Cauchy sequence with respect to $\|\cdot\|_H$.
Since $L_2(A)$is complete, $(\tilde{f}_n)$ converges in $L_2(A)$ to $\tilde{f}$ with respect to $\|\cdot\|_A$and, therefore, it also converges with respect to $\|\cdot\|_H$. We have

$$\|\tilde{f}\|_H = \lim_{n\to\infty} \|\tilde{f}_n\|_H = \lim_{n\to\infty} \|f_n\|_H = \|f\|_H .$$

The continuity of Mellin transform and the Plancherel formula for the Mellin transform, where $f \in L_2(A)$, imply that the mapping $f \to \tilde{f}$ is a Hilbert space isomorphism of $L_2(A)$ onto $L_2(A)$. The theorem is, therefore, completely proved.□

2 Mellin-Plancherel Transform for Boehmian Spaces $B \in (L_2(A), \Delta)$ and $B \in (L_2(A), \tilde{\Delta})$

To define Boehmian spaces, which may be referred to [3, 6, 7, 8] where G be an additive commutative semigroup and $S \subseteq G$ a sub semigroup (S is multiplicative commutative semigroup) having a mapping $*$ from $G \times S$ to G, and we consider A=$\mathbb{C}^n$ for some n. Let $G = L_2(A)$ and $S = \mathcal{D}(A)$. For $f \in G, g \in S$, we define $f * g$ (see Theorem 1.2).

Lemma 2.1. (i) *If $g_1, g_2 \in S$, then $g_1 * g_2 \in S$.*
(ii) *If $f, g \in G$ and $h \in S$, then $(f + g) * h = f * h + g * h$,*
(iii) $f * g = g * f$, *for all $f, g \in S$*
(iv) *If $f \in G, g, h \in S$, then $(f * g) * h = f * (g * h)$.*
Proofs of (i)-(iv) are simple analogues of those given in [7].

Connection between convergence and multiplication are defined as

(i) if $\lim \alpha_n = \alpha$ and $\delta \in S$, then $\alpha_n\delta = \alpha\delta$

(ii) if $\lim \alpha_n = \alpha$ and $(\delta_1, \delta_2, \ldots) \in \Delta$, then $\alpha_n\delta_n = \alpha$

Convergence in this space is the δ - convergence, Δ is the family of delta sequence. Convergence of delta sequence can be referred to [7] and defined as
A sequence of Boehmian x_n is δ-convergent to a Boehmian x and we write $\Delta - \lim x_n = x$ if there exists a delta sequence (δ_k) such that $x_n\delta_k, x\delta_k \in G$ for all $k, n \in N$ and $\lim\limits_{n\to\infty} x_n\delta_k = x\delta_k$ for each $k \in N$.
A sequence of Boehmian x_n is Δ-convergent to a Boehmian x and we write $\delta - \lim x_n = x$ if there exists a delta sequence (δ_k) such that $(x_n - x)\delta_n \in G$ for each $n \in N$ and $\lim\limits_{n\to\infty} (x_n - x)\delta_n = 0$.
Each delta sequence possessing these properties will be called Δ-convergence of the factor x_n .

Definition 2.1. *A sequence of A – valued functions $(\delta_n) \in S$ is said to be in Δ if*
(i) $\int_{\mathrm{R}} \delta_n(x)dx = e$
(ii) $\int_{\mathrm{R}} \|\delta_n(x)\|\, dm(x) \leq M$ *, for some $M \in R$ and for all n ,*
(iii) $supp\ \delta_n \to 0$ *as $n \to \infty$.*

Theorem 2.1. *Let $f, g \in G$ and $(\delta_i) \in \Delta$ be such that $f * \delta_i = g * \delta_i$ for all i = 1, 2. Then $f = g$ in $L_2(A)$.*

Proof of this theorem is almost similar to the proof, that we write, for the theorem as under

We have $f * \delta_i \to f$ in $L_2(A)$. Let supp $\delta_i \subseteq K$ for all i. Let

$$\|f * \delta_i - f\|_2^2 \leq \int_{\mathrm{R}} \left(\int_K \|f(x-y) - f(x)\|\ \|\delta_i(y)\|\, dm(y) \right)^2 dm(x).$$

Using Theorem 1.2, we have

$\|f*\delta_i - f\|_2^2 \leq M \int_{|y|<\eta} \|\delta_i(y)\| \; \|f_y - f\|_2^2 \, dm(y) < \varepsilon^2$, for large i .

Similarly, $g*\delta_i \to g$ in $L_2(A)$as $i \to \infty$. The proof of the theorem now follows by taking L_2 limits in the equality $f*\delta_i = g*\delta_i$. □

Theorem 2.2. *Let* $\delta = (\delta_1, \delta_2, \delta_3, \ldots)$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots)$ *be in* Δ. *Then* $\delta * \varepsilon =(\delta_1 * \varepsilon_1, \delta_2 * \varepsilon_2, \delta_3 * \varepsilon_3, \ldots) \in \Delta$.

The proof of the theorem may be referred to [5, Theorem 3.4, p.1335].

Theorem 2.3. (i) *If* $\lim\limits_{n\to\infty} f_n = f$ *in* $L_2(A)$, *then for* $\delta \in S$, $\lim\limits_{n\to\infty} f_n * \delta = f*\delta$.
(ii) *If* $\lim\limits_{n\to\infty} f_n = f$ *in* $L_2(A)$, *then for* $(\delta_n) \in \Delta$, $\lim\limits_{n\to\infty} f_n * \delta_n = f$.

Incidentally, proofs of (so called properties) (i) and (ii) of the theorem is a very straight forward approach through Theorem 1.2 and Theorem 2.1 of this paper.
Thus, using Theorems 2.1, 2.2 and 2.3, respectively, we have the Boehmian space in the canonical sense, using $L_2(A)$and Δ. This space is denoted by $B(L_2(A), \Delta)$.

Theorem 2.4. *The mapping* $f \to [f*\delta_i/\delta_i]$, *where* $(\delta_i) \in \mathcal{D}(A)$, *is a continuous imbedding of* $L_2(A)$ *into* $B(L_2(A), \Delta)$.

Proof. The mapping is one-to-one since $[f*\delta_i/\delta_i] = [g*\delta_i/\delta_i]$ implies $(f*\delta_i)*\delta_j = (g*\delta_i)*\delta_j$ for all i, j, and in particular , $\delta_i * \delta_i = \delta_i^2$. Thus, we have$(f*\delta_i^2) = (g*\delta_i^2)$. Using Lemma 2.1 and Theorems 2.1 and 2.2, respectively, we have $f = g$.
Considering $f_n \to 0$ in $L_2(A)$, we have $x_n = [f_n*\delta_i/\delta_i] \xrightarrow{\delta} 0$ in $B(L_2(A), \Delta)$. From Theorem 2.3, we have $x_n * \delta_i = f_n * \delta_i \to 0$ in $L_2(A)$. The proof is completed. □

We follow the convention , that the set of all sequences $(\tilde{\delta}_i)$ such that $(\delta_i) \in \Delta$, will be denoted by $\tilde{\Delta}$.

Lemma 2.2. *If* $f \in L_2(A), g \in \mathcal{D}(A)$, *then* $M(f*g) = M(f) \cdot M(g)$.

Using Plancherel theorem and the convolution formula for the Mellin transform, the lemma can easily be proved, which is quite similar to [5, Lemma 3.7, p.1336] .
Consider another Boehmian space, which contains $L_2(A)$and $S_1 = \tilde{S} = \{\tilde{\delta}/\delta \in S\}$, where $S = \mathcal{D}(A)$. For $f \in G, \tilde{\delta} \in S_1$ we define $(f\tilde{\delta})(x) = f(x)\tilde{\delta}(x), \forall x \in R$.

Lemma 2.3. *If* $f \in G$ *and* $\tilde{\delta} \in S_1$, *then* $f\tilde{\delta} \in G$.

Proof. We know $f\tilde{\delta}$ is Borel measurable and

$$\int_R \|f\tilde{\delta}\|_A^2 \, dm(t) = \int_R \|f(t)\|_A^2 \, \|\tilde{\delta}(t)\|_A^2 \, dm(t)$$

$$\leq \int_R \|f(t)\|_A^2 \, \|\tilde{\delta}(t)\|_1^2 \, dm(t) \quad (\because \forall t, \|\tilde{\delta}(t)\|_A \leq \|\tilde{\delta}(t)\|_1)$$

$$= \|f\|_2^2 \, \|\delta\|_1^2 < \infty.$$

Hence $f\tilde{\delta} \in G$. □

Lemma 2.4. *The mapping* $(f, \tilde{\delta}) \to f\tilde{\delta}$ *from* $G \times S \to G$ *satisfies the following properties*
(i) if $\tilde{\delta}_1, \tilde{\delta}_2 \in S_1$, then $\tilde{\delta}_1\tilde{\delta}_2 \in S_1$
(ii) if $f, g \in G$ and $\tilde{\delta} \in S_1$, then $(f+g)\tilde{\delta} = f\tilde{\delta} + g\tilde{\delta}$.
(iii) $\tilde{\delta}_1\tilde{\delta}_2 = \tilde{\delta}_2\tilde{\delta}_1$ for $\tilde{\delta}_1, \tilde{\delta}_2 \in S_1$,
(iv) if $f \in G$ and $\tilde{\delta}, \tilde{\varepsilon} \in S_1$, then $(f\tilde{\delta})\tilde{\varepsilon} = f(\tilde{\delta}\tilde{\varepsilon})$.

Proof. Since A is commutative Banach algebra and properties defined in Lemma 2.1 proves the results (i) – (iv). $\square$

Lemma 2.5. *Let $f, g \in G$ and $(\tilde{\delta}_i) \in \tilde{\Delta}$ such that $f\tilde{\delta}_i = g\tilde{\delta}_i$ for all i. Then $f = g$ in $L_2(A)$.*

Using the Plancherel theorem and the definitions of the Mellin and Fourier transforms and also using [5, Lemma 3.8, Lemma 3.9 and Lemma 3.11, p. 1337-38] the lemma can easily be proved. For $\tilde{\Delta}$, refer to [7].

Lemma 2.6. (i) *If $f_n \to f$ in $L_2(A)$ and $\tilde{\delta} \in S_1$, then $f_n\tilde{\delta} \to f\tilde{\delta}$ in $L_2(A)$.*
(ii) If $f_n \to f$ in $L_2(A)$ and $(\tilde{\delta}_n) \in \tilde{\Delta}$, then $f_n\tilde{\delta} \to f\tilde{\delta}$ in $L_2(A)$.

Proof. (i) Since $\tilde{\delta}(t)$ is function of t is bounded, we have the result.
(ii) With the help of Theorem 1.4 and Theorem 2.3, we complete the proof. $\square$

Lemma 2.7. *The mapping $i : f \to \left[\frac{f\tilde{\delta}_i}{\tilde{\delta}_i}\right]$, $(\tilde{\delta}_i) \in \tilde{\Delta}$ is continuous imbedding of $L_2(A)$ into $B(L_2(A), \tilde{\Delta})$.*

Proofs of the above lemmas may be seen in [5, Lemma 3.12 and Lemma 3.13, p. 1338-39] . The above lemmas show the convergence conditions of the space G and $B(L_2(A), \tilde{\Delta})$ can be regarded true, as well, for a Boehmian space.

Definition 2.2. *Let $x = [f_n/\varphi_n] \in B(L_2(A), \Delta)$. The Mellin transform of x is $[\tilde{f}_n/\tilde{\varphi}_n] \in B(L^2(A), \tilde{\Delta})$, which is denoted by $\tilde{x}$.*

The Mellin transform is well defined. If $x = [f_n/\varphi_n] = [g_n/\xi_n]$, where $f_n, g_n \in L_2(A)$ and $\varphi_n, \xi_n \in \Delta$, then $f_n * \xi_n = g_n * \varphi_n$. Invoking the Plancerel transform on both the sides and using Theorem 1.4 and Theorem 2.3 and the Lemma 2.2, we get $\tilde{f}_n\tilde{\xi}_n = \tilde{g}_n\tilde{\phi}_n$. Thus,

$$[\tilde{f}_n/\tilde{\varphi}_n] = [\tilde{g}_n/\tilde{\xi}_n] \in B(L_2(A, \tilde{\Delta}).$$

Theorem 2.5. *Let $F : B(L_2(A, \Delta) \to B(L_2(A, \tilde{\Delta})$ be defined by $F(x) = \tilde{x}$. Then F is a continuous one-to-one map from $B(L_2(A, \Delta)$ onto $B(L_2(A), \tilde{\Delta})$.*

Proof. Let $(x_n) \xrightarrow{\delta} 0$ in $B(L_2(A, \Delta)$, $x_n = [f_{n,i}/\varphi_i]$, since $\tilde{x}_n = [\tilde{f}_{n,i}/\tilde{\varphi}_i] \xrightarrow{\delta} 0$ in $B(L_2(A), \tilde{\Delta})$. By hypothesis, for each fixed i as $n \to \infty$, $(f_{n,i}) \to 0$ in $L_2(A)$ with respect to $\|\cdot\|_2$. Thus, for each fixed i , as $n \to \infty$, $(f_{n,i}) \to 0$ in $L_2(A)$ with respect to the norm $\|\cdot\|_H$.
By Theorem 1.4, for each fixed i , as $n \to \infty$, $(\tilde{f}_{n,i}) \to 0$ in $L_2(A)$ with respect to $\|\cdot\|_H$. Therefore, for each fixed i , as $n \to \infty$, $(\tilde{f}_{n,i}) \to 0$ in $L_2(A)$ with respect to $\|\cdot\|_2$. Thus, $(\tilde{x}_n) \xrightarrow{\delta} 0$ in $B(L_2(A), \tilde{\Delta})$.
Now to prove the map F to be one-to-one, consider $\tilde{x}_1 = \tilde{x}_2$, which gives $[\tilde{f}_n/\tilde{\varphi}_n] = [\tilde{g}_n/\tilde{\xi}_n]$, and thereby, as a consequence, $\tilde{f}_n\tilde{\xi}_n = \tilde{g}_n\tilde{\varphi}_n$. By Lemma 2.2, we get $(\tilde{f}_n * \tilde{\xi}_n) = (\tilde{g}_n * \tilde{\varphi}_n)$. Since Plancherel theorem is one-to-one, $(f_n * \xi_n) = (g_n * \varphi_n)$ implies $x_1 = x_2$, which justifies the map F is onto. Since Plancherel transform is an onto mapping, by Theorem 1.4, given $y = [g_n/\xi_n]$ in $B(L_2(A), \tilde{\Delta})$, if $x = [f_n/\varphi_n]$, where $\tilde{f}_n = g_n$ and $\tilde{\phi}_n = \xi_n$, then it verifies that $x \in B(L_2(A), \Delta)$ and $\tilde{x} = y$. The theorem is thus proved. $\square$

Lemma 2.8. *If $x_1, x_2 \in B(L_2(A), \Delta)$, then*
(i) $(\tilde{x}_1 + \tilde{x}_2) = \tilde{x}_1 + \tilde{x}_2$.
(ii) $\widetilde{(\lambda\, x)} = \lambda\tilde{x}, \quad \lambda \in \mathbb{C}$, where addition and multiplication, for Boehmians, are defined as usual.

Proof . By virtue of the Definition 2.2, cited above, the lemma can easily be proved. $\square$

Conclusion. The Theorems 2.1 to 2.5, of Section 2, indeed, show that the Plancherel theorem is one-to-one continuous linear mapping from $B(L_2(A), \Delta)$ onto $B(L_2(A), \tilde{\Delta})$, i.e.,

(i) The function $f \in L_2(A)$ can be identified with the element $x = [f * \delta_i/\delta_i] \in B(L_2(A), \Delta)$, where (δ_i) is any delta sequence in Δ. Its Plancherel theorem, as a Boehmian, is given by

$$[M(f * \delta_i)/\tilde{\delta}_i] = [\tilde{f}\tilde{\delta}_i/\tilde{\delta}_i].$$

The Boehmian $B(L_2(A), \Delta)$is simply the identification of $\tilde{f}$ in $B(L_2(A), \tilde{\Delta})$. Thus, the Plancherel theorem on $B(L_2(A), \Delta)$ is, indeed, an extension of the Plancherel theorem on $L_2(A)$.

(ii). If $x = [f_n/\varphi_n] \in B(L_2(A), \Delta)$ and $y = [g_n/\xi_n]$, $g_n \in D(A)$, $\xi_n \in \Delta$, then we define

$$x * y = [(f_n * g_n)/(\varphi_n * \xi_n)].$$

In this case $M(x * y) = M(x)M(y)$ holds true due to Lemma 2.1.

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