Palestine Journal of Mathematics

Vol. 4 (Spec. 1) (2015), 490–495

Unions of Dominant Chains of Pairwise Disjoint, Completely Isolated Subsemigroups

Karen A. Linton and Ronald C. Linton

Communicated by Ayman Badawi

MSC 2010 Classifications: 20M99

Keywords and phrases: semigroup; subsemigroup; isolated-subsemigroups; inverse semigroup; Green relations; isomorphism; homomorphism, disjoint.

The authors wish to express their gratitude to Dr. Berit Givens, who tirelessly answered numerous emails regarding LaTeX issues.

Abstract. A subsemigroup T of a semigroup S is *completely isolated* provided that $S = T$ or $S\backslash T$ is a semigroup. We investigate the collection of semigroups that are unions of chains of pairwise disjoint, completely isolated subsemigroups, $\{S_\alpha\}$. For a particular class $\mathbb{W}$ of semigroups S in this collection, we show that the Green's equivalence classes $\mathcal{L}$ and $\mathcal{R}$ of S are exactly the pairwise disjoint unions of the corresponding equivalence classes for each of the subsemigroups, S_α. In addition, for several properties, we show that if each of the completely isolated subsemigroups, S_α, has property P, then S also has property P. Furthermore, we demonstrate that $\mathbb{W}$ is closed with respect to taking subgroups, homomorphic images, and forming inflations and invariants. We also show that each finite member of $\mathbb{W}$ can be represented as a homomorphic image of a subsemigroup T of a finite inverse semigroup, where T is also a member of $\mathbb{W}$.

1 Introduction

In 1957, in [7], Tamura notes that if A is a totally ordered set indexing a collection of arbitrary semigroups, $\{S_\alpha : \alpha \in A\}$, and if we let $S = \cup_{\alpha \in A} S_\alpha$, then we can define an associative operation $*$ on S by

$$x * y = \begin{cases} xy & \text{if } x, y \in S_\alpha \\ x & \text{if } x \in S_\alpha, y \in S_\beta \text{ and } \alpha > \beta \\ y & \text{if } x \in S_\alpha, y \in S_\beta \text{ and } \beta > \alpha \end{cases} \tag{1.1}$$

In [4], Ljapin attributes some of the earliest study of isolation of subsemigroups to P.G. Kontorovich, who used the term *isolated ideals*, and defines a subsemigroup T of the semigroup S to be *completely isolated* provided $xy \in T$ implies $x \in T$ and $y \in T$ for all $x, y \in S$. In all subsequent works, however, semigroups with that property are typically referred to as *convex* [5] and the following alternative definition of completely isolated subsemigroup is used. (For the purpose of this article, we also adopt this more common definition.)

Definition 1.1. Let S be a semigroup with subsemigroup T. We say T is *completely isolated* provided $xy \in T$ implies $x \in T$ or $y \in T$, for all $x, y \in S$.

Using the well-known result that T is completely isolated if and only if $T = S$ or $S\backslash T$ is a semigroup, we see that Tamura's definition (1) guarantees that each S_α is completely isolated. In [6], Redei calls S *breakable* if every non-empty subset is a subsemigroup. Recall that a semigroup S is called a *left zero semigroup* if there exists some $x \in S$ such that $xy = x$ for all $y \in S$. Right zero semigroups are defined analogously. Redei proves the following.

Theorem 1.2. *A semigroup S is breakable if and only if $S = \cup_{\alpha \in A} S_\alpha$, where*

(i) A is a totally ordered set,

(ii) the S_α are pairwise disjoint,

(iii) each S_α is either a left zero semigroup or right zero semigroup, and

(iv) if $a \in S_\alpha$ and $b \in S_\beta$ with $\alpha < \beta$, then $ab = b$.

Here again, each S_α is completely isolated. Interest in such subsemigroups continues in recent research. Recall, for $N = \{1, 2, 3, \ldots, n\}$, we use T_n to denote the semigroup of all transformations on N, that is the collection of all permutations on N under composition of functions.

Definition 1.3. For $a \in T_n$, we use $(T_n, *_a)$ to denote the *variant* of T_n induced by a, that is the semigroup T_n with the sandwich operation $*_a$ defined by $\beta *_a \gamma = \beta a \gamma$, for all $\beta, \gamma \in T_n$.

Arguments by Mazorchuk and Tsyaputa in [5] provide a seven-part classification of all completely isolated subsemigroups of T_n.

Definition 1.4. A semigroup S is *inverse* if for every $x \in S$, there exists a unique $y \in S$ such that $x = xyx$ and $y = yxy$.

Adopting the notation introduced by Tsyaputa in [8], we use IS_n to denote the inverse symmetric semigroup of all injective partial transformations on $N = \{1, \ldots, n\}$. Further results for IS_n have been obtained by Tsyaputa [8] as follows: Recall that a semigroup element, e, is an idempotent provided $e^2 = e$. Fix an idempotent $\alpha \in IS_n$ and let $A = dom(\alpha)$. Let C_A denote the set of all elements from IS_n that map A onto A and are arbitrarily defined on $N \backslash A$, that is, $C_A = \{\beta \in IS_n : \beta(A) = A\}$.

Theorem 1.5. *The only completely isolated subsemigroups of* $(IS_n, *_n)$ *are* $(IS_n, *_n)$, C_A, *and* $(IS_n, *_n) \backslash C_A$.

Building on these concepts, and with the operation $*$ defined in (1), we offer definitions of a few new terms. We begin by noting that if $\alpha > \beta$, the product $x * y$ will always result in an element in S_α.

Definition 1.6. The semigroup S_α *dominates* semigroup S_β provided $S_\alpha * S_\beta = S_\alpha$, if $\alpha > \beta$. We refer to the collection $\{S_\alpha\}$ of such semigroups as a *dominant chain of completely isolated semigroups*. Furthermore, if $(S, *)$ is a union of pairwise disjoint semigroups and $*$ satisfies (1), we refer to $(S, *)$ as a dominant chain of completely isolated semigroups. The S_α are referred to as *links* of the chain.

We let $\mathbb{D}$ denote the class of all semigroups that are dominant chains of completely isolated semigroups.

We now relax the definition of the product of elements in distinct subsemigroups in (1) to provide a weaker form of dominance. Notice that in the following definition, each S_α is completely isolated.

Definition 1.7. Let $\{S_\alpha\}$ be a collection of pairwise disjoint subsemigroups of some semigroup $(S, *)$ such that $S = \cup_{\alpha \in A} S_\alpha$. Suppose further that $*$ satisfies the property that $S_\alpha * S_\beta \subseteq S_\alpha$ if $\alpha > \beta$. Then S is called a *weakly dominant chain of subsemigroups*.

We let $\mathbb{W}$ denote the class of all semigroups that are weakly dominant chains of semigroups. Obviously, $\mathbb{D} \subseteq \mathbb{W}$. The construction described in Proposition 14 demonstrates that the inclusion is actually proper.

With these definitions in hand, we now describe the intent of this paper: We wish to investigate the relationship between the structure and properties of any member of $\mathbb{W}$ and those of each of the completely isolated subsemigroups S_α in the corresponding chain. (Note: Henceforth, we will drop the reference to the indexing set A when its presence is not required.)

2 Uniqueness of Representation

Definition 2.1. Suppose that $S = \cup_{\alpha \in A} S_\alpha$ and $S \in \mathbb{D}$, such that each $S_\alpha \notin \mathbb{D}$. Then we say that the chain $\{S_\alpha\}$ is $\mathbb{D}$-*irreducible*. Similarly, if $S = \cup_{\alpha \in A} S_\alpha$ and $S \in \mathbb{W}$, such that each $S_\alpha \notin \mathbb{W}$, we say that the chain $\{S_\alpha\}$ is $\mathbb{W}$-*irreducible*.

Theorem 2.2. *If S is a* $\mathbb{W}$-irreducible *member of* $\mathbb{W}$ *that can be written both as the union of* $\{S_\alpha : \alpha \in A\}$ *and as the union of* $\{T_\beta : \beta \in B\}$*, then there exists a bijection,* φ*, between the indexing sets A and B such that* $S_\alpha = T_{\varphi(\alpha)}$ *for each* $\alpha \in A$.

Proof. Let $\beta \in B$, and consider $T_\beta = T_\beta \cap S = T_\beta \cap (\cup_{\alpha \in A} S_\alpha) = \cup_{\alpha \in A} (T_\beta \cap S_\alpha)$. Since each T_β is $\mathbb{W}$-irreducible, for exactly one $\alpha \in A$, $T_\beta \cap S_\alpha$ is nonempty. It follows that $T_\beta = T_\beta \cap S_\alpha$ or $T_\beta = S_\alpha$, for some $\alpha \in A$. Repeating this argument for each S_α establishes a bijection from B to A and completes the proof. □

Henceforth, we assume semigroups selected from $\mathbb{W}$ are $\mathbb{W}$-irreducible.

3 Green's Relations

We begin by reviewing two of Green's relations for semigroups, namely $\mathcal{L}$ and $\mathcal{R}$.

Definition 3.1. Given a semigroup S and $x, y \in S$, we say x and y are *$\mathcal{L}$-related* if they generate the same principal left ideal (that is, $S^1x = Sx \cup \{x\} = S^1y = Sy \cup \{y\}$), in which case we write $x\mathcal{L}y$. Similarly, if x and y generate the same principal right ideal, (i.e., $xS^1 = xS \cup \{x\} = yS^1 = yS \cup \{y\}$), then x and y are said to be *$\mathcal{R}$-related*, and we write $x\mathcal{R}y$.

Suppose that $S = \cup_{\alpha \in A} S_\alpha \in \mathbb{W}$, and let $a \in S$. Consider the principal left ideal $S^1a = Sa \cup \{a\}$. If $a \in S_\alpha$ for some $\alpha \in A$, then $S^1a = (Sa \cup \{a\}) \cup (\cup_{\beta \in A}\{S_\beta : \beta > \alpha\})$, since $\{a\} \subseteq S_\alpha$ and S_α weakly dominates S_β for $\beta < \alpha$. In general, it follows then that if $a\mathcal{L}b$, for some $a, b \in S$, then a and b are contained in the same S_α. Furthermore, $S^1a = S^1b$ if and only if $S_\alpha a \cup \{a\} = S_\beta b \cup \{b\}$, that is, $S^1a = S^1b$ if and only if $a\mathcal{L}_\alpha b$. This proves the following result for Green's $\mathcal{L}$-relation. (A parallel argument verifies the result for Green's $\mathcal{R}$-relation.)

Theorem 3.2. *If $S = \cup_{\alpha \in A} S_\alpha \in \mathbb{W}$, then $a\mathcal{L}b$ if and only if $a\mathcal{L}_\alpha b$ for some $\alpha \in A$, and $a\mathcal{R}b$ if and only if $a\mathcal{R}_\alpha b$ for some $\alpha \in A$.*

Corollary 3.3. *If $S = \cup_{\alpha \in A} S_\alpha \in \mathbb{W}$, then $\mathcal{L} = \cup_{\alpha \in A} \mathcal{L}_\alpha$, then and $\mathcal{R} = \cup_{\alpha \in A} \mathcal{R}_\alpha$.*

4 Shared Properties

A natural point of interest is whether a member $S = \cup_{\alpha \in A} S_\alpha$ of $\mathbb{D}$ or $\mathbb{W}$ shares any particular property with all of its links. Consider the following.

Definition 4.1. Let $S = \cup_{\alpha \in A} S_\alpha \in \mathbb{D}$. We say a semigroup property $\mathcal{P}$ is a *contagious* chain property provided: If every chain link, S_α, has property $\mathcal{P}$, then S also has property $\mathcal{P}$. If S has property $\mathcal{P}$ implies every link, S_α, also has property $\mathcal{P}$ we say that $\mathcal{P}$ is a *hereditary* chain property. If $S = \cup_{\alpha \in A} S_\alpha \in \mathbb{W}$. We say a semigroup property $\mathcal{P}$ is *weakly contagious* provided: If every chain link S_α, then S also has property $\mathcal{P}$. Finally, if a chain $S \in \mathbb{W}$ has property $\mathcal{P}$ implies that every link of S has property $\mathcal{P}$, we say $\mathcal{P}$ is a *weakly hereditary* chain property..

For example, because of the symmetry of the definition of $*$ in (1), clearly if $S = \cup_{\alpha \in A} S_\alpha \in \mathbb{D}$, then S is commutative if and only if each S_α is commutative. Therefore, commutativity is both a contagious and a hereditary chain property. However, the following result demonstrates a technique for creating a noncommutative semigroup that is the union of a weakly dominant chain of two isomorphic commutative semigroups.

Proposition 4.2. *Suppose that S_1 and S_2 are isomorphic disjoint commutative semigroups. Let $\varphi : S_1 \to S_2$ be an isomorphism, and let $S = S_1 \cup S_2$. Define $*$ on S as follows.*

$$x * y = \begin{cases} \varphi(x)y & \text{if } x \in S_1, y \in S_2 \\ \varphi^{-1}(x)y & \text{if } x \in S_2, y \in S_1 \\ xy & \text{if } x, y \in S_i \text{ for } i = 1, 2 \end{cases} \tag{4.1}$$

Then $(S, *)$ is a noncommutative semigroup.

Proof. The demonstration that the operation defined in (2) is associative requires a case-by-case argument. For the sake of brevity, we demonstrate only one such case: Let $x_1, x_1{}' \in S_1$ and $x_2 \in S_2$. Consider the product $(x_1 * x_2) * x_1{}'$. Since φ^{-1} is a homomorphism, and since $\varphi(x_1), x_2 \in S_2$, we have $(x_1 * x_2) * x_1{}' = (\varphi(x_1)x_2) * x_1{}' = \varphi^{-1}((\varphi(x_1)x_2))x_1{}' = (\varphi^{-1}(\varphi(x_1))\varphi^{-1}(x_2))x_1{}'$ $= (x_1\varphi^{-1}(x_2))x_1{}' \in S_1$. Similarly, $x_1 * (x_2 * x_1{}') = x_1 * (\varphi^{-1}(x_2)x_1{}') = x_1(\varphi^{-1}(x_2)x_1{}') \in S_1$ Hence, $(x_1 * x_2) * x_1{}' = x_1 * (x_2 * x_1{}')$. Notice that $x_1 * x_2 = \varphi(x_1)x_2 \in S_2$, while $x_2 * x_1 = \varphi^{-1}(x_2)x_1 \in S_1$. This result, together with the fact that S_1 and S_2 are disjoint, gives us the noncommutativity of $(S, *)$. □

Thus, noncommutativity is not weakly hereditary and commutativity is not weakly contagious.

We note that considerable research has been conducted in the area of the decomposition of semigroups by semilattices of subsemigroups. Recall that a commutative semigroup $(S, *)$ is a semilattice provided every element is idempotent. (The reader is referred to works of Clifford and Preston [2] and Howie [3] for details.). Since chains are a special case of semilattices, the following result is known, but is stated for completeness.

Proposition 4.3. *Let $S = \cup_{\alpha \in A} S_\alpha \in \mathbb{W}$. Then the following statements hold.*

(i) S is regular if and only if each S_α is regular

(ii) S is completely regular if and only if each S_α is completely regular

(iii) S is inverse if and only if each S_α is inverse

We now demonstrate the impact of comparability of elements in chains by recalling the following definition.

Definition 4.4. Let S be an *inverse* semigroup with semilattice E of idempotents. Then S is *E-unitary* if, for all $e \in E$ and $x \in S$, $ex \in E \Rightarrow x \in E$.

Theorem 4.5. *Suppose that $S = \cup_{\alpha \in A} S_\alpha \in \mathbb{W}$. Then S is E-unitary if and only if each S_α is E-unitary.*

Proof. First, suppose that S is E-unitary. We can write $E = \cup_{\alpha \in A} E_\alpha$, where each E_α denotes the semilattice of idempotents of S_α. The set $\{E_\alpha\}$ is pairwise disjoint, and, by Proposition 15, each S_α is an inverse subsemigroup. The result now follows easily, even when the indexing set is a semilattice. Hence, the property of being E-unitary is weakly hereditary.
Now suppose that each S_α is E-unitary and that $e \in E = \cup_{\alpha \in A} E_\alpha$, for some $e \in E$ and $x \in S$. It follows that $ex \in E_\alpha$, for some $\alpha \in A$. Suppose that $e \in E_\beta = S_\beta \cap E$ and $x \in S_\delta$ for some $\beta, \delta \in A$. Because the indexing set A is a chain, and since $ex \in E_\alpha$, it follows that either $\alpha = \beta$ or $\alpha = \delta$. If $\alpha = \delta$., then the desired result is immediate. If $\beta = \alpha$, then the result follows directly from the fact that each S_α is E-unitary. Whereupon, the property of being E-unitary is weakly contagious.

□

The following example demonstrates that the previous result cannot be improved upon. We rely on the following construction method offered by Yamada in [9]: Suppose that the indexing set for $\{S_\alpha\}$ is the semilattice $A = \{\alpha, \beta, \gamma\}$ defined by the relations: $\alpha^2 = \alpha, \beta^2 = \beta, \gamma^2 = \gamma$, and $\gamma = \alpha\beta = \beta\alpha = \alpha\gamma = \gamma\alpha = \beta\gamma = \gamma\beta$. If $S = \cup_{\alpha \in A} S_\alpha$, we select one idempotent e_δ from each S_δ and then define an associative binary operation $\star$ on S as follows:

$$x_\delta \star y_\lambda = \begin{cases} x_\delta y_\lambda & \text{if } \delta = \lambda \\ x_\delta e_\delta & \text{if } \delta > \lambda (i.e., \delta\lambda = \delta \text{ and } \delta \neq \lambda) \\ e_\lambda y_\lambda & \text{if } \delta < \lambda (i.e., \delta\lambda = \lambda \text{ and } \delta \neq \lambda) \\ e_{\delta\lambda} & \text{if } \delta \neq \lambda, \delta\lambda \neq \delta, \text{ and } \delta\lambda \neq \lambda \end{cases} \tag{4.2}$$

Proposition 4.6. *Suppose that A is the semilattice defined above, and that $(S, \star) = \cup_{\alpha \in A} S_\alpha$ is a semigroup decomposition by A, where the operation $\star$ on S is defined as above. Then $(S, \star)$ is not E-unitary even if each S_α is E-unitary.*

Proof. Select a non-idempotent x_α from S_α, and an idempotent e_β from S_β. Then $x_\alpha \star e_\beta = e_{\alpha\beta} = e_\gamma$ is idempotent in S, but x_α is not. □

5 Characteristics of Members of $\mathbb{W}$ and $\mathbb{D}$

Proposition 5.1. *The class $\mathbb{W}$ is closed under the formation of homomorphic images.*

Proof. Let $S = \cup_{\alpha \in A} \{S_\alpha\} \in \mathbb{W}$ and suppose $\varphi : S \to T$ is a surmorphism from S onto the semigroup T. Let $T_\alpha = T \cap \varphi(S_\alpha)$ for all α. Then T is the disjoint union of the subsemigroups T_α. Suppose now that $x \in T_\alpha$ and $y \in T_\beta$, for some $\alpha > \beta$. Then $x = \varphi(x')$ and $y = \varphi(y')$, for some $x' \in S_\alpha$ and $y' \in S_\beta$. Therefore, $xy = \varphi(x')\varphi(y') = \varphi(x'y') \in T \cap \varphi(S_\alpha) = T_\alpha$ since S_α weakly dominates S_β. It follows then that T_α weakly dominates T_β whenever $\alpha > \beta$, whereupon $T \in \mathbb{W}$. □

The following result follows from a straightforward set-theoretic argument, and is stated without proof.

Proposition 5.2. *The class $\mathbb{W}$ is closed under the formation of subsemigroups.*

Definition 5.3. Suppose that T is a subsemigroup of S. Then S is an *inflation* of T if there is a function $\varphi : S \to T$ such that

(i) $\varphi(x) = x$, for all $x \in T$ and

(ii) $xy = \varphi(x)\varphi(y)$, for all $x, y \in S$.

Theorem 5.4. *Suppose that S is an inflation of T with function $\varphi : S \to T$ satisfying the conditions of the definition above. If $T \in \mathbb{W}$, then $S \in \mathbb{W}$.*

Proof. If we suppose that $T = \cup_{\alpha \in A} T_\alpha$, then we define $S_\alpha = \varphi^{-1}(T_\alpha)$ for each $\alpha \in A$. Clearly each S_α is closed and the collection $\{S_\alpha\}$ is pairwise disjoint. To show that S_α weakly dominates S_β if T_α weakly dominates T_β suppose that $x \in S_\alpha$ and $y \in S_\beta$. Then $xy = \varphi(x)\varphi(y) \in T_\alpha \subseteq \varphi^{-1}(T_\alpha) = S_\alpha$, since $x \in S_\alpha = \varphi^{-1}(T_\alpha)$, and $y \in S_\beta = \varphi^{-1}(T_\beta)$ implies that $\varphi(x) \in \varphi(S_\alpha) = T_\alpha$ and $\varphi(y) \in \varphi(S_\beta) = T_\beta$. □

We now turn our attention to the characterization of variants $(S, *_\alpha)$ of a semigroup $(S, *)$.

Definition 5.5. For any $\alpha \in A$ and $a \in S_\alpha$ we define $S^a_\gamma = (S_\gamma, \circ)$ with the operation $\circ$ defined by $x \circ y = a$ for $x, y \in S_\gamma$. Further, we extend this definition to $S^a = \cup_{\gamma \in A}\{S^a_\gamma : \gamma < \alpha\}$ by $x_\delta \circ y_\lambda = a$ for $x_\delta \in S_\delta$ and $y_\lambda \in S_\lambda$.

Clearly, S^a is a semigroup. Furthermore, S^a is $\mathbb{D}$-irreducible since it contains no dominant chains.

Theorem 5.6. *Suppose that $(S, *)$ is the disjoint union of a weakly dominant chain of subsemigroups $\{S_\gamma\}$. Let $a \in S_\alpha$ for some α. Then the variant $(S, *_a) = (\cup_{\gamma \in A}(S_\gamma, *) : \gamma > \alpha) \cup (S_\alpha, *_a) \cup S^a$ is also in $\mathbb{D}$.*

Proof. First, we define an associative operation on $\cup_{\gamma \in A}\{(S_\gamma, \star_a) : \gamma > \alpha\}$ as follows.

$$x_\delta \star y_\lambda = \begin{cases} x *_a y & \text{if } x, y \in (S_\gamma, *_a) \\ x & \text{if } x \in (S_\gamma, *_a), y \in (S_\delta, *_a) \text{ and } \gamma > \delta \\ y & \text{if } x \in (S_\gamma, *_a), y \in (S_\delta, *_a) \text{ and } \delta > \gamma \end{cases} \tag{5.1}$$

It follows that $(\cup_{\gamma \in A}\{(S_\gamma, *_a) : \gamma > \alpha\}, \star) \cong (\cup_{\gamma \in A} S_\gamma, *_a) = (S, *_a)$. Consider the following observations: First, $(S_\gamma, *_a) \cong (S_\gamma, *)$ via the identity map whenever $\gamma > \alpha$, since in this case, $x *_a y = x * a * y = x * y$. Second, $(S_\gamma, *_a) \cong S^a_\gamma = (S_\gamma, \circ)$ via the identity map whenever $\gamma < \alpha$, since in this case, $x *_a y = x * a * y = a = x \circ y$. It follows that: $(S_\gamma, *_a) \cong (\cup_{\gamma \in A}(S_\gamma, *_a), \star)$ $\cong (\cup_{\gamma \in A}\{(S_\gamma, *_a) : \gamma > \alpha\}) \cup (S_\alpha, *_a) \cup (\cup_{\gamma \in A}\{S^a_\gamma : \gamma < \alpha\}) = (\cup_{\gamma \in A}\{(S_\gamma, *_a) : \gamma > \alpha\}) \cup (S_\alpha, *_a) \cup S^a$. This completes the proof. □

6 Finite Semigroups from $\mathbb{D}$

In [1], Ash shows that if S is a finite semigroup with commuting idempotents, then there exists a finite inverse semigroup I, a subsemigroup T of I, and a surmorphism from T onto S. He suggests that this result could be viewed as a structure theorem for finite semigroups with commuting idempotents. Although the structure of the semigroups in $\mathbb{D}$ is known, we now consider the consequences of applying Ash's result to any finite semigroup S in $\mathbb{D}$, whose idempotents commute.

Theorem 6.1. *Let the finite semigroup S be the union of a dominant chain $\{S_i : 1 \leq i \leq n\}$. Suppose that for each i, all idempotents in S_i commute. Then there is a finite inverse semigroup I, a subsemigroup T of I, and a surmorphism $\varphi : T \to S$. Furthermore, T is a disjoint union of a dominant chain.*

Proof. Since idempotents commute in each S_i and since $\{S_i : 1 \leq i \leq n\}$ is a dominant chain, it follows that idempotents commute in S. Hence, by Theorem 2 in [1], S is a homomorphic image of a subsemigroup T of some finite inverse semigroup I. If φ is this homomorphism, then let $T_i = \varphi^{-1}(S_i)$ for each i. It follows that $T = \cup_{i=1}^n T_i$. Since members of the set $\{S_i : 1 \leq i \leq n\}$ are pairwise disjoint, it follows that members of the set $\{T_i : 1 \leq i \leq n\}$ are also disjoint. In order to show that T_i dominates T_j whenever $i > j$, let $x_i \in T_i$ and $x_j \in T_j$. Then $\varphi(x_i x_j) = \varphi(x_i)\varphi(x_j) \in S_i$, and we have $x_i x_j = x_i \in \varphi^{-1}(S_i) = T_i$. Therefore, T is a disjoint union of a dominant chain. □

References

[1] Ash, C., *Finite semigroups with commuting idempotents*, J. Austral. Math. Soc. (Series A), **43**(1987), pp. 81-90.

[2] Clifford, A.H., and G.B. Preston, "The Algebraic Theory of Semigroups: Volume I.", American Mathematical Society, Providence, RI, 1967.

[3] Howie, J.M., "Fundamentals of Semigroup Theory", Oxford University Press, 1995.

[4] Ljapin, E.S., "Semigroups", Translations of Mathematical Monographs Volume 3, American Mathematical Society, Providence, RI, 1963.

[5] Mazorchuk, V., G. Tsyaputa, *Isolated subsemigroups in the variants of* T_n, Acta Math. Univ. Comenianae, **77**(2008), pp. 63-84.

[6] Redei, L., "Algebra, Volume 1", Pergamon Press, Oxford-London, 1967.

[7] Tamura, T., *The theory of construction of finite semigroups. II.*, Osaka Mathematical Journal, **9**(1957), pp. 1- 42.

[8] Tsyaputa, G., *Isolated and nilpotent subsemigroups in the variants of* IS_n, Algebra and Discrete Mathematics, **1**(2006), pp. 89-97.

[9] Yamada, M., *Compositions of semigroups*, Kodai Mathematical Journal. Volume 8, **3**(1956), pp. 107-111.

Author information

Karen A. Linton, Everett Community College, Mathematics Department, 2000 Tower Street, Everett, WA 98201, USA.
E-mail: `klinton@everettcc.edu`

Ronald C. Linton, Department of Mathematics, Columbus State University, 4225 University Avenue, Columbus, GA 31970, USA.
E-mail: `linton ronald@columbusstate.edu`

Received: January 12, 2015.

Accepted: February 12, 2015.