Palestine Journal of Mathematics

Vol. 5(1) (2016) , 12–16

ON 3-PRIME NEAR-RINGS WITH GENERALIZED DERIVATIONS

A. Boua, L. Oukhtite and A. Raji

Communicated by N. Mahdou

MSC 2010 Classifications: Primary 16N60; Secondary 16Y30.

Keywords and phrases: 3-prime near-rings, generalized derivations, commutativity.

Abstract We prove some theorems in the setting of a 3-prime near-ring admitting a suitably constrained generalized derivation, thereby extending some known results on derivations. Moreover, we give an example proving that the hypothesis of 3-primeness is necessary.

1 Introduction

A left near-ring is a set N with two operations $+$ and $\cdot$ such that $(N,+)$ is a group not necessarily abelian and $(N,\cdot)$ is a semigroup satisfying the left distributive law $x\cdot(y+z)=x\cdot y+x\cdot z$ for all $x,y,z\in N$. In this paper N will be a zero-symmetric *left* near-ring, and usually N will be 3-prime, that is, it will have the property that $xNy=\{0\}$ for all $x,y\in N$ implies $x=0$ or $y=0$. The symbol $Z(N)$ will denote the multiplicative center of N. A near-ring N is called zero-symmetric if $0x=0$ for all $x\in N$ (recall that left distributivity yields $x0=0$); and N is called 2-torsion free if $2x=0$ implies $x=0$ for all $x\in N$. An additive mapping $d:N\longrightarrow N$ is said to be a derivation if $d(xy)=xd(y)+d(x)y$ for all $x,y\in N$, or equivalently, as noted in [15], that $d(xy)=d(x)y+xd(y)$ for all $x,y\in N$. An additive mapping $F:N\longrightarrow N$ is said to be a generalized derivation on N if there exists a derivation $d:N\longrightarrow N$ such that $F(xy)=F(x)y+xd(y)$ holds for all $x,y\in N$. (Note that this definition differs from the one given by Ö. Gölbasi in [12]; for us, a generalized derivation is exactly a right generalized derivation, but is not a left generalized derivation). Clearly, every derivation on a near-ring is a generalized derivation. But the converse statement does not hold in general. We will write for all $x,y\in N$, $[x,y]=xy-yx$ and $x\circ y=xy+yx$ for the Lie products and Jordan products, respectively. We note that for left near-rings, $x(-y)=-xy$ for all $x,y\in N$.

During the last two decades, there has been a great deal of work concerning the relationship between the commutativity of a ring R and the existence of certain specified derivations of R (see [1], [2], [3], [4], [11], [13], [14] and [15]). In 1992 Daif and Bell [11] demonstrated that a prime ring R must be commutative if it admits a derivation d such that $d([x,y])=[x,y]$ for all $x,y\in R$. In [14] M. A. Quadri, M. Shadab Khan and N. Rehman generalized this result in the case of a generalized derivation instead of a derivation, also the researchers A. Boua and L. Oukhtite extended this result to 3-prime near-rings (see [7], [8]). In the present paper, our aim is to establish conditions under which a near-ring becomes a commutative ring. More precisely, we shall attempt to generalize some known results for rings with derivations or near-rings with derivations to near-rings with generalized derivations.

2 Preliminary results

We begin our discussion with the following lemmas which are essential for developing the proof of the main theorem:

Lemma 2.1. *Let N be a 3-prime near-ring.*

i) [4, Lemma 1.2 (iii)] *If $z\in Z(N)-\{0\}$ and $xz\in Z(N)$, then $x\in Z(N)$.*

ii) [11, Lemma 2] *Let d be a derivation on N. If $x\in Z(N)$, then $d(x)\in Z(N)$.*

Lemma 2.2. *Let N be a 3-prime near-ring.*

i) [4, Lemma 1.3 (i)] *If $x\in N$ and $Nx=\{0\}$, then $x=0$.*

ii) [4, Lemma 1.5] *If $N\subseteq Z(N)$, then N is a commutative ring.*

Lemma 2.3. [4, Lemma 1.1] *If N is an arbitrary left near-ring and d is a derivation, then*

$$(d(x)y + xd(y))z = d(x)yz + xd(y)z \ \text{ for all } \ x, y, z \in N.$$

Lemma 2.4. [2, Corollary 4.1] *Let N be a 2-torsion free 3-prime near-ring. If N admits a nonzero derivation d such that $d([x, y]) = 0$ for all $x, y \in N$, then N is a commutative ring.*

Remark. Lemma 2.4 remains true if we remove the assumption " 2-torsion free" and the proof is often the same .

Lemma 2.5. [5, Lemma 1.3] *Let N be a near-ring. If N admits a generalized derivation F associated with a derivation d, then*

$$\big(F(x)y + xd(y)\big)z = F(x)yz + xd(y)z \ \text{ for all } \ x, y, z \in N.$$

3 Polynomial conditions with generalized derivation

This section is devoted to studying the commutativity of a near-ring admitting a nonzero generalized derivation F satisfying the properties $F([x, y]) = 0$, $F([x, y]) = [x, y]$ for all $x, y \in N$. As a consequence of the results obtained in this section, we generalize Theorem 2 due to Daif and Bell in [11], Theorem 2.1 due to M. A. Quadri, M. Shadab Khan and N. Rehman in [14], Corollary 4.1 due to M. Ashraf and S. Ali in [1], and Theorem 2.2 due to A. Boua and L. Oukhtite in [7].

Theorem 3.1. *Let N be a 3-prime near-ring. If N admits a nonzero generalized derivation F associated with a derivation d, then the following assertions are equivalent:*

i) $F([x, y]) = 0$ *for all* $x, y \in N$.

ii) *N is a commutative ring.*

Proof. It is obvious that $ii)$ implies $i)$.
$i) \Rightarrow ii)$
$a)$ If $d = 0$, then $F(xy) = F(x)y$ for all $x, y \in N$; and since $F(xy - yx) = 0$, we have

$$F(x)y - F(y)x = 0 \ \text{ for all } \ x, y \in N.$$

Taking $y = [u, v]$, we get $F(x)[u, v] = 0$ for all $x, u, v \in N$; and replacing x by xw gives $F(x)w[u, v] = 0$ for all $x, w, u, v \in N$. Since N is 3-prime and $F \neq 0$, we conclude that $[u, v] = 0$ for all $u, v \in N$ and therefore N is a commutative ring by Lemma 2.2(ii).
$b)$ Assume that $d \neq 0$, we are given that

$$F([x, y]) = 0 \ \text{ for all } \ x, y \in N. \tag{3.1}$$

Replacing x and y by $[u, v]$ and $[u, v]y$ respectively in (3.1) and using (3.1), we get

$$[u, v]d\big(\big[[u, v], y]\big]\big) = 0 \ \text{ for all } \ u, v, y \in N. \tag{3.2}$$

Substituting $[u, v]y$ for y in (3.2) and invoking (3.2), we arrive at

$$[u, v]d([u, v])\big[[u, v], y\big] = 0 \ \text{ for all } \ u, v, y \in N,$$

so that,

$$[u, v]d([u, v])[u, v]y = [u, v]d([u, v])y[u, v] \ \text{ for all } \ u, v, y \in N. \tag{3.3}$$

Taking yz instead of y in (3.3), we obtain

$$[u, v]d([u, v])y[u, v]z = [u, v]d([u, v])yz[u, v] \ \text{ for all } \ u, v, y, z \in N,$$

that is

$$[u, v]d([u, v])N\big[[u, v], z\big] = \{0\} \ \text{ for all } \ u, v, z \in N.$$

In light of the 3-primeness of N, the last expression implies that

$$[u, v]d([u, v]) = 0 \ \text{ or } \ [u, v] \in Z(N) \ \text{ for all } \ u, v \in N. \tag{3.4}$$

If there are two elements $u_0, v_0 \in N$ such that $[u_0, v_0] \in Z(N)$, then

$$\begin{aligned} 0 &= F\big([[u_0, v_0]u_0, v_0]\big) \\ &= F([u_0, v_0][u_0, v_0]) \\ &= [u_0, v_0]d([u_0, v_0]), \end{aligned}$$

and in this case, (3.4) becomes

$$[u, v]d([u, v]) = 0 \text{ for all } u, v \in N. \tag{3.5}$$

Application of (3.5), (3.2) yields that

$$[u, v]^2 d(y) = [u, v]d(y)[u, v] + [u, v]yd([u, v]) \text{ for all } u, v, y \in N. \tag{3.6}$$

Replacing y by $y[u, v]$ in (3.6) and using again (3.2), we find that

$$\begin{aligned} [u, v]^2 yd([u, v]) + [u, v]^2 d(y)[u, v] &= [u, v]d(y[u, v])[u, v] \\ &= [u, v]d([u, v]y)[u, v] \\ &= [u, v]^2 d(y)[u, v] \text{ for all } u, v, y \in N. \end{aligned}$$

Hence $[u, v]^2 yd([u, v]) = 0$ for all $u, v, y \in N$, that is

$$[u, v]^2 N d([u, v]) = \{0\} \text{ for all } u, v \in N.$$

Since N is 3-prime, the last equation gives

$$[u, v]^2 = 0 \text{ or } d([u, v]) = 0 \text{ for all } u, v \in N. \tag{3.7}$$

Suppose there exist two elements $u_0, v_0 \in N$ such that $[u_0, v_0]^2 = 0$, so by (3.6), we have

$$[u_0, v_0]d(y)[u_0, v_0] + [u_0, v_0]yd([u_0, v_0]) = 0 \text{ for all } y \in N. \tag{3.8}$$

On the other hand, we have $F\big([[u_0, v_0], y]\big) = 0$ for all $y \in N$, which implies that

$$F([u_0, v_0]y) = F(y[u_0, v_0]) \text{ for all } y \in N.$$

By the defining property of F, we get

$$[u_0, v_0]d(y) = F(y)[u_0, v_0] + yd([u_0, v_0]) \text{ for all } y \in N. \tag{3.9}$$

Right multiplying (3.9) by $[u_0, v_0]$ and invoking Lemma 2.5, we find that

$$[u_0, v_0]d(y)[u_0, v_0] = yd([u_0, v_0])[u_0, v_0] \text{ for all } y \in N. \tag{3.10}$$

Using (3.5) again, we have

$$\begin{aligned} d([u_0, v_0])[u_0, v_0] &= d([u_0, v_0])[u_0, v_0] + [u_0, v_0]d([u_0, v_0]) \\ &= d([u_0, v_0]^2) \\ &= 0 \end{aligned}$$

And therefore, (3.10) implies that $[u_0, v_0]d(y)[u_0, v_0] = 0$ for all $y \in N$, and by a return to the equation (3.8), we obtain

$$[u_0, v_0]yd([u_0, v_0]) = 0 \text{ for all } y \in N,$$

which can be rewritten as

$$[u_0, v_0]Nd([u_0, v_0]) = \{0\}.$$

By the 3-primeness of N, we arrive at $d([u_0, v_0]) = 0$, and according to (3.7) we conclude that $d([u, v]) = 0$ for all $u, v \in N$ and our result follows by Lemma 2.4. □

Theorem 3.2. *Let N be a 3-prime near-ring. If N admits a nonzero generalized derivation F associated with a nonzero derivation d, then the following assertions are equivalent:*

i) *$F([x, y]) = [x, y]$ for all $x, y \in N$.*

ii) *N is a commutative ring.*

Proof. It is clear that $ii) \Rightarrow i)$.
$i) \Rightarrow ii)$ We assume

$$F([x,y]) = [x,y] \text{ for all } x,y \in N. \tag{3.11}$$

Taking $[u,v]$ and $[u,v]y$ instead of x and y respectively in (3.11) and using the same techniques which are introduced into the proof of the previous theorem, we arrive at

$$[u,v]d([u,v]) = 0 \text{ or } [u,v] \in Z(N) \text{ for all } u,v \in N. \tag{3.12}$$

If there are $u_0, v_0 \in N$ such that $[u_0, v_0] \in Z(N)$, then

$$\begin{aligned} \big[[u_0,v_0]u_0, v_0\big] &= F\big(\big[[u_0,v_0]u_0, v_0\big]\big) \\ &= F([u_0,v_0][u_0,v_0]) \\ &= \big[[u_0,v_0]u_0, v_0\big] + [u_0,v_0]d([u_0,v_0]). \end{aligned}$$

The last expression implies that $[u_0,v_0]d([u_0,v_0]) = 0$, hence (3.12) becomes

$$[u,v]d([u,v]) = 0 \text{ for all } u,v \in N. \tag{3.13}$$

Since (3.13) is the same as (3.5), by the same arguments as used after (3.5) in the proof of Theorem 3.1, we obtain

$$[u,v]^2 = 0 \text{ or } d([u,v]) = 0 \text{ for all } u,v \in N. \tag{3.14}$$

Suppose there are $u_0, v_0 \in N$ such that $[u_0,v_0]^2 = 0$.
By (3.11), we have

$$F\Bigg(\big[[u_0,v_0], y\big]\Bigg) = \big[[u_0,v_0], y\big] \text{ for all } y \in N,$$

which can be rewritten

$$F\big([u_0,v_0]y\big) - F\big(y[u_0,v_0]\big) = [u_0,v_0]y - y[u_0,v_0] \text{ for all } y \in N.$$

Using (3.11), the last expression implies that

$$[u_0,v_0]d(y) - yd([u_0,v_0]) - F(y)[u_0,v_0] = -y[u_0,v_0] \text{ for all } y \in N. \tag{3.15}$$

Putting $[r,s]$ instead of y in (3.15), we obtain

$$[u_0,v_0]d([r,s]) = [r,s]d([u_0,v_0]) \text{ for all } r,s \in N. \tag{3.16}$$

Left multiplying (3.16) by $[u_0,v_0]$, we find that

$$[u_0,v_0][r,s]d([u_0,v_0]) = 0 \text{ for all } r,s \in N,$$

so that,

$$\big([u_0,v_0]rs - [u_0,v_0]sr\big)d([u_0,v_0]) = 0 \text{ for all } r,s \in N. \tag{3.17}$$

Taking $[u_0,v_0]s$ instead of s in (3.17), we get

$$[u_0,v_0]r[u_0,v_0]sd([u_0,v_0]) = 0 \text{ for all } r,s \in N.$$

Hence,

$$[u_0,v_0]N\Bigg([u_0,v_0]sd([u_0,v_0])\Bigg) = \{0\} \text{ for all } s \in N. \tag{3.18}$$

In the light of the 3-primeness of N, (3.18) shows that

$$[u_0,v_0] = 0 \text{ or } [u_0,v_0]Nd([u_0,v_0]) = \{0\}.$$

Once again N is 3-prime, the last expression yields

$$[u_0,v_0] = 0 \text{ or } d([u_0,v_0]) = 0$$

which implies that $d([u_0,v_0]) = 0$. By (3.14), we conclude that $d([u,v]) = 0$ for all $u,v \in N$ and by Lemma 2.4, we get the required result. □

The following example demonstrates that N to be 3-prime is essential in the hypotheses of the above theorems.

Example. Let S be a near-ring. Let us define $N, d, F : N \to N$ by:

$$N = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y \in S \right\}, \quad d \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } F = d.$$

It is clear that N is not 3-prime, and N admits a nonzero generalized derivation F associated with a nonzero derivation d. Moreover, it is easy to verify that F satisfies the properties:

$$F([A, B]) = 0, \quad F([A, B]) = [A, B]$$

for all $A, B \in N$. However N is not a commutative ring.

References

[1] M. Ashraf and N. Rehman, On commutativity of rings with derivations, *Result. Math.* **42** , 3-8 (2002).

[2] M. Ashraf and A. Shakir, On (σ, τ)-derivations of prime near-rings-II , *Sarajevo J. Math.,* **4** (16), 23-30 (2008).

[3] K. I. Beidar, Y. Fong and X. K. Wang, Posner and Herstein theorems for derivations of 3-prime near-rings, *Comm. Algebra,* **24** (5), 1581-1589 (1996).

[4] H. E. Bell, On derivations in near-rings II , *Kluwer Academic Publishers Netherlands*, 191-197 (1997).

[5] H. E. Bell, On prime near-rings whith generalized derivations, *Int. J. Math. and Math. Sci.,* (2008), Article ID 490316, 5 pages.

[6] H. E. Bell, A. Boua and L. Oukhtite, Differential identities on semigroup ideals of right near-rings, *Asian Eur. J. Math.,* **6** (4) (2013) 1350050 (8 pages).

[7] A. Boua and L. Oukhtite, Derivations on prime near-rings, *Int. J. Open Probl. Comput. Sci. Math.,* **4** (2), 162-167 (2011).

[8] A. Boua and L. Oukhtite, Generalized derivations and commutativity of prime near-rings, *J. Adv. Res. Pure Math.,* **3**, 120-124 (2011).

[9] A. Boua, H. E. Bell and L. Oukhtite, On derivations of prime near-rings, *Afr. Diaspora J. Math.,* **14**, 65-72 (2012).

[10] A. Boua and L. Oukhtite, On commutativity of prime near-rings with derivations, *Southeast Asian Bull. Math.,* **37**, 325-331 (2013).

[11] M. N. Daif and H. E. Bell, Remarks on derivations on semiprime rings, *Internat. J. Math & Math Sci.,* **15**, 205-206 (1992).

[12] Ö. Gölbasi, On generalized derivations of prime near-rings, *Hacet. J. Math. Stat.,* **35** (2), 173-180 (2006).

[13] B. Hvala, Generalized derivations in rings, *Comm. Algebra,* **26**, 1147-1166 (1998).

[14] M. A. Quadri, M. Shadab Khan and N. Rehman,Generalized derivations and com- muttivity of prime rings, *Indian J. pure appl. Math.,* **39**, 1393-1396 (2003).

[15] X. K. Wang, Derivations in prime near-rings, *Proc. Amer. Math. Soc.,* **121**, 361-366 (1994).

Author information

A. Boua, Ibn Zohr University, Faculty of Sciences,
Department of Mathematics, P.O.Box 8106, Agadir, Morocco.
E-mail: `karimoun2006@yahoo.fr`

L. Oukhtite, Moulay Ismaïl University, Department of Mathematics
FST Errachidia, P.O.Box 509-Boutalamine, Morocco.
E-mail: `oukhtitel@hotmail.com`

A. Raji, Moulay Ismaïl University, Department of Mathematics
FST Errachidia, P.O.Box 509-Boutalamine, Morocco.
E-mail: `rajiabd2@gmail.com`

Received: December 23, 2014.

Accepted: February 14, 2015