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SOME IDEAL CONVERGENT SEQUENCE SPACES OF FUZZY REAL NUMBERS

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Abstract. In this article we introduce some new sequence spaces of fuzzy numbers using I-convergence and study some basic topological and algebraic properties of these spaces. Also we investigate the relations related to these spaces and some of their properties like solidity, symmetricity, convergence free etc. and prove some inclusion results.

1 Introduction

The concept of fuzzy sets was initially introduced by Zadeh[6]. It has a wide range of application in almost all branches of study, in particular in science, where mathematics is used. Now the notion of fuzzyness is used by many researcher in cybernetics, artificial intelligence, expert systems and fuzzy control, pattern recognition, operation research, decision making, image analysis, projectiles, probability theory, agriculture, weather forecasting etc. It has attracted many researcher on sequence space and summability theory to introduce different types of fuzzy sequence spaces and to study their different properties. Our study is based on the linear spaces of sequences of fuzzy numbers which are very important for higher level studies in quantum mechanics, particle physics and statistical mechanics etc.

The notion of I-convergence was initially introduced by Kostyrko et.al[9]. Later on it was further investigated from the sequence space point of view and linked with the summability theory by Salat et.al[11], Tipathy and Hazarika[1][2][3], Kumar and Kumar[12], Savas[5], Nanda[4], Kamthan and Gupta[10], M.Sen[7]. In this article we introduce the sequence spaces ${}_Fc^I$, ${}_Fc_0^I$, ${}_Fl_\infty^I$ of fuzzy numbers defined by I- convergence. We study some basic topological and algebraic properties of these spaces.We also investigate the relations related to these spaces and some of their properties viz. solidity, symmetry, convergence free etc. and prove some inclusion results.

Let X be a non-empty set. Then a family of sets $I \subset 2^X$ is said to be an ideal if I is additive, i.e.$A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e. $A \in I$,$B \subset A \Rightarrow B \in I$. A non-empty family of sets $F \subset 2^X$ is said to be a filter on X iff i) $\phi \notin F$, ii) for all $A, B \in F \Rightarrow A \cap B \in F$ iii) $A \in F$, $A \subset B \Rightarrow B \in F$. An ideal $I \subset 2^X$ is called non-trivial if $I \subset 2^X$. A non-trivial ideal I is called admissible iff $I \neq 2^X$. A non-trivial ideal I is maximal if there does not exist any non-trivial ideal $J \neq I$, containing I as a subset. For each ideal I there is a filter $F(I)$ corresponding to I i.e $F(I) = \{K \subseteq$N: $K^c \in I\}$, where $K^c = N - K$. Further details on ideals of $I \subset 2^X$ can be found in Kostyrko et.al[9].

Lemma 1.1:[8] If $I \subset 2^N$ is a maximal ideal then for each $A \in N$, we have either $A \in I$ or $N - A \in I$.

Example:[8] Let $I = I_F = \{A \subseteq N$: A is finite$\}$.Then I_F is non trivial admissible ideal of N and the corresponding convergence coincides with ordinary convergence.

Example:[8] Let $I = I_\delta = \{A \subseteq N : \delta(A) = 0\}$ where $\delta(A)$ denotes the asymptotic density of A. Then I_δ is a non-trivial admissible ideal of N and the corresponding convergence coincide with statistical convergence.

Definition 1.2: A sequence $(X_k) \in w$ is said to be I- convergent to a number L if for every $\epsilon > 0$, $\{k \in N : |X_k - L| > \epsilon\} \in I$ and we write $I - limX_k = L$.

Definition 1.3: A sequence $(X_k) \in w$ is said to be I- null if for every $\epsilon > 0$, $\{k \in N : |X_k| > \epsilon\} \in I$ and we write $I - limX_k = 0$.

Definition 1.4: A sequence $(X_k) \in w$ is said to be I- bounded if there exists $M > 0$ such that $\{k \in N : |X_k| > M\} \in I$

2 Fuzzy number and its algebra

Now we shall give a brief introduction about the sequences of fuzzy real numbers. Let D be the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line R. For $X, Y \in D$ we define $X \leq Y$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$ and $d(X,Y)$= max$\{|x_1 - y_1|, |x_2 - y_2|\}$ where $X = [x_1, x_2]$ and $Y = [y_1, y_2]$. Then it can be shown that (D, d) is a complete metric space. Also the relation $' <'$ is a partial order relation on D. A fuzzy number X is a fuzzy subset of the real line R, i.e, a mapping $X : R \rightarrow I = [0, 1]$ associating each real number t with its grade of membership $X(t)$. A fuzzy number X is normal if there exists $t_0 \in R$ such that $X(t_0) = 1$. A fuzzy number X is upper semi continuous if for each $\epsilon > 0$, $X^{-1}([0, a+\epsilon))$ is open in the usual topology for all $a \in [0, 1]$.

Let $R(I)$ denote the set of all fuzzy numbers which are upper semi continuous and have a compact support ,i.e, if $X \in R(I)$ then for any $\alpha \in [0, 1]$, $[X]^\alpha$ is compact where $[X]^\alpha = \{t \in R : X(t) \geq \alpha\}$.

The set R of all real numbers can be embedded into $R(I)$ if we define $\bar{r}(t)$=$\begin{cases} 1 & \text{for } r \neq t \\ 0 & \text{for } r = t \end{cases}$

The additive identity and multiplicative identity of $R(I)$ are $\bar{0}$ and $\bar{1}$ respectively. The arithmetic operators on $R(I)$ are defined as follows:

Let $X, Y \in R(I)$ and the α-level set $[X]^\alpha = [x_1^\alpha, x_2^\alpha]$ and $[Y]^\alpha = [y_1^\alpha, y_2^\alpha]$ and $\alpha \in [0, 1]$. Then we define

$[X \oplus Y]^\alpha$ =$[x_1^\alpha + y_1^\alpha, x_2^\alpha + y_2^\alpha]$

$[X \ominus Y]^\alpha$=$[x_1^\alpha - y_1^\alpha, x_2^\alpha - y_2^\alpha]$

$[X \otimes Y]^\alpha$=$[min\{x_i^\alpha y_i^\alpha\}, max\{x_i^\alpha y_i^\alpha\}]$,$i = 1, 2$

$[X^{-1}]^\alpha$=$[(x_2^\alpha)^{-1}, (x_1^\alpha)^{-1}]$, $x_i^\alpha > 0$ for all $\alpha \in [0, 1]$

For $r \in R$ and $X \in R(I)$, the product rX is defined as rX(t)=$\begin{cases} X(r^{-1}t) & \text{for r} \neq 0 \\ 0 & \text{for r=0} \end{cases}$

The absolute value $|X|(t)$ is defined by $|X|(t)=\begin{cases} max\{X(t), X(-t)\} & \text{for } t>0 \\ 0 & \text{for } t\leq 0 \end{cases}$

Let us define a mapping $\overline{d} : R(I) \times R(I) \longrightarrow R^{+} \bigcup\{0\}$ by $\overline{d}(X, Y)$=sup $d([X]^{\alpha}, [Y]^{\alpha})$.

It can be shown that $(R(I), \overline{d})$is a complete metric space.

A sequence of fuzzy numbers $X = (X_k)$ is said to be $I - convergent$ to a fuzzy number X_0 if for each $\epsilon > 0, A(\epsilon) = \{k \in N : \overline{d}(X_k, X_0) > \epsilon\} \in I$. The fuzzy number X_0 is called the $I - limit$ of the sequence (X_k) of fuzzy number and we write $I - limX_k = X_0$.

A sequence of fuzzy numbers $X = (X_k)$ is said to be $I - bounded$ if $\exists\ M > 0$ such that $\{k \in N : \overline{d}(X_k, \overline{0}) > M\} \in I$.

Let E_F denote the sequence spaces of fuzzy numbers.

A sequence space E_F is said to be solid (or normal) if $(Y_k) \in E_F$ whenever $(X_k) \in E_F$ and $|Y_k| \leq |X_k|$ for all $k \in N$.

A sequence space E_F is said to be symmetric if $(X_k) \in E_F \Rightarrow (X_{\pi(k)}) \in E_F$ where π is a permutation of N.

A sequence space E_F is said to be monotone if E_F contains the canonical pre image of all its step spaces.

Lemma 2.1:[10] A sequence space E_F is normal implies that it is monotone.

3 Main result

In this section we shall introduce the following new sequence spaces of fuzzy numbers and examine some properties of the resulting sequence spaces.

Let I be an admissible ideal of N and $X = (X_k)$ be a sequence of fuzzy numbers. We define the following sequence spaces of fuzzy number.

${}_Fc^I$=$\{X_k \in E_F : A(\epsilon) = \{k \in N : \overline{d}(X_k, L) \geq \epsilon\} \in I$ for $\epsilon > 0$ and $L \in R(I)$.

${}_Fc_0^I$=$\{X_k \in E_F : A(\epsilon) = \{k \in N : \overline{d}(X_k, \overline{0}) \geq \epsilon\} \in I$ for $\epsilon > 0$ and $L \in R(I)$.

${}_Fl_\infty^I$=$\{X_k \in E_F : \exists M > 0 : \{k \in N : \overline{d}(X_k, \overline{0}) \geq M\} \in I\}$.

${}_Fl_\infty$=$\{X_k \in E_F$:$Sup_k\ \overline{d}(X_k, \overline{0}) < \infty\}$.

From definition it is obvious that ${}_Fc_0^I \subset_F l_\infty^I \subset_F l_\infty$

Example 3.1: Let $X_k(t) = \overline{1}$ for $k = 2^n$,n=1,2,...

otherwise, $X_k(t) = \begin{cases} \frac{k}{3}(t-2)+1 & \text{for } t \in [\frac{2k-3}{2}, 2] \\ \frac{-k}{3}(t-2)+1 & \text{for } t \in [2, \frac{2k+3}{2}] \\ 0 \text{ , otherwise} & \end{cases}$

$$[X_k]^\alpha = \begin{cases} [1,1] & ,k=2^n, n=1,2,... \\ 2-\frac{3}{k}(1-\alpha), 2+\frac{3}{k}(1-\alpha) & ,otherwise \end{cases}$$

$$\overline{d}(X_k,\overline{0}) = Sup_\alpha d([X_k]^\alpha,[0]^\alpha) = Sup_\alpha\{\max \begin{cases} |1-0|,|1-0| & ,k=2^n, n=1,2,... \\ |2-\frac{3}{k}(1-\alpha)|,|2+\frac{3}{k}(1-\alpha)| & ,k\neq 2^n \end{cases}$$

$sup_k\ \overline{d}(X_k,\overline{0}) = 5$

Thus $(X_k) \in_F l^\infty$. But(X_k) is not I-convergent.

Remark 3.2: If $I = I_F$ then the sequence spaces${}_Fc^I$, ${}_Fc_0^I$, ${}_Fl_\infty^I$ coincide with the sequence spaces ${}_Fc$, ${}_Fc_0$, ${}_Fl_\infty$ which were studied by Tripathy and Nanda [4], Savas [5], Das and Choudhury [8] and many others.

Theorem 3.3: The spaces ${}_Fc^I$, ${}_Fc_0^I$, ${}_Fl_\infty^I$ are linear space.

Proof: Let $X=(X_k)$ and $Y=(Y_k)$ be any element of ${}_Fc_0^I$ and α,β be any scalar.Then $A(\epsilon)=\{k\in N:\overline{d}(X_k,\overline{0})\geq\frac{\epsilon}{2}\}\in I.$

$B(\epsilon)=\{k\in N:\overline{d}(Y_k,\overline{0})\geq\frac{\epsilon}{2}\}\in I.$

$\overline{d}(\alpha X_k+\beta Y_k,\overline{0})\leq|\alpha|\overline{d}(X_k,\overline{0})+|\beta|\overline{d}(Y_k,\overline{0})$

Now $C(\epsilon)$=$\{k\in N:\overline{d}(\alpha X_k+\beta Y_k,\overline{0})\geq\epsilon\}\subseteq\{k\in N:|\alpha|\overline{d}(X_k,\overline{0})\geq\frac{\epsilon}{2}\}\bigcup\{k\in N:|\beta|\overline{d}(Y_k,\overline{0})\geq\frac{\epsilon}{2}\}$ =$\{k\in N:\overline{d}(X_k,\overline{0})\geq\frac{\epsilon}{2|\alpha|}\}\bigcup\{k\in N:\overline{d}(Y_k,\overline{0})\geq\frac{\epsilon}{2|\beta|}\}\subseteq A(\frac{\epsilon}{2|\alpha|})\bigcup B(\frac{\epsilon}{2|\beta|})\in I$.

Hence ${}_Fc_0^I$ is a linear space. Similarly ${}_Fc^I$ and ${}_Fl_\infty^I$ are linear spaces. ■

Theorem 3.4: The spaces ${}_Fc_0^I$ and ${}_Fl_\infty^I$ are normal and monotone.

Proof: Let $X=(X_k)$ be any element of ${}_Fc_0^I$ and $Y=(Y_k)$ be any sequence such that $\overline{d}(Y_k,\overline{0})\leq\overline{d}(X_k,\overline{0})$ for all $k\in N$. Then for all $\epsilon>0$, $\{k\in N:\overline{d}(Y_k,\overline{0})\geq\epsilon\}\subseteq\{k\in N:\overline{d}(X_k,\overline{0})\geq\epsilon\}\in I$. Hence $Y=(Y_k)\in_F c_0^I$. Thus the spaces ${}_Fc_0^I$ is normal and hence monotone. Similarly ${}_Fl_\infty^I$ is normal and monotone. ■

Theorem 3.5: If I is not maximal ideal then the space ${}_Fc^I$ is neither normal nor monotone.

Example: Let us consider a sequence of fuzzy number

$$X_k(t)=\begin{cases}\frac{1+t}{2} & ,-1\leq t\leq 1\\ \frac{3-t}{2} & ,1\leq t\leq 3\\ 0 & \text{, otherwise}\end{cases}$$

Then $(X_k)\in_F c^I$. Since I is not maximal, by $lemma 1.1$, there exists a subset K of N such that $K\notin I$ and $N-K\notin I$. Let us define a sequence $Y=(Y_k)$ by

$$Y_k=\begin{cases} X_k & \text{, k}\in K\\ 0 & \text{, otherwise}\end{cases}$$

Then (Y_k) belongs to the canonical pre image of the k-step spaces of ${}_Fc^I$. But $(y_k)\notin_F c^I$. Hence ${}_Fc^I$ is not monotone. Therefore by lemma 2.1, ${}_Fc^I$ is not normal.

Proposition 3.6: If I is neither maximal nor $I=I_F$ then the spaces ${}_Fc^I$ and ${}_Fc_0^I$ are not symmetric.

Example 3.7: Let us consider a sequence of fuzzy number defined by $X=(X_k)$, where

$$X_k(t)=\begin{cases} 1+t & ,\text{-}1\leq t\leq 0 \\ 1-t & ,0\leq t\leq 1 \end{cases}$$

Then for $k \in A \in I$(an infinite set), $(X_k) \in_F c^I$. Let $K \subset N$ be such that $K \notin I$ and $N - K \notin I$. Let us consider a sequence space $Y = (Y_k)$, a rearrangement of the sequence (X_k) defined by

$$Y_k=\begin{cases} X_k & ,\text{k}\in K \\ 0 & ,\text{ otherwise} \end{cases}$$

Then $(Y_k) \notin_F c^I$. Hence ${}_Fc^I$ is not symmetric. Similarly ${}_Fc_0^I$ is not symmetric.

Theorem 3.8: The spaces ${}_Fc^I$, ${}_Fc_0^I$, ${}_Fl_\infty^I$ are sequence algebra.

Proof: Let X_k and Y_k be two elements of ${}_Fc^I$. For $\alpha \in [0,1]$, let $X_k^\alpha, Y_k^\alpha, X_0^\alpha, Y_0^\alpha$ be the α level set of X_k, Y_k, X_0, Y_0 respectively. Since $d(X_k^\alpha Y_k^\alpha, X_0^\alpha Y_0^\alpha)\leq C_1 d(X_k^\alpha, X_0^\alpha)+C_2 d(X_k^\alpha, X_0^\alpha)$, therefore we have $\overline{d}(X_kY_k, X_0Y_0)\leq C_1\overline{d}(X_k, X_0) + C_2\overline{d}(Y_k, Y_0)$. Let $\epsilon > 0$ be given.Then
$A(\frac{\epsilon}{2}) = \{k \in N : \overline{d}(X_k, X_0 \geq \frac{\epsilon}{2}\} \in I$
$B(\frac{\epsilon}{2}) = \{k \in N : \overline{d}(Y_k, Y_0 \geq \frac{\epsilon}{2}\} \in I.$
$C(\epsilon) = \{k \in N : \overline{d}(X_kY_k, X_0Y_0 \geq \epsilon\}$
To prove the result it is sufficient to prove that $C(\epsilon) \subset A(\epsilon_1)\bigcup B(\epsilon_2)$.
Now $\{k \in N : \overline{d}(X_kY_k, X_0Y_0) \geq \epsilon\} \subseteq C_1\{k \in N : \overline{d}(X_k, X_0) \geq \frac{\epsilon}{2}\}\bigcup C_2\{k \in N : \overline{d}(Y_k, Y_0 \geq \frac{\epsilon}{2}\}$
$C(\epsilon) \subset \{k \in N : \overline{d}(X_k, X_0) \geq \frac{\epsilon}{2C_1}\}\bigcup\{k \in N : \overline{d}(Y_k, Y_0) \geq \frac{\epsilon}{2C_2}\}$. i.e $C(\epsilon) \subseteq A(\epsilon_1)\bigcup B(\epsilon_2)$, where $\epsilon_1 = \frac{\epsilon}{2C_1}, \epsilon_2 = \frac{\epsilon}{2C_2}$.
The other results can be shown similarly. ■

Theorem 3.9: The spaces ${}_Fc^I$, ${}_Fc_0^I$, are not convergence free in general.

Proof: Let us consider a sequence of fuzzy number defined by $X_k(t)=\begin{cases} \frac{1+t}{2} & ,-1\leq t\leq 1 \\ \frac{3-t}{2} & ,1\leq t\leq 3 \\ 0 & ,\text{ otherwise} \end{cases}$

Then $X_k(t) \in_F c^I$. Let $Y_k(t)=\frac{1}{k}$ for all $k \in N$.Then $Y_k(t) \in_F c^I$.But $X_k = 0$ does not implies $Y_k = 0$.Hence ${}_Fc^I$ is not convergence free. Similarly ${}_Fc_0^I$ is not convergence free.■

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