Palestine Journal of Mathematics

Vol. 4(1) (2015) , 21–29

On new general integral inequalities for quasi-convex functions and their applications

Imdat İşcan

Communicated by Ayman Badawi

MSC 2010 Classifications: 26A51, 26D15.

Keywords and phrases: quasi-convex function, Simpson's inequality, Hermite-Hadamard's inequality, midpoint inequality, trapezoid inequality.

Abstract. In this paper, we give a unified approach to establish midpoint, trapezoid, and Simpson's inequalities for functions whose derivatives in absolute value at certain power are quasi-convex. Some applications to special means of real numbers are also given.

1 Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. See [1, 3, 4, 6, 7, 9, 10, 11], the results of the generalization, improvement and extention of the famous integral inequality (1.1).

The notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a, b] \to \mathbb{R}$ is said quasi-convex on $[a, b]$ if

$$f(tx + (1-t)y) \leq \sup\{f(x), f(y)\},$$

for any $x, y \in [a, b]$ and $t \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [7]).

The following inequality is well known in the literature as Simpson's inequality .

Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\left\|f^{(4)}\right\|_\infty = \sup\limits_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then the following inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

In recent years many authors have studied error estimations for Simpson's inequality; for refinements, counterparts, generalizations and new Simpson's type inequalities, see [2, 5, 12, 13, 14]

In [7], Ion introduced two inequalities of the right hand side of Hadamard's type for quasi-convex functions, as follow:

Theorem 1.1. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). If $|f'|$ is quasi-convex on $[a, b]$, then the following inequality holds true*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\sup\{|f'(a)|, |f'(b)|\}. \tag{1.2}$$

Theorem 1.2. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{p/(p-1)}$ is quasi-convex on $[a, b]$, then the following inequality* *holds true*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2\,(p+1)^{p/(p-1)}}\left(\sup\left\{|f'(a)|^{\frac{p}{p-1}},|f'(b)|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}. \tag{1.3}$$

In [3], Alomari et al. established some new upper bound for the right -hand side of Hadamard's inequality for quasi-convex mappings, which is the better than the inequality had done in [7]. The authors obtained the following results:

Theorem 1.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is an quasi-convex on $[a,b]$, for $p > 1$, then the following inequality holds:*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{4\,(p+1)^{1/p}}\left[\left(\sup\left\{\left|f'(\frac{a+b}{2})\right|^{\frac{p}{p-1}},|f'(b)|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right.\\ &\left.+\left(\sup\left\{\left|f'(\frac{a+b}{2})\right|^{\frac{p}{p-1}},|f'(a)|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right].\end{aligned} \tag{1.4}$$

Theorem 1.4. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is an quasi-convex on $[a,b]$, for $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{8}\left[\left(\sup\left\{\left|f'(\frac{a+b}{2})\right|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\sup\left\{\left|f'(\frac{a+b}{2})\right|^q,|f'(a)|^q\right\}\right)^{\frac{1}{q}}\right].\end{aligned} \tag{1.5}$$

In this paper, in order to provide a unified approach to establish midpoint inequality, trapezoid inequality and Simpson's inequality for functions whose derivatives in absolute value at certain power are quasi-convex, we need the following lemma:

Lemma 1.5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0,1]$. Then the following equality holds:*

$$\lambda\left(\alpha f(a)+(1-\alpha)f(b)\right)+(1-\lambda)f(\alpha a+(1-\alpha)b)-\frac{1}{b-a}\int_a^b f(x)dx \tag{1.6}$$

$$\begin{aligned}&= (b-a)\left[\int_0^{1-\alpha}(t-\alpha\lambda)f'(tb+(1-t)a)\,dt\right.\\ &\left.+\int_{1-\alpha}^1(t-1+\lambda(1-\alpha))f'(tb+(1-t)a)\,dt\right].\end{aligned}$$

A simple proof of equality can be given by performing an integration by parts in the integrals from the right side and changing the variable (see [8]).

2 Main results

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$ and $\alpha, \lambda \in [0,1]$. If $|f'|^q$ is quasi-convex on $[a,b]$, $q \geq 1$, then the following inequality holds:*

$$\left|\lambda\left(\alpha f(a)+(1-\alpha)f(b)\right)+(1-\lambda)f(\alpha a+(1-\alpha)b)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \begin{cases}(b-a)(\gamma_2+\upsilon_2)A^{\frac{1}{q}} & \alpha\lambda\leq 1-\alpha\leq 1-\lambda(1-\alpha)\\ (b-a)(\gamma_2+\upsilon_1)A^{\frac{1}{q}} & \alpha\lambda\leq 1-\lambda(1-\alpha)\leq 1-\alpha\\ (b-a)(\gamma_1+\upsilon_2)A^{\frac{1}{q}} & 1-\alpha\leq\alpha\lambda\leq 1-\lambda(1-\alpha)\end{cases} \tag{2.1}$$

where

$$\gamma_1 = (1-\alpha)\left[\alpha\lambda - \frac{(1-\alpha)}{2}\right],\ \gamma_2 = (\alpha\lambda)^2 - \gamma_1\ ,$$

$$\begin{aligned} \upsilon_1 &= \frac{1-(1-\alpha)^2}{2} - \alpha\left[1-\lambda(1-\alpha)\right], \\ \upsilon_2 &= \frac{1+(1-\alpha)^2}{2} - (\lambda+1)(1-\alpha)\left[1-\lambda(1-\alpha)\right], \end{aligned}$$

and

$$A = \sup\left\{|f'(a)|^q, |f'(b)|^q\right\}.$$

Proof. Suppose that $q \geq 1$. From Lemma 1.5 and using the well known power mean inequality, we have

$$\begin{aligned} &\left|\lambda(\alpha f(a) + (1-\alpha) f(b)) + (1-\lambda) f(\alpha a + (1-\alpha) b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ \leq\ & (b-a)\left[\int_0^{1-\alpha} |t-\alpha\lambda|\,|f'(tb+(1-t)a)|\,dt + \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)|\,|f'(tb+(1-t)a)|\,dt\right] \\ \leq\ & (b-a)\left\{\left(\int_0^{1-\alpha}|t-\alpha\lambda|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^{1-\alpha}|t-\alpha\lambda|\,|f'(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\right. \\ &+\left.\left(\int_{1-\alpha}^1|t-1+\lambda(1-\alpha)|\,dt\right)^{1-\frac{1}{q}}\left(\int_{1-\alpha}^1|t-1+\lambda(1-\alpha)|\,|f'(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\right\}. \end{aligned} \tag{2.2}$$

Since $|f'|^q$ is quasi-convex on $[a,b]$, we know that for $t \in [0,1]$

$$|f'(tb+(1-t)a)|^q \leq \sup\left\{|f'(a)|^q, |f'(b)|^q\right\},$$

hence, by simple computation

$$\int_0^{1-\alpha}|t-\alpha\lambda|\,dt = \begin{cases} \gamma_2, & \alpha\lambda \leq 1-\alpha \\ \gamma_1, & \alpha\lambda \geq 1-\alpha \end{cases}, \tag{2.3}$$

$$\int_{1-\alpha}^1|t-1+\lambda(1-\alpha)|\,dt = \begin{cases} \upsilon_1, & 1-\lambda(1-\alpha) \leq 1-\alpha \\ \upsilon_2, & 1-\lambda(1-\alpha) \geq 1-\alpha \end{cases}, \tag{2.4}$$

Thus, using (2.3) and (2.4) in (2.2), we obtain the inequality (2.1). This completes the proof. □

Corollary 2.2. *Under the assumptions of Theorem 2.1 with $q = 1$, the inequality (2.1) reduced to the following inequality*

$$\left|\lambda(\alpha f(a) + (1-\alpha) f(b)) + (1-\lambda) f(\alpha a + (1-\alpha) b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq$$

$$\begin{cases} (b-a)(\gamma_2+\upsilon_2)\sup\{|f'(a)|, |f'(b)|\} & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ (b-a)(\gamma_2+\upsilon_1)\sup\{|f'(a)|, |f'(b)|\} & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ (b-a)(\gamma_1+\upsilon_2)\sup\{|f'(a)|, |f'(b)|\} & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases},$$

where where $\gamma_1,\ \gamma_2,\ \upsilon_1$ and υ_2 are defined as in Theorem 2.1.

Corollary 2.3. *Under the assumptions of Theorem 2.1 with $\alpha=\frac{1}{2}$ and $\lambda=\frac{1}{3}$, from the inequality (2.1) we get the following Simpson type inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ & (b-a)\left(\frac{5}{36}\right)\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}.\end{aligned}$$

Corollary 2.4. *Under the assumptions of Theorem 2.1 with $\alpha=\frac{1}{2}$ and $\lambda=0$, from the inequality (2.1) we get the following midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}.$$

Corollary 2.5. *Under the assumptions of Theorem 2.1 with $\alpha=\frac{1}{2}$ and $\lambda=1$, from the inequality (2.1) we get the following trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}.$$

which is the same of the inequality (1.2) for $q=1$.

Using Lemma 1.5 we shall give another result for convex functions as follows.

Theorem 2.6. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$ and $\alpha,\lambda\in[0,1]$. If $|f'|^q$ is quasi-convex on $[a,b]$, $q>1$, then the following inequality holds:*

$$\left|\lambda\left(\alpha f(a)+(1-\alpha)f(b)\right)+(1-\lambda)f(\alpha a+(1-\alpha)b)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq(b-a)\tag{2.5}$$

$$\times\left(\frac{1}{p+1}\right)^{\frac{1}{p}}A^{\frac{1}{q}}\begin{cases}\left[(1-\alpha)^{\frac{1}{q}}\varepsilon_1^{\frac{1}{p}}+\alpha^{\frac{1}{q}}\varepsilon_3^{\frac{1}{p}}\right], & \alpha\lambda\leq1-\alpha\leq1-\lambda(1-\alpha)\\ \left[(1-\alpha)^{\frac{1}{q}}\varepsilon_1^{\frac{1}{p}}+\alpha^{\frac{1}{q}}\varepsilon_4^{\frac{1}{p}}\right], & \alpha\lambda\leq1-\lambda(1-\alpha)\leq1-\alpha\\ \left[(1-\alpha)^{\frac{1}{q}}\varepsilon_2^{\frac{1}{p}}+\alpha^{\frac{1}{q}}\varepsilon_3^{\frac{1}{p}}\right], & 1-\alpha\leq\alpha\lambda\leq1-\lambda(1-\alpha)\end{cases}$$

where

$$A=\sup\left\{|f'(a)|^q,|f'(b)|^q\right\},$$

$$\begin{aligned}\varepsilon_1&=(\alpha\lambda)^{p+1}+(1-\alpha-\alpha\lambda)^{p+1},\ \varepsilon_2=(\alpha\lambda)^{p+1}-(\alpha\lambda-1+\alpha)^{p+1},\\ \varepsilon_3&=[\lambda(1-\alpha)]^{p+1}+[\alpha-\lambda(1-\alpha)]^{p+1},\ \varepsilon_4=[\lambda(1-\alpha)]^{p+1}-[\lambda(1-\alpha)-\alpha]^{p+1},\end{aligned}$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and by Hölder's integral inequality, we have

$$\begin{aligned}&\left|\lambda\left(\alpha f(a)+(1-\alpha)f(b)\right)+(1-\lambda)f(\alpha a+(1-\alpha)b)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ &(b-a)\left[\int_0^{1-\alpha}|t-\alpha\lambda|\,|f'(tb+(1-t)a)|\,dt+\int_{1-\alpha}^1|t-1+\lambda(1-\alpha)|\,|f'(tb+(1-t)a)|\,dt\right]\\ \leq\ &(b-a)\left\{\left(\int_0^{1-\alpha}|t-\alpha\lambda|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^{1-\alpha}|f'(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\right.\end{aligned}$$

$$+\left(\int_{1-\alpha}^{1}|t-1+\lambda(1-\alpha)|^{p}\,dt\right)^{\frac{1}{p}}\left(\int_{1-\alpha}^{1}|f'(tb+(1-t)a)|^{q}\,dt\right)^{\frac{1}{q}}\Bigg\}. \tag{2.6}$$

Since $|f'|^q$ is quasi-convex on $[a,b]$, for $\alpha\in[0,1]$, we get

$$\int_{0}^{1-\alpha}|f'(tb+(1-t)a)|^{q}\,dt=(1-\alpha)\sup\left\{|f'(a)|^{q},|f'(b)|^{q}\right\} \tag{2.7}$$

Similarly, for $\alpha\in[0,1]$, we have

$$\int_{1-\alpha}^{1}|f'(tb+(1-t)a)|^{q}\,dt=\alpha\sup\left\{|f'(a)|^{q},|f'(b)|^{q}\right\}. \tag{2.8}$$

By simple computation

$$\int_{0}^{1-\alpha}|t-\alpha\lambda|^{p}\,dt=\begin{cases}\frac{(\alpha\lambda)^{p+1}+(1-\alpha-\alpha\lambda)^{p+1}}{p+1}, & \alpha\lambda\le 1-\alpha\\ \frac{(\alpha\lambda)^{p+1}-(\alpha\lambda-1+\alpha)^{p+1}}{p+1}, & \alpha\lambda\ge 1-\alpha\end{cases}, \tag{2.9}$$

and

$$\int_{1-\alpha}^{1}|t-1+\lambda(1-\alpha)|^{p}\,dt=\begin{cases}\frac{[\lambda(1-\alpha)]^{p+1}+[\alpha-\lambda(1-\alpha)]^{p+1}}{p+1}, & 1-\alpha\le 1-\lambda(1-\alpha)\\ \frac{[\lambda(1-\alpha)]^{p+1}-[\lambda(1-\alpha)-\alpha]^{p+1}}{p+1}, & 1-\alpha\ge 1-\lambda(1-\alpha)\end{cases}, \tag{2.10}$$

thus, using (2.7)-(2.10) in (2.6), we obtain the inequality (2.5). This completes the proof. □

Corollary 2.7. *Under the assumptions of Theorem 2.6 with $\alpha=\frac{1}{2}$ and $\lambda=\frac{1}{3}$, from the inequality (2.5) we get the following Simpson type inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|$$

$$\le\frac{b-a}{6}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left(\sup\left\{|f'(a)|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}}.$$

Corollary 2.8. *Under the assumptions of Theorem 2.6 with $\alpha=\frac{1}{2}$ and $\lambda=0$, from the inequality (2.5) we get the following midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\le\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\sup\left\{|f'(a)|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}}.$$

Corollary 2.9. *Let the assumptions of Theorem 2.6 hold. Then for $\alpha=\frac{1}{2}$ and $\lambda=1$, from the inequality (2.5) we get the following trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\le\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\sup\left\{|f'(a)|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}}.$$

Theorem 2.10. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I^{\circ}$ with $a<b$ and $\alpha,\lambda\in[0,1]$. If $|f'|^{q}$ is quasi-convex on $[a,b]$, $q>1$, then the following inequality holds:*

$$\left|\lambda(\alpha f(a)+(1-\alpha)f(b))+(1-\lambda)f(\alpha a+(1-\alpha)b)-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\le(b-a) \tag{2.11}$$

$$\times\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\begin{cases}\left[(1-\alpha)^{\frac{1}{q}}B^{\frac{1}{q}}\varepsilon_{1}^{\frac{1}{p}}+\alpha^{\frac{1}{q}}C^{\frac{1}{q}}\varepsilon_{3}^{\frac{1}{p}}\right], & \alpha\lambda\le 1-\alpha\le 1-\lambda(1-\alpha)\\ \left[(1-\alpha)^{\frac{1}{q}}B^{\frac{1}{q}}\varepsilon_{1}^{\frac{1}{p}}+\alpha^{\frac{1}{q}}C^{\frac{1}{q}}\varepsilon_{4}^{\frac{1}{p}}\right], & \alpha\lambda\le 1-\lambda(1-\alpha)\le 1-\alpha\\ \left[(1-\alpha)^{\frac{1}{q}}B^{\frac{1}{q}}\varepsilon_{2}^{\frac{1}{p}}+\alpha^{\frac{1}{q}}C^{\frac{1}{q}}\varepsilon_{3}^{\frac{1}{p}}\right], & 1-\alpha\le\alpha\lambda\le 1-\lambda(1-\alpha)\end{cases}$$

where

$$\begin{aligned} B &= \sup\left\{|f'(a)|^q, |f'(\alpha a + (1-\alpha)b)|^q\right\}, \\ C &= \sup\left\{|f'(b)|^q, |f'(\alpha a + (1-\alpha)b)|^q\right\}, \end{aligned}$$

, $\frac{1}{p}+\frac{1}{q}=1$, *and* $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3,\ \varepsilon_4$ *numbers are defined as in Theorem 2.6.*

Proof. From Lemma 1.5 and by Hölder's integral inequality, we have the inequality (2.6). Since $|f'|^q$ is quasi-convex on $[a,b]$, for all $t \in [0,1]$ and $\alpha \in [0,1)$ we get

$$|f'(ta + (1-t)[\alpha a + (1-\alpha)b])|^q \le B = \sup\left\{|f'(a)|^q, |f'(\alpha a + (1-\alpha)b)|^q\right\}$$

then

$$\int_0^1 |f'(ta + (1-t)[\alpha a + (1-\alpha)b])|^q\,dt = \frac{1}{(1-\alpha)(b-a)} \int_a^{(1-\alpha)b+\alpha a} |f'(x)|^q\,dx \le B. \tag{2.12}$$

By the inequality (2.12), we get

$$\begin{aligned} \int_0^{1-\alpha} |f'(tb + (1-t)a)|^q\,dt &= (1-\alpha)\left[\frac{1}{(1-\alpha)(b-a)} \int_a^{(1-\alpha)b+\alpha a} |f'(x)|^q\,dx\right] \\ &\le (1-\alpha)B. \end{aligned} \tag{2.13}$$

The inequality (2.13) also holds for $\alpha = 1$. Since $|f'|^q$ is quasi-convex on $[a,b]$, for all $t \in [0,1]$ and $\alpha \in (0,1]$ we have

$$|f'(tb + (1-t)[\alpha a + (1-\alpha)b])|^q \le C = \sup\left\{|f'(b)|^q, |f'(\alpha a + (1-\alpha)b)|^q\right\}$$

then

$$\int_0^1 |f'(tb + (1-t)[\alpha a + (1-\alpha)b])|^q\,dt = \frac{1}{\alpha(b-a)} \int_{(1-\alpha)b+\alpha a}^{b} |f'(x)|^q\,dx \le C. \tag{2.14}$$

By the inequality (2.14), we get

$$\begin{aligned} \int_{1-\alpha}^{1} |f'(tb + (1-t)a)|^q\,dt &= \alpha\left[\frac{1}{\alpha(b-a)} \int_{(1-\alpha)b+\alpha a}^{b} |f'(x)|^q\,dx\right] \\ &\le \alpha C. \end{aligned} \tag{2.15}$$

The inequality (2.15) also holds for $\alpha = 0$. Thus, using (2.9), (2.10), (2.13) and (2.15) in (2.6), we obtain the inequality (2.11). This completes the proof. □

Corollary 2.11. *Under the assumptions of Theorem 2.10 with* $\alpha = \frac{1}{2}$ *and* $\lambda = \frac{1}{3}$*, from the inequality (2.11) we get the following Simpson type inequality*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\le \frac{b-a}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}} \left\{\left(\sup\left\{\left|f'(\frac{a+b}{2})\right|^q, |f'(a)|^q\right\}\right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\sup\left\{\left|f'(\frac{a+b}{2})\right|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Corollary 2.12. *Under the assumptions of Theorem 2.10 with $\alpha = \frac{1}{2}$ and $\lambda = 1$, from the inequality (2.11) we get the following trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4(p+1)^{1/p}}\left[\left\{\left(\sup\left\{\left|f'(\frac{a+b}{2})\right|^q, |f'(a)|^q\right\}\right)^{\frac{1}{q}} + \left\{\left(\sup\left\{\left|f'(\frac{a+b}{2})\right|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}}\right]. $$

which is the same of the inequality (1.4).

Corollary 2.13. *Under the assumptions of Theorem 2.10 with $\alpha = \frac{1}{2}$ and $\lambda = 0$, from the inequality (2.11) we get the following midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4(p+1)^{1/p}}\left[\left\{\left(\sup\left\{\left|f'(\frac{a+b}{2})\right|^q, |f'(a)|^q\right\}\right)^{\frac{1}{q}} + \left\{\left(\sup\left\{\left|f'(\frac{a+b}{2})\right|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}}\right],$$

which is the better than the inequality in [1, Corollary 8].

3 Some applications for special means

Let us recall the following special means of arbitrary real numbers a, b with $a \neq b$ and $\alpha \in [0,1]$:

(i) The weighted arithmetic mean

$$A_\alpha(a,b) := \alpha a + (1-\alpha)b,\ a,b \in \mathbb{R}.$$

(ii) The unweighted arithmetic mean

$$A(a,b) := \frac{a+b}{2},\ a,b \in \mathbb{R}.$$

(iii) The weighted harmonic mean

$$H_\alpha(a,b) := \left(\frac{\alpha}{a} + \frac{1-\alpha}{b}\right)^{-1},\ \ a,b \in \mathbb{R}\setminus\{0\}.$$

(iv) The unweighted harmonic mean

$$H(a,b) := \frac{2ab}{a+b},\ \ a,b \in \mathbb{R}\setminus\{0\}.$$

(v) The Logarithmic mean

$$L(a,b) := \frac{b-a}{\ln b - \ln a},\ \ a,b > 0,\ a \neq b\ .$$

(vi) Then n-Logarithmic mean

$$L_n(a,b) := \left(\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}\right)^{\frac{1}{n}},\ n \in \mathbb{N},\ a,b \in \mathbb{R},\ a \neq b.$$

Proposition 3.1. *Let $a, b \in \mathbb{R}$ with $a < b$, and $n \in \mathbb{N},\ n \ge 2$. Then, for $\alpha, \lambda \in [0,1]$ and $q \ge 1$,we have the following inequality:*

$$|\lambda A_\alpha(a^n, b^n) + (1-\lambda)A_\alpha^n(a,b) - L_n^n(a,b)|$$
$$\le \begin{cases} n(b-a)(\gamma_2+\upsilon_2)E^{\frac{1}{q}} & \alpha\lambda \le 1-\alpha \le 1-\lambda(1-\alpha) \\ n(b-a)(\gamma_2+\upsilon_1)E^{\frac{1}{q}} & \alpha\lambda \le 1-\lambda(1-\alpha) \le 1-\alpha \\ n(b-a)(\gamma_1+\upsilon_2)E^{\frac{1}{q}} & 1-\alpha \le \alpha\lambda \le 1-\lambda(1-\alpha) \end{cases},$$

where

$$E = \sup\left\{|a|^{(n-1)q}, |b|^{(n-1)q}\right\},$$

$\gamma_1,\ \gamma_2,\ \upsilon_1$ *and* υ_2 *are defined as in Theorem 2.1.*

Proof. The assertion follows from Theorem 2.1, for $f(x)=x^{n},\ x\in\mathbb{R}$. □

Proposition 3.2. *Let $a,b\in\mathbb{R}$ with $a<b$, and $n\in\mathbb{N},\ n\geq 2$. Then, for $\alpha,\lambda\in[0,1]$ and $q>1$,we have the following inequality:*

$$\left|\lambda A_{\alpha}\left(a^{n},b^{n}\right)+(1-\lambda)A_{\alpha}^{n}(a,b)-L_{n}^{n}(a,b)\right|\leq(b-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}n$$

$$\times\begin{cases}\left[(1-\alpha)^{\frac{1}{q}}F^{\frac{1}{q}}\varepsilon_{1}^{\frac{1}{p}}+\alpha^{\frac{1}{q}}G^{\frac{1}{q}}\varepsilon_{3}^{\frac{1}{p}}\right], & \alpha\lambda\leq 1-\alpha\leq 1-\lambda(1-\alpha)\\ \left[(1-\alpha)^{\frac{1}{q}}F^{\frac{1}{q}}\varepsilon_{1}^{\frac{1}{p}}+\alpha^{\frac{1}{q}}G^{\frac{1}{q}}\varepsilon_{4}^{\frac{1}{p}}\right], & \alpha\lambda\leq 1-\lambda(1-\alpha)\leq 1-\alpha\\ \left[(1-\alpha)^{\frac{1}{q}}F^{\frac{1}{q}}\varepsilon_{2}^{\frac{1}{p}}+\alpha^{\frac{1}{q}}G^{\frac{1}{q}}\varepsilon_{3}^{\frac{1}{p}}\right], & 1-\alpha\leq\alpha\lambda\leq 1-\lambda(1-\alpha)\end{cases},$$

where

$$\begin{aligned}F&=\sup\left\{|a|^{(n-1)q},|A_{\alpha}(a,b)|^{(n-1)q}\right\},\\ G&=\sup\left\{|b|^{(n-1)q},|A_{\alpha}(a,b)|^{(n-1)q}\right\},\end{aligned}$$

$\frac{1}{p}+\frac{1}{q}=1$, *and $\varepsilon_{1},\ \varepsilon_{2},\ \varepsilon_{3},\ \varepsilon_{4}$ numbers are defined as in Theorem 2.6.*

Proof. The assertion follows from Theorem 2.10, for $f(x)=x^{n},\ x\in\mathbb{R}$. □

Proposition 3.3. *Let $a,b\in\mathbb{R}$ with $a<b,\ 0\notin[a,b]$. Then, for $\alpha,\lambda\in[0,1]$ and $q\geq 1$, we have the following inequality:*

$$\begin{aligned}&\left|\lambda H_{\alpha}^{-1}(a,b)+(1-\lambda)A_{\alpha}^{-1}(a,b)-L^{-1}(a,b)\right|\\ &\leq\begin{cases}(b-a)(\gamma_{2}+\upsilon_{2})K^{\frac{1}{q}} & \alpha\lambda\leq 1-\alpha\leq 1-\lambda(1-\alpha)\\ (b-a)(\gamma_{2}+\upsilon_{1})K^{\frac{1}{q}} & \alpha\lambda\leq 1-\lambda(1-\alpha)\leq 1-\alpha\\ (b-a)(\gamma_{1}+\upsilon_{2})K^{\frac{1}{q}} & 1-\alpha\leq\alpha\lambda\leq 1-\lambda(1-\alpha)\end{cases},\end{aligned}$$

where

$$K=\sup\left\{a^{-2q},b^{-2q}\right\},$$

$\gamma_{1},\ \gamma_{2},\ \upsilon_{1}$, *and* υ_{2} *are defined as in Theorem 2.1..*

Proof. The assertion follows from Theorem 2.1., for $f(x)=\frac{1}{x},\ x\in(0,\infty)$. □

Proposition 3.4. *Let $a,b\in\mathbb{R}$ with $0<a<b$. Then, for $\alpha,\lambda\in[0,1]$ and $q>1$, we have the following inequality:*

$$\left|\lambda H_{\alpha}^{-1}(a,b)+(1-\lambda)A_{\alpha}^{-1}(a,b)-L^{-1}(a,b)\right|\leq(b-a)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\times\begin{cases}\left[(1-\alpha)^{\frac{1}{q}}M^{\frac{1}{q}}\varepsilon_{1}^{\frac{1}{p}}+\alpha^{\frac{1}{q}}N^{\frac{1}{q}}\varepsilon_{3}^{\frac{1}{p}}\right], & \alpha\lambda\leq 1-\alpha\leq 1-\lambda(1-\alpha)\\ \left[(1-\alpha)^{\frac{1}{q}}M^{\frac{1}{q}}\varepsilon_{1}^{\frac{1}{p}}+\alpha^{\frac{1}{q}}N^{\frac{1}{q}}\varepsilon_{4}^{\frac{1}{p}}\right], & \alpha\lambda\leq 1-\lambda(1-\alpha)\leq 1-\alpha\\ \left[(1-\alpha)^{\frac{1}{q}}M^{\frac{1}{q}}\varepsilon_{2}^{\frac{1}{p}}+\alpha^{\frac{1}{q}}N^{\frac{1}{q}}\varepsilon_{3}^{\frac{1}{p}}\right], & 1-\alpha\leq\alpha\lambda\leq 1-\lambda(1-\alpha)\end{cases},$$

where

$$\begin{aligned}M&=\sup\left\{a^{-2q},A_{\alpha}(a,b)^{-2q}\right\},\\ N&=\sup\left\{b^{-2q},A_{\alpha}(a,b)^{-2q}\right\},\end{aligned}$$

$\frac{1}{p}+\frac{1}{q}=1$, *and $\varepsilon_{1},\ \varepsilon_{2},\ \varepsilon_{3}$, and ε_{4} are defined as in Theorem 2.10.*

Proof. The assertion follows from Theorem 2.10, for $f(x)=\frac{1}{x}, x\in(0,\infty)$. □

References

[1] M. Alomari and M. Darus, Some Ostrowski type inequalities for quasi-convex functions with applications to special means, RGMIA Preprint, **13**(2) (2010) article No. 3. [http://rgmia.org/papers/v13n2/quasi-convex.pdf].

[2] M. Alomari, M. Darus, S.S. Dragomir, New inequalities of Simpson's Type for s-convex functions with applications, RGMIA Res. Rep. Coll., **12**(4) (2009) Article 9. [Online http://ajmaa.org/RGMIA/v12n4.php].

[3] M.W. Alomari, M.Darus, U.S. Kirmaci, Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, Comput. Math. Appl., **59** (2010), 225-232.

[4] M.W. Alomari, M.Darus, U.S. Kirmaci, Some inequalities of Hermite-Hadamard type for s-convex functions, Acta Math. Sci., **31**(4) (2011), 1643-1652.

[5] M. Alomari, S. Hussain, Two inequalities of Simpson type for quasi-convex functions and applications, Appl. Math. E-Notes, **11** (2011), 110-117.

[6] S.S. Dragomir and C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.

[7] D.A. Ion, Some estimates onthe Hermite-Hadamard inequality through quasi-convex functions, Annals of University of Craiova, Math. Comp. Sci. Ser. **34** (2007), 82-87.

[8] İ. İşcan, A new generalization of some integral inequalities and their applications, International Journal of Engineering and Applied sciences, **3**(3) (2013), 17-27.

[9] İ. İşcan, Hermite-Hadamard type inequalities for functions whose derivatives are (α, m)-convex, International Journal of Engineering and Applied Sciences, **2**(3) (2013), 69-78.

[10] İ. İşcan, Generalization of different type integral inequalities via fractional integrals for functions whose second derivatives absolute values are quasi-convex, Konuralp j. Math., 1(2) (2013), 67–79.

[11] U.S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comput. **147** (2004), 137-146.

[12] M.Z. Sarikaya, N. Aktan, On the generalization of some integral inequalities and their applications, Math. Comput. Modelling, **54** (2011), 2175-2182.

[13] M.Z. Sarikaya, E. Set, M.E. Özdemir, On new inequalities of Simpson's type for s-convex functions, Comput. Math. Appl., **60** (2010), 2191-2199.

[14] M.Z. Sarikaya, E. Set, M.E. Özdemir, On new inequalities of Simpson's type for convex functions, RGMIA Res. Rep. Coll., **13**(2) (2010), Article 2.

Author information

Imdat İşcan, Department of Mathematics, Faculty of Arts and Sciences, Giresun University, 28100, Giresun, Turkey.
E-mail: imdat.iscan@giresun.edu.tr; imdati@yahoo.com

Received: July 13, 2014.

Accepted: October 20, 2013.