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Some more combinatorics results on Nagata Extensions

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Abstract. We show that the length of a ring extension $R \subseteq S$ is preserved under the formation of the Nagata extension $R(X) \subseteq S(X)$. A companion result holds for the Dobbs-Mullins invariant. D. Dobbs and the authors proved elsewhere that the cardinal number of the set $[R, S]$ of subextensions of $R \subseteq S$ is preserved under the formation of Nagata extension when $|[R(X), S(X)]|$ is finite. We show that in the only pathological case, namely $R \subseteq S$ is subintegral, then $|[R, S]|$ is preserved if and only if it is either infinite or finite and $R \subseteq S$ is arithmetic; that is, $[R, S]$ is locally a chain. The last section gives properties of arithmetic extensions and their links with Prüfer extensions.

1 Introduction and Notation

We consider the category of commutative and unital rings and first give some notation and definitions, needed for explaining the subject of the paper. Let $R \subseteq S$ be a (ring) extension. The set of all R-subalgebras of S is denoted by $[R, S]$ and the integral closure of R in S by $\overline{R}$. As usual, Spec(R), Max(R) and Min(R) are the sets of prime ideals, maximal ideals and minimal prime ideals of a ring R. Moreover, Tot(R) denotes the total quotient ring of a ring R.

The support of an R-module E is $\mathrm{Supp}_R(E) := \{P \in \mathrm{Spec}(R) \mid E_P \neq 0\}$, and $\mathrm{MSupp}_R(E) := \mathrm{Supp}_R(E) \cap \mathrm{Max}(R)$ is also the set of all maximal elements of $\mathrm{Supp}_R(E)$. If E is an R-module, $\mathrm{L}_R(E)$ is its length. If $R \subseteq S$ is a ring extension and $P \in \mathrm{Spec}(R)$, then S_P is both the localization $S_{R\setminus P}$ as a ring and the localization at P of the R-module S. We denote by $(R : S)$ the conductor of $R \subseteq S$. Finally, $\subset$ denotes proper inclusion and $|X|$ the cardinality of a set X.

The extension $R \subseteq S$ is said to have FIP (for the "finitely many intermediate algebras property") if $[R, S]$ is finite. A *chain* of R-subalgebras of S is a set of elements of $[R, S]$ that are pairwise comparable with respect to inclusion. An extension $R \subseteq S$ is called a *chained* extension if $[R, S]$ is a chain. We say that the extension $R \subseteq S$ has FCP (for the "finite chain property") if each chain in $[R, S]$ is finite. It is clear that each extension that satisfies FIP must also satisfy FCP. Dobbs and the authors characterized FCP and FIP extensions [5]. Minimal (ring) extensions, introduced by Ferrand-Olivier [8], are an important tool of the paper. Recall that an extension $R \subset S$ is called *minimal* if $[R, S] = \{R, S\}$. The key connection between the above ideas is that if $R \subseteq S$ has FCP, then any maximal (necessarily finite) chain of R-subalgebras of S, $R = R_0 \subset R_1 \subset \cdots \subset R_{n-1} \subset R_n = S$, with *length* $n < \infty$, results from juxtaposing n minimal extensions $R_i \subset R_{i+1}$, $0 \leq i \leq n-1$. For any extension $R \subseteq S$, the *length* of $[R, S]$, denoted by $\ell[R, S]$, is the supremum of the lengths of chains of R-subalgebras of S. It should be noted that if $R \subseteq S$ has FCP, then there *does* exist some maximal chain of R-subalgebras of S with length $\ell[R, S]$ [6, Theorem 4.11].

In passing we also consider a condition weaker than FCP on an extension $R \subset S$, recently explored by Ayache and Dobbs in [2]: there is a finite maximal chain in $[R, S]$ from R to S (condition FMC for some authors). Then [2, Theorem 4.12] combined with [8, Proposition 4.2] and [5, Theorem 4.2] yields the following result, which may be useful to detect an FCP extension.

Proposition 1.1. *Let $R \subset S$ be an FMC extension of rings. Then $R \subset S$ satisfies FCP if and only if the length of the R-module $\overline{R}/R$ is finite, or equivalently $R \subseteq \overline{R}$ has FCP.*

We note here that $\overline{R} \subseteq S$ has FIP when $R \subseteq S$ has FCP [5, Theorem 6.3].

Let R be a ring and $R[X]$ the polynomial ring in the indeterminate X over R. (Throughout, we use X to denote an element that is indeterminate over all relevant coefficient rings.) Also, let $C(p)$ denote the content of any polynomial $p(X) \in R[X]$. Then $\Sigma_R := \{p(X) \in R[X] \mid C(p) = R\}$ is a saturated multiplicatively closed subset of $R[X]$, each of whose elements is a non-zero-divisor of $R[X]$. The *Nagata ring of* R is defined to be $R(X) := R[X]_{\Sigma_R}$.

Let $R \subseteq S$ be an extension. It was shown in [6, Theorem 3.9] that $R(X) \subseteq S(X)$ has FCP if and only if $R \subseteq S$ has FCP. One aim of this paper is to show that, when $R \subseteq S$ has FCP, then $\ell[R,S] = \ell[R(X), S(X)]$, a question addressed in [6, Remark 4.18(b)].

We begin to show that this property holds for FCP field extensions in Section 2. The main result is gotten in Section 3 where, after several steps involving the integral closure and the t-closure of an FCP extension, we prove in Theorem 3.3 that, when $R \subseteq S$ has FCP, then $\ell[R,S] = \ell[R(X), S(X)]$. We also introduce the Dobbs-Mullins invariant of an extension $R \subseteq S$ as being the supremum $\Lambda(S/R)$ of the lengths of residual extensions of $R \subseteq S$, considered as ring extensions [4]. We show in Theorem 3.7 that $\Lambda(S/R) = \Lambda(S(X)/R(X))$.

We will have to consider the following material.

Definition 1.2. Let $R \subseteq S$ be an integral extension. Then $R \subseteq S$ is called **infra-integral** [17] (resp. **subintegral** [19]) if all its residual extensions $R_P/PR_P \to S_Q/QS_Q$, (with $Q \in \mathrm{Spec}(S)$ and $P := Q \cap R$) are isomorphisms (resp. and the spectral map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is bijective). An extension $R \subseteq S$ is called **t-closed** (cf. [17]) if the relations $b \in S$, $r \in R$, $b^2 - rb \in R$, $b^3 - rb^2 \in R$ imply $b \in R$. The t-**closure** ${}_S^tR$ of R in S is the smallest R-subalgebra B of S such that $B \subseteq S$ is t-closed and the greatest $B' \in [R,S]$ such that $R \subseteq B'$ is infra-integral.

The canonical decomposition of an arbitrary ring extension $R \subset S$ is $R \subseteq {}_S^+R \subseteq {}_S^tR \subseteq \overline{R} \subseteq S$, where ${}_S^+R$ is the seminormalization of R in S (see [19]).

The other aim is achieved in Section 4. It consists to improve a characterization of the transfer of the FIP property for subintegral extensions of Nagata rings (see [6, Theorem 3.30]). We consider only this (pathological) case because in the canonical decomposition of a ring extension, the subintegral part $R \subseteq {}_S^+R$ is the only obstruction for $R(X) \subseteq S(X)$ having FIP [6, Theorem 3.21]. This leads us to introduce extensions $R \subseteq S$ such that $R_M \subseteq S_M$ is a chained extension for each $M \in \mathrm{Supp}_R(S/R)$. Such extensions are called *arithmetic*, the definition being reminiscent of arithmetic rings. Note that $\mathrm{Supp}(S/R)$ can be replaced with one of the following subsets $\mathrm{Spec}(R)$, $\mathrm{Max}(R)$, $\mathrm{MSupp}(S/R)$), since the natural map $[R,S] \to [R_P, S_P]$ is surjective for each $P \in \mathrm{Spec}(R)$. We show in Theorem 4.2 that if $R \subset S$ is a subintegral extension, then $R(X) \subset S(X)$ has FIP if and only if $R \subset S$ has FIP and is arithmetic.

For an FCP extension $R \subseteq S$, it will be convenient to consider $\mathrm{MSupp}(S/R)$. Observe that an FCP extension $R \subseteq S$ is arithmetic if and only if $R_M \to S_M$ can be factored into a unique finite sequence of minimal morphisms, for each $M \in \mathrm{MSupp}(S/R)$.

Moreover, if $R \subseteq T \subseteq S$ is an arithmetic extension, then so are $R \subseteq T$ and $T \subseteq S$. Let $R \subseteq S$ be an extension with conductor $C := (R : S)$. It is clear that $R \subseteq S$ is arithmetic if and only if $R/C \subseteq S/C$ is arithmetic.

The paper ends with Section 5, that contains results on arithmetic extensions.

The following notions and results are also deeply involved in our study.

Theorem 1.3. *[8, Théorème 2.2 and Lemme 3.2] Let $A \subset B$ be a minimal extension. Then, there is some $M \in \mathrm{Max}(A)$, called the* **crucial (maximal) ideal** *of $A \subset B$, such that $A_P = B_P$ for each $P \in \mathrm{Spec}(A) \setminus \{M\}$. We denote this ideal M by $\mathcal{C}(A,B)$.*

Moreover, $A \subset B$ is either an integral (finite) extension, or a flat epimorphism, these two conditions being mutually exclusive.

There are three types of minimal integral extensions, given by the following theorem.

Theorem 1.4. *[5, Theorem 2.2] Let $R \subset T$ be an extension and set $M := (R : S)$. Then $R \subset T$ is minimal and finite if and only if $M \in \mathrm{Max}(R)$ and one of the following three conditions holds:*

(a) **inert case***: $M \in \mathrm{Max}(T)$ and $R/M \to T/M$ is a minimal field extension;*

(b) **decomposed case***: There exist $M_1, M_2 \in \mathrm{Max}(T)$ such that $M = M_1 \cap M_2$ and the natural maps $R/M \to T/M_1$ and $R/M \to T/M_2$ are both isomorphisms;*

(c) **ramified case***: There exists $M' \in \mathrm{Max}(T)$ such that ${M'}^2 \subseteq M \subset M'$, $[T/M : R/M] = 2$, and the natural map $R/M \to T/M'$ is an isomorphism.*

Decomposed and ramified minimal extensions are infra-integral while inert minimal extensions are not. Ramified minimal extensions are subintegral.

The next lemma will be used later. Let $\mathcal{P}$ be a property holding for a class $\mathcal{C}$ of ring extensions, stable under subextensions (i.e. $R \subseteq S$ in $\mathcal{C}$ and $[U, V] \subseteq [R, S]$ imply $U \subseteq V$ in $\mathcal{C}$). We say that $\mathcal{P}$ admits a closure in $\mathcal{C}$ if the following conditions (i), (ii), (iii) and (iv) hold for any extension $R \subset S$ in $\mathcal{C}$:

(i) For any tower of extensions $R \subseteq U \subseteq S$, then $R \subseteq S$ has $\mathcal{P}$ if and only if $R \subseteq U$ and $U \subseteq S$ have $\mathcal{P}$.

(ii) There exists a largest subextension $T \in [R, S]$ such that $R \subseteq T$ has $\mathcal{P}$.

(iii) No subextension $U \subseteq V$ of $T \subseteq S$ has $\mathcal{P}$.

(iv) $T = R$ when $R \subset S$ is a composite of finitely many minimal extensions which do not satisfy $\mathcal{P}$.

Such a T is unique, is called the $\mathcal{P}$-closure of R in S and is denoted by $R^{\mathcal{P}}$. Some instances are the separable closure in the class of algebraic field extensions and the t-closure in the class of integral ring extensions.

Lemma 1.5. *Let $\mathcal{P}$ be a property of ring extensions admitting a $\mathcal{P}$-closure in a class $\mathcal{C}$ of ring extensions. If an FCP extension $R \subseteq S$ belongs to $\mathcal{C}$ and $R^{\mathcal{P}}$ is its $\mathcal{P}$-closure, then, $\ell[R, S] = \ell[R, R^{\mathcal{P}}] + \ell[R^{\mathcal{P}}, S]$.*

Proof. Obviously, $\ell[R, S] \geq \ell[R, R^{\mathcal{P}}] + \ell[R^{\mathcal{P}}, S]$. We prove by induction on $n := \ell[R, S] \geq 1$ that there exists a maximal chain from R to S with length n containing $R^{\mathcal{P}}$. If $n = 1$, then $R \subset S$ is a minimal extension, so that either $R^{\mathcal{P}} = R$, or $R^{\mathcal{P}} = S$. Assume now that $n > 1$ and that the induction hypothesis holds for any $n' < n$. We may assume that $R \neq R^{\mathcal{P}}$. Let $R = R_0 \subset R_1 \subset \cdots \subset R_{n-1} \subset R_n = S$ be a maximal chain of subextensions with length n. The induction hypothesis applied to the extension $R_1 \subseteq S$ (with length $n-1$) gives that there exists a maximal chain $R_1 = R'_1 \subset \cdots \subset R'_{n-1} \subset R'_n = S$ with length $n-1$ containing $R_1^{\mathcal{P}}$, so that $R_1 \subseteq R_1^{\mathcal{P}}$ satisfies $\mathcal{P}$.

If $R \subset R_1$ satisfies $\mathcal{P}$, then, $R \subset R_1 \subseteq R_1^{\mathcal{P}}$ satisfies $\mathcal{P}$ by (i), so that $R_1^{\mathcal{P}} = R^{\mathcal{P}}$. It follows that we get a maximal chain from R to S with length n containing $R^{\mathcal{P}}$.

Assume that $R \subset R_1$ does not satisfy $\mathcal{P}$. If $R_1 \neq R_1^{\mathcal{P}}$, then $R_1 \subset R'_2$ satisfies $\mathcal{P}$ because $R'_2 \subseteq R_1^{\mathcal{P}}$. Let R' be the $\mathcal{P}$-closure of the extension $R \subset R'_2$. We have $R' \neq R'_2$ because $R \subset R'_2$ does not satisfy $\mathcal{P}$ by (i). Assume that $R \neq R'$. Because of the length of $R'_2 \subset \cdots \subset R'_{n-1} \subset R'_n = S$, we get that $R \subset R'$ is minimal and satisfies $\mathcal{P}$. For the same reason, $R' \subset R'_2$ is minimal. Let R'' be the $\mathcal{P}$-closure of the extension $R' \subset S$ (with length $n-1$). We have $R'' = R^{\mathcal{P}}$. The induction hypothesis gives that there exists a maximal chain from R' to S with length $n-1$ containing R'', so that there exists a maximal chain from R to S with length n containing $R^{\mathcal{P}}$. Now, assume that $R = R'$. By (iii), no subextension of $R \subset R'_2$ satisfies $\mathcal{P}$, a contradiction, since $R_1 \subset R'_2$ satisfies $\mathcal{P}$.

At last, assume that $R_1 = R_1^{\mathcal{P}}$, then, $R = R^{\mathcal{P}}$ by (iv). Indeed, $R \subset S$ is composed of minimal subextensions, each of them not satisfying $\mathcal{P}$.

To end, $\ell[R, S] = \ell[R, R^{\mathcal{P}}] + \ell[R^{\mathcal{P}}, S]$. □

We recover in particular that $\ell[R, S] = \ell[R, \overline{R}] + \ell[\overline{R}, S]$ [6, Theorem 4.11].

Remark 1.6. For the reverse order, there is some companion result that can be written if after all it reveals useful.

We end by recalling some useful characterizations of the support of an FCP extension.

Lemma 1.7. *[5, Remark 6.14 (b), Theorem 6.3] Let $R \subseteq S$ be an integrally closed FMC extension. Then,* $\mathrm{Supp}(S/R) = \{P \in \mathrm{Spec}(R) \mid PS = S\}$.

Lemma 1.8. *[5, Corollary 3.2] Suppose that there is a maximal chain $R = R_0 \subset \cdots \subset R_i \subset \cdots \subset R_n = S$ of extensions, where $R_i \subset R_{i+1}$ is minimal with crucial ideal M_i for each $i = 0, \ldots, n-1$ (i.e. $R \subseteq S$ has FMC). Then* $\mathrm{Supp}(S/R)$ *is a finite set; in fact,* $\mathrm{Supp}(S/R) = \{M_i \cap R \mid i = 0, \ldots, n-1\}$.

2 Preliminary results about FCP field extensions

We first observe that an FCP field extension $K \subset L$ is finite, whence algebraic. It follows that $L(X) \simeq L \otimes_K K(X)$ by [6, Lemma 3.1], so that $[L : K] = [L(X) : K(X)]$. Moreover, a minimal field extension is clearly either separable, or purely inseparable (see for instance [16]) and the degree of a minimal purely inseparable extension of a field K is equal to the characteristic of K.

Proposition 2.1. *Let $K \subset L$ be an FCP field extension and let K_s be the separable closure of K in L. Then, $\ell[K, L] = \ell[K, K_s] + \ell[K_s, L]$.*

Proof. We use Lemma 1.5, where $\mathcal{P}$ is the property to be a separable extension and $K^{\mathcal{P}} = K_s$ is the separable closure ([3, Ch. V, Proposition 13, p. 42]). □

So, it is enough to consider the situation for FCP separable extensions and FCP purely inseparable extensions.

Proposition 2.2. *Let $K \subset L$ be an FCP separable field extension. Then, $\ell[K, L] = \ell[K(X), L(X)]$.*

Proof. Since $K \subset L$ has FCP, its degree is finite. As a finite separable extension has a primitive element, it has FIP. We infer from [7, Propositions 9 and 11] that there is an order-isomorphism $[K, L] \to [K(X), L(X)]$, given by $T \mapsto T(X)$. It follows that any maximal chain of $K \subset L$ leads to a maximal chain of $K(X) \subset L(X)$. Conversely, any maximal chain of $K(X) \subset L(X)$ comes from a maximal chain of $K \subset L$, giving $\ell[K, L] = \ell[K(X), L(X)]$. □

Proposition 2.3. *Let $K \subset L$ be an FCP purely inseparable field extension. Then, $\ell[K, L] = \ell[K(X), L(X)]$.*

Proof. Since $K \subset L$ is an FCP purely inseparable field extension, K is a field of characteristic a prime number p and $[L : K]$ is a power of p, say p^n. It follows that there is only one maximal chain composing $K \subset L$, and it has length n, and leads to a maximal chain composing $K(X) \subset L(X)$ with length n, which is also purely inseparable, with $K(X)$ of characteristic p and $[L(X) : K(X)] = p^n$. Then, $n = \ell[K, L] = \ell[K(X), L(X)]$. □

Proposition 2.4. *Let $K \subset L$ be an FCP field extension and let K_s be the separable closure of K in L. Then, $K_s(X)$ is the separable closure of $K(X)$ in $L(X)$.*

Proof. We got in the proof of Proposition 2.2 that $K \subseteq K_s$ has FIP, and so has a primitive element α, which is separable over K. Then, α is also a primitive element of the extension $K(X) \subset K_s(X)$, and is separable over $K(X)$. It follows that $K(X) \subset K_s(X)$ is a separable extension.

Moreover, $K_s \subseteq L$ is purely inseparable. Then, any element of L is purely inseparable over K_s, so that any element of $L(X)$ is purely inseparable over $K_s(X)$. Hence, $K_s(X) \subseteq L(X)$ is purely inseparable, giving that $K_s(X)$ is the separable closure of $K(X)$ in $L(X)$. □

We can now state the result for FCP fields extensions.

Theorem 2.5. *Let $K \subset L$ be an FCP field extension. Then, $\ell[K, L] = \ell[K(X), L(X)]$.*

Proof. Let K_s be the separable closure of K in L. Then, $K_s(X)$ is the separable closure of $K(X)$ in $L(X)$ by Proposition 2.4. Applying Proposition 2.1 twice, Proposition 2.2 and Proposition 2.3, we get that $\ell[K(X), L(X)] = \ell[K(X), K_s(X)] + \ell[K_s(X), L(X)] = \ell[K, K_s] + \ell[K_s, L] = \ell[K, L]$. □

This gives the result needed for the next section.

Corollary 2.6. *Let $R \subset S$ be an FCP t-closed extension. Then, $\ell[R, S] = \ell[R(X), S(X)]$.*

Proof. From [6, Lemmata 3.3 and 3.15], we get that $R(X) \subset S(X)$ is an FCP t-closed extension, with $\{MR(X) \mid M \in \mathrm{MSupp}(S/R)\} = \mathrm{MSupp}(S(X)/R(X))$. Then, [6, Proposition 4.6 and Lemma 3.16 proof] give that $\ell[R(X), S(X)] = \sum[\ell[R_M(X), S_M(X)] | M \in \mathrm{MSupp}(S/R)]$ and $\sum[\ell[R_M, S_M] | M \in \mathrm{MSupp}(S/R)] = \ell[R, S]$. Hence we can reduce the proof to the case of a quasi-local ring (R, M). Since $M = (R : S) \in \mathrm{Max}(S)$ by [6, Lemma 3.17], we get $MR(X) = (R(X) : S(X)) = MS(X)$. Now $\ell[R(X), S(X)] = \ell[R(X)/MR(X), S(X)/MR(X)] = \ell[(R/M)(X), (S/M)(X)]$ are consequences of [5, Proposition 3.7]. We then observe that $\ell[R, S] = \ell[R/M, S/M] = \ell[(R/M)(X), (S/M)(X)]$ in view of Theorem 2.5 and [5, Proposition 3.7]. □

3 On the lengths of FCP extensions of Nagata rings

To introduce this section, we give the following lemma.

Lemma 3.1. *Let $R \subset S$ be an integral extension and consider a maximal chain $\mathcal{C}$ of R-subextensions of S defined by $R = R_0 \subset \cdots \subset R_i \subset \cdots \subset R_n = S$, where $R_i \subset R_{i+1}$ is minimal for each $i \in \{0, \ldots, n-1\}$. Then,*

(1) $R \subset S$ is infra-integral if and only if, for each $i \in \{0, \ldots, n-1\}$, $R_i \subset R_{i+1}$ is either ramified or decomposed.

(2) $R \subset S$ is t-closed if and only if $R_i \subset R_{i+1}$ is inert for each $i \in \{0, \ldots, n-1\}$.

(3) If in addition (R, M) is a quasi-local ring and the conditions of (2) hold, then $M = (R : S)$ and (S, M) is a quasi-local ring.

Proof. (1) is obvious, because all the residual field extensions are isomorphisms.

(2) Assume that $R \subset S$ is t-closed. Then $R_i \subset R_{i+1}$ is inert for each $i \in \{0, \ldots, n-1\}$ in view of [5, Lemma 5.6]. Conversely, if $R_i \subset R_{i+1}$ is inert for each $i \in \{0, \ldots, n-1\}$, and so t-closed, then $R \subset S$ is obviously t-closed.

(3) Moreover, if (R, M) is quasi-local, [6, Lemma 3.17] shows that M is the only maximal ideal of S. □

We can now see how the t-closure is involved in the length of an integral FCP extension.

Proposition 3.2. *Let $R \subset S$ be an integral FCP extension, then $\ell[R, S] = \ell[R, {}_S^tR] + \ell[{}_S^tR, S]$.*

Proof. Use Lemma 1.5 and Lemma 3.1, where $\mathcal{P}$ is the property to be an infra-integral extension, and $R^{\mathcal{P}} = {}_S^tR$ is the t-closure of R in S. □

We are now in position to give a positive answer to [6, Remark 4.18(b)].

Theorem 3.3. *Let $R \subset S$ be an FCP extension. Then, $\ell[R, S] = \ell[R(X), S(X)]$.*

Proof. Let $R \subset S$ be an FCP extension. We begin to notice that the t-closure of $R(X)$ in $S(X)$ is ${}_{S(X)}^tR(X) = ({}_S^tR)(X)$ by [6, Lemma 3.15]. Moreover, in [6, Remark 4.18 (b)], we proved that $\ell[R, S] = \ell[R(X), S(X)]$ if and only if $\ell[R, \overline{R}] = \ell[R(X), \overline{R}(X)]$. It follows that we can assume that $R \subset S$ is an integral FCP extension. But, Proposition 3.2 gives that $\ell[R, S] = \ell[R, {}_S^tR] + \ell[{}_S^tR, S]$, and, in the same way, $\ell[R(X), S(X)] = \ell[R(X), ({}_S^tR)(X)] + \ell[({}_S^tR)(X), S(X)]$. Now, $\ell[{}_S^tR, S] = \ell[({}_S^tR)(X), S(X)]$ by Corollary 2.6. To end, $\ell[R, {}_S^tR] = \ell[R(X), ({}_S^tR)(X)]$ [6, Proposition 4.7]. □

Corollary 3.4. *Let $R \subset S$ be an FCP extension and n a positive integer. Then, $\ell[R, S] = \ell[R(X_1, \ldots, X_n), S(X_1, \ldots, X_n)]$.*

We end this section by some considerations about the length of FCP extensions $R \subseteq S$ with respect to their residual extensions. Following Dobbs and Mullins [4], we define $\Lambda(S/R)$ to be the supremum of the lengths of residual extensions of $R \subseteq S$, considered as ring extensions.

Proposition 3.5. *Let $R \subset S$ be an FCP extension. Then $\Lambda(S/R) = \Lambda(\overline{R}/{}_S^tR)$.*

Proof. We first observe that an FCP extension $R \subseteq S$ is strongly affine, that is each of the R-algebras $T \in [R, S]$ is of finite type. Since $R \subseteq S$ is a composite of minimal morphisms that are either flat epimorphisms or integral morphisms, $R \subseteq T$ is an INC extension for $T \in [R, S]$ and hence a quasi-finite extension. Moreover, the residual extensions of each minimal morphism $T \subset U$, with $T, U \in [R, S]$ are either isomorphisms or minimal field extensions, induced by inert minimal morphisms. Then in the canonical decomposition $R \subseteq {}_S^tR \subseteq \overline{R} \subseteq S$, the extension $\overline{R} \subseteq S$ is a flat epimorphism by the Zariski Main Theorem. Therefore the residual extensions of $R \subseteq S$ identify with the residual extensions of ${}_S^tR \subseteq \overline{R}$ and the components of maximal chains in $[{}_S^tR, \overline{R}]$ need to be minimal inert extensions by Lemma 3.1(2). The above discussion shows that for an FCP extension $R \subseteq S$, then $\Lambda(S/R) = \Lambda(\overline{R}/{}_S^tR)]$. □

So, it is enough to consider an FCP integral t-closed extension $R \subset S$.

Proposition 3.6. *Let $R \subset S$ be an FCP integral t-closed extension. Then $\Lambda(S/R) = \sup_{M \in \mathrm{MSupp}(S/R)} \ell[R_M, S_M]$ and $\ell[R,S] \leq n\Lambda(S/R)$, where $n := |\mathrm{MSupp}(S/R)|$.*

Proof. We get $\sum[\ell[R_M, S_M] | M \in \mathrm{MSupp}(S/R)] = \ell[R,S]$ $(*)$ by [6, Proposition 4.6]. Assume first that (R,M) is a quasi-local ring, and so $(R:S) = M$ by Lemma 3.1. Then, $\ell[R,S] = \ell[R/M, S/M] = \Lambda(S/R)$ by [5, Proposition 3.7]. Now, in the general case, set $\mathrm{MSupp}(S/R) := \{M_1, \dots, M_n\}$. Consider a maximal chain of R-subextensions of S defined by $R = R_0 \subset \cdots \subset R_i \subset \cdots \subset R_p = S$, where $R_i \subset R_{i+1}$ is minimal inert for each $i \in \{0, \dots, p-1\}$. In view of Lemma 1.8, we have, $\{M_1, \dots, M_n\} = \{\mathcal{C}(R_i, R_{i+1}) \cap R \mid i \in \{0, \dots, p-1\}\} = \{(R_i : R_{i+1}) \cap R \mid i \in \{0, \dots, p-1\}\}$. An easy induction using [5, Lemma 3.3], shows that we can exhibit R-subextensions of S such that $R = R'_0 \subset \cdots \subset R'_j \subset \cdots \subset R'_n = S$, $R'_j \subset R'_{j+1}$ is t-closed for each $j \in \{0, \dots, n-1\}$ and satisfies $(R'_j : R'_{j+1}) \cap R = M_{j+1}$. This is obvious for $j = 0$. But, since $R \subset R'_1$ is t-closed and integral, for each $j \in \{2, \dots, n\}$, there is a unique $M'_j \in \mathrm{Max}(R'_1)$ lying above M_j, and we have $\mathrm{MSupp}(S/R'_1) = \{M'_2, \dots, M'_n\}$. Then, for each $j \in \{1, \dots, n\}$, we have $[R_{M_j}, S_{M_j}] = [(R'_{j-1})_{M_j}, (R'_j)_{M_j}]$ since $R_{M_j} = (R'_{j-1})_{M_j}$ and $(R'_j)_{M_j} = S_{M_j}$. It follows that $\ell[R_{M_j}, S_{M_j}] = \Lambda(S_{M_j}/R_{M_j})$, so that $\ell[R,S] = \sum_{j=1}^n \ell[R_{M_j}, S_{M_j}] = \sum_{j=1}^n \Lambda(S_{M_j}/R_{M_j}) = \sum_{j=1}^n \Lambda((R'_j)_{M_j}/(R'_{j-1})_{M_j}) = \sum_{j=1}^n \Lambda(R'_j/R'_{j-1})$ because $(R'_j)_M = (R'_{j-1})_M$ for any $M \neq M_j$. To end, let $Q \in \mathrm{Spec}(S)$ and set $P := Q \cap R$. If $P \notin \mathrm{MSupp}(S/R)$, we get that $R_P = S_P = S_Q$, so that $k(P) = k(Q)$. If $P \in \mathrm{MSupp}(S/R)$, then Q is the only prime ideal of S lying over P, so that $S_P = S_Q$ and $[k(P), k(Q)] = [R/P, S/Q]$. It follows that $\Lambda(S/R) = \sup_{j \in \{1,\dots,n\}} \Lambda(S_{M_j}/R_{M_j}) = \sup_{j \in \{1,\dots,n\}} \ell[R_{M_j}, S_{M_j}]$.

For each $M \in \mathrm{MSupp}(S/R)$, we have $\ell[R_M, S_M] = \Lambda(S_M/R_M) \leq \Lambda(S/R)$, so that $(*)$ gives $\ell[R,S] \leq n\Lambda(S/R)$. ◻

Coming back to the Nagata ring extension, we get the following theorem.

Theorem 3.7. *Let $R \subset S$ be an FCP extension. Then $\Lambda(S/R) = \Lambda(S(X)/R(X))$.*

Proof. We get $\Lambda(S(X)/R(X)) = \Lambda(\overline{R(X)}/{}_{S(X)}^{t}R(X))$ and $\Lambda(S/R) = \Lambda(\overline{R}/{}_{S}^{t}R)$ from Proposition 3.5 and $\overline{R(X)} = \overline{R}(X)$ and ${}_{S(X)}^{t}R(X) = ({}_{S}^{t}R)(X)$ from [6, Proposition 3.8 and Lemma 3.15]. To make easier the reading,we set $R' := {}_{S}^{t}R$ and $S' := \overline{R}$. Proposition 3.6 gives

$$\Lambda(S'/R') = \sup_{M \in \mathrm{MSupp}(S'/R')} \ell[R'_M, S'_M]$$

and

$$\Lambda(S'(X)/R'(X)) = \sup_{M' \in \mathrm{MSupp}(S'(X)/R'(X))} \ell[R'(X)_{M'}, S'(X)_{M'}]$$

Now, we have the following results: $\ell[R'_M, S'_M] = \ell[R'_M(X), S'_M(X)]$, $\mathrm{MSupp}(S'(X)/R'(X)) = \{MR'(X) \mid M \in \mathrm{MSupp}(S'/R')\}$ (see the proof of Corollary 3.5) and, for $M' \in \mathrm{MSupp}(S'(X)/R'(X))$, $M \in \mathrm{MSupp}(S'/R')$ such that $M' = MR'(X)$, we have $R'(X)_{M'} = R'_M(X)$ and $S'(X)_{M'} = S'_M(X)$. Then, $\ell[R'_M, S'_M] = \ell[R'(X)_{M'}, S'(X)_{M'}]$, giving $\Lambda(S/R) = \Lambda(S(X)/R(X))$. ◻

4 On some new properties of FIP extensions

In [6, Theorem 3.30], we got the following result: Let (R,M) be a quasi-local ring and $R \subset S$ a subintegral extension. Put $R_i := R + SM^i$ and $M_i := M + SM^i$ for each $i > 0$. Then, $R(X) \subset S(X)$ has FIP if and only if $R_2 \subseteq S$ is chained and $\mathrm{L}_R((SM)/M) = n - 1$, where $n := \nu(R/(R:S))$ is the index of nilpotency of $M/(R:S)$ in $R/(R:S)$. When $|R/M| = \infty$, these conditions are equivalent to $R \subset S$ has FIP. We intend to establish a more agreeable characterization. Before that, we reprove part of [5, Lemma 5.12] under weaker assumptions that are enough for our purpose.

Lemma 4.1. *Let (R, M) be a quasi-local Artinian ring which is not a field and let n be the index of nilpotency of M in R. Let $R \subset S$ be a finite subintegral extension such that $(R : S) = 0$. Set $R_i := R + SM^i$ and $M_i := M + SM^i$ for $i \in \{0, 1, \dots, n\}$. Then $R \subset S$ has FCP. Moreover, the following conditions are equivalent:*

(1) $\mathrm{L}_R(SM/M) = n - 1$.

(2) $\mathrm{L}_R(M_i/M_{i+1}) = 1$ *for all* $i = 1, \dots, n - 1$.

(3) $R \subseteq R_1$ *is chained.*

Proof. First, we may remark that $R \subset S$ has FCP in view of [5, Theorem 4.2]. Next, (R_i, M_i) is quasi-local for all $i = 1, \dots, n$, because $R \subset S$ is subintegral and $R_i/M_i = (R + SM^i)/(M + SM^i) \cong R/[R \cap (M + SM^i)] = R/M =: K$, which is a field. Moreover, for $1 \leq i < n$, we have $M_i \neq M_{i+1}$ (for if not, we would have $SM^i \subseteq M + SM^{i+1}$ and multiplication by M^{n-i-1} would lead to $SM^{n-1} \subseteq M^{n-i} \subset R$ and $0 \neq M^{n-1} \subseteq (R : S) = 0$, an absurdity). It follows that $R_i \neq R_{i+1}$. Then $MR_i = M_{i+1} = (R_{i+1} : R_i)$; note also that $M_i^2 \subseteq M_{i+1} \subset M_i$.

(1) ⇔ (2). Since $M_1 = SM$ and $M_n = M$, we get $\sum_{i=1}^{n-1} \mathrm{L}_R(M_i/M_{i+1}) = \mathrm{L}_R(SM/M)$. Also, if $i = 1, \dots, n - 1$, then $M_i \neq M_{i+1}$, and so $\mathrm{L}_R(M_i/M_{i+1}) \geq 1$. Thus, $\mathrm{L}_R(SM/M) \geq n - 1$, with equality if and only if $\mathrm{L}_R(M_i/M_{i+1}) = 1$ for all $i = 1, \dots, n - 1$.

(2) ⇒ (3). Assume that $\mathrm{L}_R(M_i/M_{i+1}) = 1$ for all $i = 1, \dots, n-1$. Since $MM_i \subseteq M_{i+1}$ and $K = R/M$, we have $\mathrm{L}_R(M_i/M_{i+1}) = \mathrm{L}_{R/M}(M_i/M_{i+1}) = \dim_K(M_i/M_{i+1})$. It follows that $\dim_K(R_i/M_{i+1}) = \dim_K(R_i/M_i) + \dim_K(M_i/M_{i+1}) = 1 + 1 = 2$, and so we deduce from Theorem 1.3(c) that $R_{i+1} \subset R_i$ is a ramified (minimal) extension. We get a maximal chain $R = R_n \subset R_{n-1} \subset \cdots \subset R_2 \subset R_1$. We will show that there cannot exist some $T \in [R, R_1] \setminus \{R_i\}_{i=1}^n$. Deny and let $k := \max\{i \in \{1, \dots, n - 1\} \mid T \subset R_i\}$. As $T \not\subseteq R_{k+1}$, we can use FCP to find some $T' \in [T, R_k]$ such that $T' \subset R_k$ is a minimal extension. This minimal extension must be ramified because it is subintegral. Note that $T' \neq R_{k+1}$ and $M' := (T' : R_k)$ is a maximal ideal of T' with $M' \cap R = M$. As $M_{k+1} = MR_k \subseteq M'R_k = M' \subset M_k$, we have $M_{k+1} \subseteq M' \subset M_k$. Since $1 = \mathrm{L}_R(M_k/M_{k+1}) = \mathrm{L}_{R/M}(M_k/M_{k+1}) = \mathrm{L}_{R_k/M_k}(M_k/M_{k+1})$, the ideals M_{k+1} and M_k of R_k must be adjacent. Hence $M' = M_{k+1}$. But $R_{k+1} = R + M_{k+1} = R + M' \subseteq T' \subset R_k$, and so the minimality of $R_{k+1} \subset R_k$ yields that $T' = R_{k+1}$, the desired contradiction.

(3) ⇒ (2). In fact, we are going to show that if there exists $k \in \{1, \dots, n - 1\}$ such that $\mathrm{L}_R(M_k/M_{k+1}) > 1$, then $[R, R_1]$ is not linearly ordered. By [5, Proposition 4.7(a)], we have that $\mathrm{L}_R(M_k/M_{k+1}) \leq \mathrm{L}_R(M_1/M) = \mathrm{L}_R(R_1/R)$ is finite. But we have $\mathrm{L}_R(M_k/M_{k+1}) = \mathrm{L}_{R/M}(M_k/M_{k+1}) = \mathrm{L}_{R_k/M_k}(M_k/M_{k+1})$, which is finite. Thus, there exists an R_k-submodule Q of M_k containing M_{k+1} such that $\dim_K(Q/M_{k+1}) = \dim_K(M_k/M_{k+1}) - 2$. Hence $\dim_K(M_k/Q) = 2$ and M_k/Q has at least two distinct one-dimensional K-vector subspaces of the form Q'/Q and Q''/Q, where Q', Q'' are appropriate ideals of R_k that contains Q. Moreover, they are incomparable. Since Q' and Q'' contain M_{k+1}, we have $Q' \cap R = Q'' \cap R = M$. Set $T' := R + Q'$, $T'' := R + Q'' \subseteq R_k$. It follows that Q' (resp. Q'') is the unique maximal ideal of T' (resp. T''). Assume, for instance, that $T' \subset T''$. Then, $Q' \subset Q''$, a contradiction. It follows that $[R, R_1]$ is not linearly ordered. □

We can now offer a nicer form of [6, Theorem 3.30]

Theorem 4.2. *Let $R \subset S$ be a subintegral extension. The following statements are equivalent:*

(1) $R(X) \subset S(X)$ *has FIP.*

(2) $R \subset S$ *has FIP and is arithmetic.*

Proof. Under each statement, $R \subset S$ has FCP, so that $|\mathrm{MSupp}(S/R)| < \infty$ [6, Theorem 3.9] and [5, Corollary 3.2]. In view of [5, Proposition 3.7], $R \subset S$ has FIP if and only if $R_M \subset S_M$ has FIP for each $M \in \mathrm{MSupp}(S/R)$ and $R \subset S$ has FCP. In the same way, $R(X) \subset S(X)$ has FIP if and only if $R_M(X) \subset S_M(X)$ has FIP for each $M \in \mathrm{MSupp}(S/R)$ and $R(X) \subset S(X)$ has FCP, because of [6, Lemma 3.16]. It follows that we may reduce to the case where (R, M) is a quasi-local ring, so that $(R(X), MR(X))$ is a quasi-local ring. In this situation, we claim that $R(X) \subset S(X)$ has FIP if and only if $R \subset S$ has FIP and is chained.

Assume first that $R(X) \subset S(X)$ has FIP. Then, $R \subset S$ has FIP by [6, Theorem 3.30]. Moreover, $|R(X)/MR(X)| = |(R/M)(X)| = \infty$. Set $C' := (R(X) : S(X))$, $R' := R(X)/C'$, $M' :=$

$MR(X)/C'$ and $S' := S(X)/C'$. Then, $R' \subset S'$ has FIP, R' is a quasi-local Artinian ring with $(R' : S') = 0$ and $|R'/M'| = \infty$. Assume first that $M' \neq 0$, so that R' is not a field. In view of [5, Proposition 5.15], we get that $[R', S']$ is a chain. Assume now that $M' = 0$, so that R' is an infinite field. Since $R' \subset S'$ has FIP, it follows from [1, Theorem 3.8 and proof of Lemma 3.6] that $[R', S']$ is a chain. In both cases $[R', S']$ is a chain, and so are $[R(X), S(X)]$ and $[R, S]$ by [6, Lemma 3.1(d)].

Conversely, assume that $R \subset S$ has FIP and is chained. Set $C := (R : S)$, $R'' := R/C$, $M'' := M/C$ and $S'' := S/C$. Then, $R'' \subset S''$ has FIP and is chained, R'' is a quasi-local Artinian ring and $(R'' : S'') = 0$. Assume that R'' is not a field. Using Lemma 4.1 and its notation, we get that $[R''_2, S'']$ is a chain, and so is $[R_2, S]$. Since $[R'', R''_1]$ is also a chain, we get that $\mathrm{L}_{R''}(S''M''/M'') = n - 1$, where n is the index of nilpotency of M'' in R''. But $\mathrm{L}_{R''}(S''M''/M'') = \mathrm{L}_R(S''M''/M'') = \mathrm{L}_R(SM/M)$, because of [13, Corollary 2 of Proposition 24, page 66]. Moreover, n is the index of nilpotency of M/C in R/C. Then, we can use [6, Theorem 3.30] to get that $R(X) \subset S(X)$ has FIP. Assume now that R'' is a field, so that $(R : S) = M$. Then, $SM = M$ gives $R_1 = R_2 = R$, and $n = 1$ implies that $\mathrm{L}_R(SM/M) = 0$ is satisfied. And [6, Theorem 3.30] gives again the result. □

Corollary 4.3. *Let $R \subseteq S$ be an FIP ring extension. Then $R(X) \subseteq S(X)$ has FIP if and only if $R \subseteq {}_S^+R$ is arithmetic. In that case $|[R(X), S(X)]| = |[R, S]|$.*

Proof. Use [6, Theorem 3.21] which states that $R(X) \subseteq S(X)$ has FIP if and only if $R \subseteq S$ and $R(X) \subseteq {}_S^+R(X)$ have FIP. Conclude with [7, Theorem 32]. □

Corollary 4.4. *Let $R \subseteq S$ be an FIP ring extension such that $|R/M| = \infty$ for each $M \in$ MSupp$({}_S^+R/R)$. Then $R(X) \subseteq S(X)$ has FIP. The result holds in particular when $|R/M| = \infty$ for each $M \in$ MSupp(S/R).*

Proof. It is enough to prove that a subintegral FIP extension $R \subset S$ such that $|R/M| = \infty$ for each $M \in$ MSupp(S/R) is arithmetic. We can suppose that the conductor of $R \subseteq S$ is zero and that R is quasi-local, with maximal ideal $M \in$ MSupp(S/R). It follows that (R, M) is a quasi-local Artinian ring by [5, Theorem 4.2]. Assume that R is not a field. Then, $[R, S]$ is a chain by [5, Proposition 5.15]. If R is an infinite field, S is of the form $R[\alpha]$, for some $\alpha \in S$ which satisfies $\alpha^3 = 0$ [1, Theorem 3.8 (3)], since $R \subset S$ is subintegral. Then, $[R, S]$ is linearly ordered by the proof of [1, Lemma 3.6 (b)]. □

Corollary 4.5. *Let $R \subseteq S$ be an extension, then $R(X_1, \dots, X_n) \subseteq S(X_1, \dots, X_n)$ has FIP for each integer $n \geq 0$ if and only if $R(X) \subseteq S(X)$ has FIP.*

We come back to the example given in [6, Example 3.12], which shows that the arithmetic condition is necessary in Theorem 4.2.

Example 4.6. Let K be a finite field and $T := K[Y]/(Y^4)$. As T is a finite-dimensional vector space over K, it follows from [1, Theorem 3.8 (b)] that the extension $K \subset T$ has FIP. Consider the extension $K(X) \subset T(X)$. We proved in [6, Example 3.12] that $K(X) \subset T(X)$ cannot have FIP because $K(X)$ is an infinite field and $T(X)$ contains an element whose index of nilpotency is 4 since $T \to T(X)$ is injective. Another proof of this result can be given by Theorem 4.2. Indeed, consider the coset $y := Y + (Y^4) \in T = K[Y]/(Y^4)$. Put $S_1 := K[y^2]$ and $S_2 := K[y^3]$. We get that $K \subset T$ is a subintegral extension which has FIP, but S_1 and S_2 are incomparable and $K \subseteq T$ is not arithmetic. So, $K(X) \subset T(X)$ cannot have FIP.

Remark 4.7. If $R \subseteq S$ is not subintegral it may be that the arithmetic condition be superfluous. We proved that a seminormal extension $R \subseteq S$ has FIP if and only if $R(X) \subseteq S(X)$ has FIP [6, Corollary 3.20]. It is easy to exhibit seminormal FIP extensions $R \subset S$ with R quasi-local and $R \subseteq S$ non arithmetic (see Example 5.13(5)).

In the next section we examine the first properties of arithmetical extensions. The study will be strongly completed in a forthcoming paper.

5 Elementary properties of arithmetical extensions

Using the language and results of Knebusch and Zhang in [11], we are able to get a characterization of some arithmetic extensions. We note here that chained ring extensions $R \subseteq S$ are called λ-extensions by Gilbert [9]. Knebusch and Zhang defined Prüfer extensions in [11]. It is now well known that $R \subseteq S$ is Prüfer if and only if (R, S) is a normal pair. We refer the reader to [11] for the properties of Prüfer extensions, noting only here that a ring extension $R \subseteq S$ is Prüfer if and only if $R \subseteq T$ is a flat epimorphism for each $T \in [R, S]$. We recall some properties of a flat epimorphism $f : A \to B$ (see [12, Chapter IV]):
Scholium

(1) Spec$(B) \to$ Spec(A) is injective

(2) f is essential; that is, for any ring morphism $g : B \to C$, such that $g \circ f$ is injective, then g is injective.

(3) Each ideal J of B is of the form $J = f^{-1}(J)B$.

(4) If f is injective and f is factored $A \to C \to B$, then $C \to B$ is a flat epimorphism, if it is injective.

(5) The class of flat epimorphisms is stable under base changes.

We refer the reader to [11] for the meaning of a Prüfer-Manis extension, called also a PM-extension. The following proposition will be completed by Theorem 5.17.

Proposition 5.1. *Let $R \subseteq S$ be an integrally closed extension. Then $R \subseteq S$ is arithmetic if and only if $R \subseteq S$ is locally Prüfer-Manis.*

Proof. Use [11, Theorem 3.1, p. 187] □

Proposition 5.2. *Let $R \subseteq S$ be an FMC extension.*

(1) Assume that $R \subseteq S$ is arithmetic and integrally closed. Then $\mathrm{Supp}_{R_P}(S_P/R_P)$ *is a chain for each $P \in$ Spec(R).*

(2) Assume that $R \subseteq S$ is chained, then $|\mathrm{MSupp}_R(S/R)| = 1$.

Proof. (1) Assume that $R \subseteq S$ is an arithmetic integrally closed FMC extension. We can assume that R is local with maximal ideal M in Supp(S/R). If $R \subseteq S$ is PM, observe that the set of all prime ideals Q of R such that $QS = S$ is Supp(S/R) by Lemma 1.7 and is a chain by the proof of [11, Theorem 3.1, p. 187].

(2) Let $R \subset S$ be a chained FMC extension. Let $M \in$ MSupp(S/R). We begin to show that there exists $R_1' \in [R, S]$ such that $R \subset R_1'$ is a minimal extension with $\mathcal{C}(R, R_1') = M$. Let $R = R_0 \subset \cdots \subset R_i \subset \cdots \subset R_n = S$ be a maximal chain of subextensions, where $R_i \subset R_{i+1}$ is minimal for each $i \in \{0, \ldots, n-1\}$. If $\mathcal{C}(R, R_1) = M$, we set $R_1' := R_1$. Assume that $\mathcal{C}(R, R_1) \neq M$, and set $k := \inf\{i \in \{1, \ldots, n\} \mid \mathcal{C}(R_{i-1}, R_i) \cap R = M\}$. Then, $k > 1$, $\mathcal{C}(R_{k-1}, R_k) \cap R = M$, and $\mathcal{C}(R_{i-1}, R_i) \cap R \neq M$ for each $i < k$. It follows that $M \notin$ MSupp(R_{k-1}/R). By [18, Lemma 1.10], there exists $R_1' \in [R, R_k]$ such that $R \subset R_1'$ is a minimal extension with $\mathcal{C}(R, R_1') = M$. We claim that $|\mathrm{MSupp}(S/R)| = 1$. Deny and let $N \in$ MSupp(S/R), $N \neq M$. The previous proof shows that there exists $R_1'' \in [R, S]$ such that $R \subset R_1''$ is a minimal extension with $\mathcal{C}(R, R_1'') = N$, so that $R_1'' \neq R_1'$, a contradiction since $R \subseteq S$ is chained. □

We will say that a ring extension $R \subseteq S$ is *quasi-Prüfer* (respectively, *quasi-Prüfer-Manis)* if $\overline{R} \subseteq S$ is Prüfer (respectively, Prüfer-Manis). We will also say that an extension $R \subseteq S$ is *pinched* at some $T \in [R, S]$ if each element of $[R, S]$ is comparable under inclusion to T.

Proposition 5.3. *Let $R \subseteq S$ be an extension. Then $R \subseteq S$ is chained if and only if $R \subseteq \overline{R}$ is chained, $R \subseteq S$ is quasi-Prüfer-Manis and $[R, S]$ is pinched at $\overline{R}$. Moreover, for each invertible element $x \in S$, we have either $x \in \overline{R}$ or $x^{-1} \in \overline{R}$.*

Proof. Use [11, Theorem 3.1, p. 187] for $\overline{R} \subseteq S$ to prove the first statement. We show the second. If $x \in S$ is invertible, then $\overline{R}[x]$ is comparable to $\overline{R}[x^{-1}]$ and $\overline{R}[x] \cap \overline{R}[x^{-1}] = \overline{R}$ because $\overline{R} \subseteq \overline{R}[x] \cap \overline{R}[x^{-1}]$ is integral [9, Lemma 1.2]. □

Remark 5.4. We can also deduce the first statement from the second by using [11, Theorem 3.13, p.195] in case $\overline{R} \subseteq S$ is a Marot extension; that is, for each $s \in S \setminus \overline{R}$, the $\overline{R}$-module $\overline{R} + \overline{R}s$ is generated over $\overline{R}$ by a set of units of S.

Lemma 5.5. *Let $R \subseteq S$ be an extension and J an ideal of S with $I := J \cap R$.*

(1) The map $T \mapsto T/(T \cap J)$ from $[R, S]$ to $[R/I, S/J]$ is surjective and order-preserving. Its restriction $[R + J, S] \to [R/I, S/J]$ is bijective and order-preserving and order-reflecting.

(2) If $R \subseteq S$ is chained, then $R/I \subseteq S/J$ is chained.

(3) If $R \subseteq S$ is arithmetic, then $R/I \subseteq S/J$ is arithmetic.

(4) If $R \subseteq S$ is Prüfer, then $R/I \subseteq S/J$ is a Prüfer extension. In particular, if N is a maximal ideal of S and $R \subseteq S$ is chained, then $R/(N \cap R)$ is a valuation domain with quotient field S/N.

Proof. To prove that (1) and (2) hold, it is enough to observe that $(R + J)/J$ is isomorphic to R/I and replacing R with $R + J$, we have to work with an extension of rings sharing the ideal J. Then (3) follows from (2), because the localization at a prime ideal of R/I is of the form R_P/I_P, where P is a prime ideal of R, and $J_P \cap R_P = I_P$.

Then (4) is a consequence of the following facts: $R \subseteq S$ is Prüfer entails that $R + J \subseteq S$ is Prüfer and then it is enough to use [11, Proposition 5.8, p.52].

For the last statement, use Proposition 5.3, because $R/(N \cap R) \subseteq S/N$ is chained by (2). □

Remark 5.6. It follows from Lemma 5.5 that a quasi-Prüfer extension $R \subseteq S$ gives a quasi-Prüfer extension $R/(J \cap R) \subseteq S/J$ for each ideal J of S and $\overline{R/(J \cap R)} = \overline{R}/(\overline{R} \cap J)$.

Let U be an absolutely flat ring. Recall that each element x of U has a unique quasi-inverse $x' \in U$, defined by $x^2x' = x$ and $x'^2x = x'$. In that case, set $e = xx'$. Then e is an idempotent and $1 - e + x$ is a unit of U, such that $(1 - e + x)^{-1} = (1 - e + x')$.

Proposition 5.7. *Let $R \subseteq S$ be a chained ring extension, such that S is zero-dimensional.*

(1) $S \cong \mathrm{Tot}(R)$ and then $\overline{R}$ is a Prüfer ring.

(2) Each $x \in S/\mathrm{Nil}(S)$ has a quasi-inverse $x' \in S/\mathrm{Nil}(S)$, such that either x or x' belongs to $\overline{R}/\mathrm{Nil}(R)$.

(3) $\overline{R} \subseteq S$ is additively regular, whence a Marot extension.

Proof. We observe that $\overline{R} \subseteq S$ is chained and then $\overline{R} \subseteq S$ is Prüfer by Proposition 5.3. It follows from [21, Corollaire 4], that S identifies with $\mathrm{Tot}(R)$ and hence $\overline{R}$ is a Prüfer ring. Since $\overline{R} \subseteq S$ is integrally closed, we have that $\mathrm{Nil}(S) = \mathrm{Nil}(\overline{R})$. Set $U := S/\mathrm{Nil}(S)$ and $T := \overline{R}/\mathrm{Nil}(\overline{R})$. We get a Prüfer extension $T \subseteq U$ by Lemma 5.5, where U is absolutely flat, whence $T \subseteq U$ is integrally closed. By the above recall and Proposition 5.3, if x is in U, then either $x \in T$ or $x' \in T$ because $1 - e$ is an idempotent of U, belonging to T. Moreover, there is some $t = 1 - e \in T$ such that $x + t$ is invertible in U. Since $\mathrm{Nil}(S) = \mathrm{Rad}(S)$, the Jacobson radical, we get that the same property holds for the extension $\overline{R} \subseteq S$. In other words, $\overline{R} \subseteq S$ is additively regular, whence a Marot extension (see [11, Remark 3.15, p. 196]). □

Gilbert proved that an integral domain R with quotient field K is such that $R \subseteq K$ is chained (R is a λ-domain with the Gilbert's terminology) if and only if $[R, K]$ is pinched at $\overline{R}$, R is a quasi-local i-domain and $R \subseteq \overline{R}$ is chained [9, Theorem 1.9]. We note that this result implies that R is a quasi-local unbranched domain, that is $\overline{R}$ is quasi-local (actually, in this case $\overline{R}$ is a valuation domain).

We intend to generalize this result to some extension. Before that we give a characterization of i-*pairs*, that are ring extension $R \subseteq S$ such that $\mathrm{Spec}(T) \to \mathrm{Spec}(R)$ is injective for each $T \in [R, S]$. We will say that a quasi-local ring R is *unbranched* in S if $\overline{R}$ is quasi-local.

Proposition 5.8. *An extension $R \subseteq S$ defines an i-pair if and only if $R \subseteq S$ is quasi-Prüfer and $R \subseteq \overline{R}$ is spectrally injective.*

Proof. One implication is given by [11, Theorem 5.2(9), p. 47]. For the converse, assume that $\mathrm{Spec}(\overline{R}) \to \mathrm{Spec}(R)$ is injective and that $R \subseteq S$ is quasi-Prüfer and let $T \in [R, S]$. To conclude, consider $U := \overline{R}T$. Then $\overline{R} \subseteq U$ is a flat epimorphism, whence spectrally injective and $T \subseteq U$ is integral. Since $R \subseteq U$ is spectrally injective, we get that $R \subseteq T$ is spectrally injective. ◻

Remark 5.9. A similar proof shows that an extension $R \subseteq S$ is quasi-Prüfer if and only if $R \subseteq S$ is an Inc-pair.

Proposition 5.10. *Let $R \subseteq S$ be an extension, such that R is quasi-local and unbranched in S. Then $R \subseteq S$ is chained if and only if $R \subseteq \overline{R}$ is chained, $R \subseteq S$ is quasi-Prüfer and $[R, S]$ is pinched at $\overline{R}$. In that case $R \subseteq S$ defines an i-pair.*

Proof. We can suppose that $\overline{R} \neq S$. Observe that $\overline{R}$ is quasi-local. So $\overline{R} \subseteq S$ is Prüfer if and only it is Prüfer-Manis [11, Theorem 1.8, p. 181] and also, if and only if $\overline{R} \subseteq S$ is chained [11, Theorem 3.1, p. 187]. The first statement is now clear.

Now, since $\overline{R}$ is quasi-local, from [5, Theorem 6.8], we deduce that there exists $Q \in \mathrm{Spec}(\overline{R})$ such that $S = \overline{R}_Q$, $Q = SQ$ and $\overline{R}/Q$ is a valuation domain. Under these conditions S/Q is the quotient field of $\overline{R}$/Q and Q is a divided ideal of $\overline{R}$; that is, comparable with any other prime ideal of $\overline{R}$. We observe that Q is the conductor of $\overline{R} \subseteq S$. Let M, M' be two prime ideals of $\overline{R}$ lying over some prime ideal P of R. If M and M' both contain Q, they are comparable and by incomparability of $R \subseteq \overline{R}$, we get that $M = M'$. If $M \subseteq Q \subseteq M'$, we get also $M = M'$. Thus there is only one case to examine: $M, M' \subset Q$. Since the flat extension $\overline{R} \subseteq S$ has the Going-Down property, Q is a minimal prime ideal in $\overline{R}$ and then $M = M'$. To conclude, it is enough to use Proposition 5.8, because $R \subseteq \overline{R}$ is spectrally injective. ◻

The following "birationnal" result is surely well-known.

Lemma 5.11. *Let R be a ring whose total quotient ring S is zero-dimensional and with integral closure $\overline{R}$. Then the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is injective and induces bijective maps $\varphi : \mathrm{Max}(S) = \mathrm{Min}(S) \to \mathrm{Min}(R)$ and $\psi : \mathrm{Min}(\overline{R}) \to \mathrm{Min}(R)$. For $M \in \mathrm{Min}(R)$, we set $M_S := \varphi^{-1}(M)$ and $M_{\overline{R}} := \psi^{-1}(M)$.*

Proof. For each injective extension $A \subseteq B$, any minimal prime ideal of A is lain over by a minimal prime ideal of B, any minimal prime ideal of B contracts to a minimal prime ideal of A when $A \subseteq B$ is flat and $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is injective when $A \subseteq B$ is a flat epimorphism (see Scholium). ◻

Theorem 5.12. *Let $R \subseteq S$ be an extension, where R is locally irreducible and S is zero-dimensional.*

(1) If $R \subseteq S$ is chained, then $\overline{R}$ is a Prüfer ring with total quotient ring S and $R \subseteq S$ defines an i-pair. Moreover, the following two conditions (*) and (**) hold:

(*) R is locally unbranched in S.

(**) R/M is a quasi-local i-domain for each $M \in \mathrm{Min}(R)$.

(2) Suppose that $R \subseteq \overline{R}$ is chained, $[R, S]$ is pinched at $\overline{R}$, $S = \mathrm{Tot}(R)$ and $\overline{R}$ is Prüfer.

(a) If (*) holds, then $R \subseteq S$ is arithmetic.

(b) If (**) holds, then $R/M \subseteq S/M_S$ is chained, for each $M \in \mathrm{Min}(R)$.

Proof. We first prove (1) and suppose that $R \subseteq S$ is chained. Then $R \subseteq S$ is quasi-Prüfer-Manis by Proposition 5.3. Hence S can be identified to $\mathrm{Tot}(R)$ in view of Proposition 5.7 and $\overline{R}$ is a Prüfer ring.

Moreover, $\mathrm{Spec}(\overline{R}) \to \mathrm{Spec}(R)$ induces a bijection $\mathrm{Min}(\overline{R}) \to \mathrm{Min}(R)$ by Lemma 5.11.

We claim that $\mathrm{Spec}(\overline{R}) \to \mathrm{Spec}(R)$ is injective. Let M, N be two prime ideals of $\overline{R}$ lying both over a prime ideal P of R and let $\mathfrak{P}$ be the unique minimal prime ideal of R contained in P. A minimal prime ideal $\mathfrak{M}$ of R, $\mathfrak{M} \subseteq M$ necessarily lies over $\mathfrak{P}$. It follows then that $\mathfrak{M} = \mathfrak{P}_{\overline{R}}$ is contained in N. Since $\mathfrak{P}_S$ is maximal, Lemma 5.5 shows that $\overline{R}/\mathfrak{P}_{\overline{R}}$ is a valuation domain and then $\mathrm{Spec}(\overline{R}/\mathfrak{P}_{\overline{R}})$ is a chain. The preceding observations yield that $\mathrm{Spec}(\overline{R}) \to \mathrm{Spec}(R)$ is injective, because $R \subseteq \overline{R}$ is an Inc-extension. Therefore, $R \subseteq S$ defines an i-pair by Proposition 5.8. From Lemma 5.5 and Remark 5.6, we deduce that $R/\mathfrak{P}$ is a quasi-local i-domain with integral closure $\overline{R}/\mathfrak{P}_{\overline{R}}$ and quotient field $S/\mathfrak{P}_S$ (see [15, Proposition 2.14]).

We now prove (2). (a) is a consequence of Proposition 5.10, since for each multiplicatively closed subset Σ of R, the map $T \mapsto T_\Sigma$ is a surjection from $[R, S]$ to $[R_\Sigma, S_\Sigma]$. Then (b) follows also from Proposition 5.10. □

In case R is an integral domain, we recover in (2)(b) the Gilbert's above-mentioned result.

Example 5.13. Arithmetic extensions appear frequently, as the reader may see below.

(1) An integrally closed FCP (whence FIP) extension $R \subseteq S$ is arithmetic. Indeed, $R_M \subset S_M$ is integrally closed and FCP for each $M \in \mathrm{MSupp}(S/R)$, so that $[R_M, S_M]$ is a chain [5, Theorem 6.10].

(2) A subintegral FIP extension $R \subset S$ such that $|R/M| = \infty$ for each $M \in \mathrm{MSupp}(S/R)$ is arithmetic. We already proved this result in the proof of Corollary 4.4.

(3) For a t-closed FIP integral extension $R \subseteq S$, Lemma 3.1(3) makes sense to say that $R_M/MR_M \subset S_M/MR_M$ is a purely inseparable field extension for each $M \in \mathrm{MSupp}(S/R)$. We assume that these hypotheses hold and show that $R \subseteq S$ is arithmetic.

We can reduce to the case where R is local with maximal ideal $M := (R : S)$. Then $[R/M, S/M]$ is a chain by [3, Proposition 2, Ch. V, page 24] and so is $[R, S]$.

(4) Let $R \subset S$ be an FIP extension. Assume that $R_M \subset S_M$ satisfies one of the above conditions (1), (2) or (3) for each $M \in \mathrm{MSupp}(S/R)$. Then $R \subseteq S$ is arithmetic.

(5) On the contrary, a seminormal and infra-integral FIP extension $R \subset S$ is never arithmetic. To see this, we can suppose that R is quasi-local with maximal ideal $M \in \mathrm{MSupp}(S/R)$ and $(R : S) = M$ by using a suitable localization. Using the proof of [6, Proposition 4.16], we get that $S/M \cong (R/M)^n$ for some positive integer n and then $[R, S]$ is not a chain.

(6) It may be asked when is a field extension $K \subseteq F$ arithmetic (chained)? To the authors knowledge, the only comprehensive study about the question is given in [20], from which we extract the following. An intermediary extension L of $K \subseteq F$ is called *reduced* if $L \neq F$ and for all $c, d \in F \setminus L$, $L(c) = L(d) \Rightarrow K(c) = K(d)$. Then $[K, L]$ is a chain if and only if each of the elements of $[K, L] \setminus \{F\}$ is reduced. In this case $K \subseteq F$ is algebraic. If $K \subseteq F$ is finite and Galois, with Galois group G, then $K \subseteq F$ is arithmetic if and only if either G is cyclic of order p^n (p a prime number and n an integer > 0) or G is isomorphic to a generalized quaternion group of order $2^n, n \geq 3$ and in this case $[K, F] = \{K, L, F\}$, with $[F : L] = 2$. Other criteria are given for separable finite extensions. Note also that if $[K, L]$ is a chain and $K \subseteq L$ algebraic, then $K \subseteq F$ is either separable or purely inseparable.

Olberding in [14] says that an extension of rings $R \subseteq S$ is *quadratic* if each intermediate R-submodule of S containing R is a ring. Other authors call Δ_0-*extension* such extensions and we will follow them. An extension $R \subseteq S$ is called *quadratic* if each $s \in S$ satisfies $P(s) = 0$ for a monic quadratic polynomial $P(X) \in R[X]$ (see for instance [10]). We call an extension $R \subseteq S$ a Δ-*extension* if $[R, S]$ is stable under addition, that is $T_1 + T_2 = T_1T_2$ for $T_1, T_2 \in [R, S]$. Note that an extension $R \subseteq S$ is a Δ_0-extension if and only if it is a quadratic Δ-extension and also that these properties localize and globalize. Actually, the proofs of [10] given for integral domains are valid for arbitrary extensions.

We first give some examples of Δ_0-extensions.

Proposition 5.14. *Let $R \subseteq S$ be a spectrally injective integral (for example, subintegral) FCP extension of rings. If the R-module S/R is locally uniserial (for example when $R \subseteq S$ is locally minimal), then $R \subseteq S$ is an arithmetic Δ_0-extension.*

Proof. We can assume that $R \subseteq S$ is an integral FCP extension of rings $R \subseteq S$, which is spectrally injective, with R quasilocal and assume that the R-module S/R is uniserial. Since $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is injective, S is quasilocal. Moreover, S/R is an Artinian R-module because $R/(R : S)$ is Artinian ([5, Theorem 4.2]) and S is an R-module of finite type. It follows from [14, Lemma 4.1] that $R \subseteq S$ is a Δ_0-extension. □

Proposition 5.15. *Let $R \subseteq S$ be a Δ-extension. Then $\overline{R} \subseteq S$ is a Prüfer extension.*

Proof. It is enough to apply [11, Theorem 1.7, p. 88]. □

Proposition 5.16. *An arithmetic extension $R \subseteq S$ is a Δ-extension and hence is quasi-Prüfer.*

Proof. In case $[R, S]$ is a chain, we have $B + C = BC = \text{Max}(B, C)$ for B, C in $[R, S]$. ◻

Theorem 5.17. *Let $R \subseteq S$ be an integrally closed extension, then $R \subseteq S$ is arithmetic if and only if $R \subseteq S$ is Prüfer and, if and only if $R \subseteq S$ is locally Prüfer-Manis.*

Proof. Assume that $R \subseteq S$ is integrally closed. If $R \subseteq S$ is arithmetic, then $R \subseteq S$ is Prüfer. Conversely, if $R \subseteq S$ is Prüfer, then $R \subseteq S$ is arithmetic. It is enough to use Proposition 5.1 and [11, Theorem 5.1, p. 46] which states that $R \subseteq S$ is locally Prüfer-Manis if $R \subseteq S$ is a Prüfer extension. ◻

Proposition 5.18. *Let $R \subseteq S$ be an arithmetic extension. Then for $B, C, D \in [R, S]$, we have $B \cap (C.D) = (B \cap C).(B \cap D)$ and $B.(C \cap D) = (B.C) \cap (B.D)$. Hence $([R, S], \cap, .)$ is a complete modular lattice.*

Proof. These equalities are locally trivial. ◻

These distributivity properties do not imply that the extension is arithmetic. See Remark 5.19.

Remark 5.19. Consider the following example. Set $R := \mathbb{Q}$, $T_1 := \mathbb{Q}(\sqrt{2})$, $T_2 := \mathbb{Q}(\sqrt[3]{2})$ and $S := \mathbb{Q}(\sqrt[6]{2})$. Then, setting $z := \sqrt[6]{2}$, $x := \sqrt[3]{2}$ and $y := \sqrt{2}$, so that $x = z^2$ and $y = z^3$, we get $S = \mathbb{Q}(z)$, $T_1 = \mathbb{Q}(y)$ and $T_2 = \mathbb{Q}(x)$. The (field) extensions $R \subset T_i$ and $T_i \subset S$, for $i = 1, 2$ are all minimal inert (ring) extensions with crucial ideal 0. Using the proof of the Primitive Element Theorem (see [3, Ch. V, Théorème 1, p. 39]), we get that $[R, S] = \{R, T_1, T_2, S\}$, so that $([R, S], \cdot, \cap)$, is a complete modular lattice, since $T_1T_2 = S$ and $T_1 \cap T_2 = R$. Indeed, the minimal polynomial of z is $X^6 - 2$, whose divisors of degree ≤ 3 in $S[X]$ are $X - z, X + z, X^2 - z^2 = X^2 - x, X^3 - z^3 = X^3 - y, X^3 + z^3 = X^3 + y, X^2 + zX + z^2, X^2 - zX + z^2, X^3 - 2zX^2 + 2z^2X - z^3, X^3 + 2zX^2 + 2z^2X + z^3$. However, $[R, S]$ is not a chain.

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