Palestine Journal of Mathematics

Vol. 3(Spec 1) (2014) , 366–373

Invertible Algebras with an augmentation map

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Dedicated to Patrick Smith and John Clark on the occasion of their 70th birthdays.

Communicated by Ayman Badawi

MSC 2010 Classifications: 13C99, 16D80, 16U80.

Keywords and phrases: group rings, crossed products, skewed group-rings, twisted group-rings, free algebras subgroup.

Abstract. We briefly survey results about Invertible Algebras (algebras having bases that consist entirely of units) and other related notions. In addition, we consider the existence of an augmentation map as a possible way in which results about group rings, the archetypical invertible algebras, may be extended to more general settings.

1 Invertible Algebras and Some Related Notions

Section 1 of this paper surveys the study of *Invertible Algebras* (over not-necessarily commutative rings) and other related notions. Section 2 introduces new ideas analogous to the augmentation map of group rings.

Invertible algebras are those algebras that satisfy the condition that they have a basis consisting entirely of units. Our brief survey also touches on a few other related concepts. The concept of invertible algebras was originally introduced in [10] and is the subject of two recent papers [8] and [11]. The somewhat related notion of fluidity is studied in the upcoming paper [9]. Other notions mentioned in this first section are *k-good rings* (c.f. [17]) and *S-rings* (c.f. [15]).

Group Rings are clearly an example of invertible algebras; their theory is very well developed and is central in many areas of mathematics. Standard references for group rings include the classics [13] and [14]. Section 2 points out a direction of research aiming to extend results about group rings to a larger family of invertible algebras. This is done by characterizing precisely those algebras that have an augmentation map.

In this paper, when we use the expression *A is an R-algebra* we deviate from the standard use of that terminology in two ways: one which narrows the net that we cast and another one that widens it. First, we do not allow a proper homomorphic image of the ring R to be contained in A; according to the definition we use in this paper, R itself is contained in A. The second difference is that R is not necessarily assumed to be contained in the center of A; in fact, we do not even assume that the ring R is commutative. Also, a feature that will be common to all those algebras considered here is that they will be free as (left-) R-modules. In other words, our setting is that we have a ring A that has a subring R such that A is a free left R-module.

Definition 1.1. Let A be an algebra over a ring R and $\mathcal{B}$ be a basis for A over R. $\mathcal{B}$ is an *invertible* basis if each element of $\mathcal{B}$ is invertible in A. If $\mathcal{B}$ is an invertible basis such that $\mathcal{B}^{-1}$, the set of the inverses of the elements of $\mathcal{B}$, also constitutes a basis then $\mathcal{B}$ is an *invertible-2*(I2) basis. An algebra with an invertible basis is an *invertible algebra* and an algebra with an I2 basis is an I2 *algebra*.

Various papers in the literature have considered properties of rings having to do with expressing their elements in terms of sums of units. See ([4], [16] and [18]), for example. In particular, S-rings and k-good are defined in ([15] and [17]) as follows:

Definition 1.2. In [17], for $k \in \mathbb{Z}^+$, Vámos calls a ring R a *k-good ring* if each element of R is the sum of k units. A related concept is that of an *S-ring*, a ring in which every element is the sum of units. (c.f. [15]).

Lemma 1.3 (Lemma 1, [15]). *Let A be a ring, let G be a group and let R be the group ring defined by A and G. Then R is an S-ring if and only if A is an S-ring.*

Lemma 1.4 (Lemma 5, [15]). *(a) In an even S-ring, zero can be written as the sum of an odd number of units.*

(b) If R is an S-ring that contains a unit u such that $u + 1$ is a unit, then R is an even S-ring.

(c) A finite product of even S-rings is an even S-ring.

(d) If R is any ring, then R_n is an even S-ring for all $n > 1$.

(e) If R is an even S-ring, and S is an S-ring, then $R \oplus S$ is an S-ring.

Proposition 1.5 (Proposition 6, [15]). *Let R be completely reducible. Then R is an S-ring if and only if the two-element field occurs at most once in the decomposition of R into completely reducible simple rings. R is an even S-ring if and only if this field does not occur at all.*

As usual, we denote the ring of $n \times n$ matrices over an arbitrary ring R by $M_n(R)$. Performing an elementary row operation on the identity matrix results in an elementary matrix.

Definition 1.6. In [17], the author defines the *elementary group of* $M_n(R)$, denoted by $E_n(R)$, as the subgroup of $GL_n(R)$ generated by elementary matrices, permutation matrices and -1.

Definition 1.7. A matrix is *strongly k-good* if it can be written as a sum of k elements of the elementary group, and a matrix ring is *strongly k-good* if every matrix is strongly k-good (see [17].)

Lemma 1.8 (Lemma 5, [17]). *Let R be an arbitrary ring and $n \geq 2$. Then any diagonal matrix in $M_n(R)$ is strongly 2-good.*

Proposition 1.9 (Proposition 6, [17]). *A proper matrix ring over an elementary divisor ring is* 2*-good. Over an Euclidean domain proper matrix rings are strongly* 2*-good.*

Proposition 1.10 (Proposition 8, [17]). *Let R be a ring, $m, n \geq 1$ and $k \geq 2$. If the matrix rings $M_n(R)$ and $M_m(R)$ are both k-good, then so is the matrix ring $M_{n+m}(R)$.*

Definition 1.11. Let R be a subring of S. R is said to be *unit closed* in S if no nonunit of R becomes a unit in S, i.e. $U(S) \cap R = U(R)$ (see [17].)

Proposition 1.12 (Theorem 13, [17]). *Every ring can be embedded as a unit closed subring in a* 2*-good ring.*

The study of k-good rings has been furthered recently in [5] which yielded the following interesting results:

Proposition 1.13 (Theorem 1, [5]). *For a right self-injective ring R, the following conditions are equivalent:*

(1) Every element of R is a sum of two units.

(2) Identity of R is a sum of two units.

(3) R has no factor ring isomorphic to $\mathbb{Z}_2$.

Proposition 1.14 (Theorem 3, [5]). *Let M_S be a quasi-continuous module with finite exchange property and $R = End_S(M)$. Then every element of R is a sum of two units if and only if no factor ring of R is isomorphic to $\mathbb{Z}_2$.*

Proposition 1.15 (Proposition 7, [5]). *If R is a right self-injective ring and G a locally finite group, then every element of RG is a sum of two units unless R has a factor ring isomorphic to $\mathbb{Z}_2$.*

The study of invertible-2 algebras naturally leads to considering questions about when the sets of inverses of linearly independent sets of units are linearly independent. This condition seems hard to satisfy and indeed the phenomena described here seem rather rare. We start by introducing appropriate terminology from [9].

Definition 1.16. (i) A linearly independent set S of units of the algebra A is said to be *fluid* if S^{-1}, the set of inverses of its elements, is also linearly independent.

(ii) An algebra A is said to be *fluid* if every linearly independent set of units S of A is fluid.

(iii) In order to prevent vacuous nonsensical consequences, we must introduce the following parameter: for an algebra A over a ring R, the *mojo* of A (mojo(A)) is the largest number of linearly independent units one can find in A. Clearly, if A is free as a module over R then $mojo(A) \leq rank_R(A)$. When A has finite rank as a free module over R then $mojo(A) = rank_R(A)$ if and only if A is an invertible algebra.

(iv) For a number $t \leq mojo(A)$, we say that the algebra A is *t-fluid* if every linearly independent set of units S with at most t elements is fluid. The *fluidity* of A (fluid(A)) is the largest t such that A is t-fluid. Clearly if $fluid(A) = rank_R(A)$ then A is invertible-2.

We start with a few immediate remarks.

Remark 1.17. (i) Every linearly independent set of units S having exactly two elements is fluid.

(ii) If $mojo(A) \geq 2$ then $fluid(A) \geq 2$.

(iii) $A = F[x]$ is an example of an F-algebra with $mojo(A) = fluid(A) = 1$.

(iv) If A is a subalgebra of B then if B is fluid then so is A.

The following example shows that it is indeed possible for the fluidity, mojo and dimension of an algebra to be different from one another.

Example 1.18. Let $A = T_3(F_2)$, the ring of upper triangular matrices over F_2. Clearly $dim(A) = 6$. It is straightforward to show $4 \leq mojo(A) \neq 6$. Furthermore, we easily find a set of 4 linearly independent units whose inverses are linearly dependent, thus $fluid(A) < 4$.

Fluid fields extensions are rather scarce, in fact, that is the subject of the following proposition from [9].

Proposition 1.19. *Let E be a field extension over F. Then $fluid(E) = 2$ and therefore, E/F is fluid if and only if the degree of E over F is 2.*

As we will show in Proposition 1.27, matrix algebras have played an important role in the study of invertible algebras. It is therefore also interesting to consider their fluidity. The next proposition from [9] announces that, in general, in spite of being $I2$, matrix algebras are far from being fluid and, in fact their fluidity is almost always 2.

Proposition 1.20. *Let R be a commutative ring such that $1 = a + b$ where a and b are units and $n \geq 3$. Then the fluidity of $M_n(R)$ is 2.*

The technical condition that the identity of R be a sum of two units is not necessary for the result to hold. This is illustrated in the following proposition. In fact, while we do not know any way to remove the hypothesis from Proposition 1.20, we also do not know at this moment any examples where the result fails.

Proposition 1.21. *Let $R = F_2$ and $n \geq 3$. Then the fluidity of $M_n(R)$ is 2.*

One of the first remarkable results about invertible algebras is the following characterization of group rings from [10].

Proposition 1.22 (Proposition 2.12, [10]). *Let A be an algebra over a ring R. A is a* ***group ring*** *if and only if A has an invertible basis that is closed under products and whose elements commute with those of R.*

Proposition 1.22 has as a corollary which strengthens a result about field extensions reported in [6] for reals over rationals and in general in [12]. Namely, Corollary 1.23 extends the result that no proper field extension has a basis that is closed under multiplication.

Corollary 1.23. *If a simple ring A is an invertible R-algebra with invertible basis $\mathcal{B} \neq 1$ then $\mathcal{B}$ is not closed under products.*

Propositon 1.22 was later generalized in [8] to include characterizations of the various crossed products as algebras having invertible bases with additional properties which are softer versions of those in Proposition 1.22.

Definition 1.24. Let A be an algebra over a ring R, and $\mathcal{B}$ an R-basis for A. If for all $v \in \mathcal{B}$ there exists $\sigma_v \in Aut(R)$ such that for all $r \in R$, $vr = \sigma_v(r)v$ then R *scalarly commutes* with $\mathcal{B}$. In the case that for all $v \in \mathcal{B}$, $\sigma_v = 1$ we naturally say R *commutes* with $\mathcal{B}$.

Definition 1.25. Let A be an algebra over a ring R and $\mathcal{B}$ be an invertible basis for A over R. If for all $v, w \in \mathcal{B}$, $\alpha vw \in \mathcal{B}$ for some $\alpha \in U(R)$ then $\mathcal{B}$ is *scalarly closed under products*.

Proposition 1.26 (Proposition 2.7, [8]). *Let A be an algebra over a ring R.*

*(i) A is a **crossed product** if and only if A has an invertible basis that is scalarly closed under products and whose elements scalarly commute with those of R.*

*(ii) A is a **skew group ring** if and only if A has an invertible basis that is closed under products and whose elements scalarly commute with those of R.*

*(iii) A is a **twisted group ring** if and only if A has an invertible basis that is scalarly closed under products and whose elements commute with those of R.*

*(iv) A is a **group ring** if and only if A has an invertible basis that is closed under products and whose elements commute with those of R.*

In [10] it was shown that over any ring R, the matrix ring $A = M_n(R)$ (for any $n \geq 1$) is an I2 R-algebra. Then the result was significantly extended in [8].

Proposition 1.27 (Proposition 3.1, [8]). *Let A be an algebra over a ring R with a basis that includes 1. Then $M_n(A)$ is an invertible algebra over R for all $n \geq 2$.*

Corollary 1.28. *Invertibility is not a Morita invariant.*

Proposition 1.29. *Let D be a division ring and let A be a semilocal D-algebra. If $D \neq F_2$ then A is invertible. If $D = F_2$ then A is invertible if and only if A does not admit an algebra epimorphism to $\mathbb{F}_2 \oplus \mathbb{F}_2$.*

Proposition 1.30 (Proposition 4.6, [8]). *Let R be a ring and let A be a free local R-algebra. Then A is invertible.*

2 Toward an analogue of the augmentation map

Group rings were one of the original motivations for our study of invertible algebras. As it has been shown in the previous sections, the class of invertible algebras is much bigger than that. It is therefore not to be expected that many of the results concerning structural properties of group rings can be extended to invertible algebras. We investigate, however, conditions that may allow some of those results to extend. Our first observation is that the idea of an augmentation map can sometimes be extended to a more general setting.

We start by describing properties that will allow the *augmentation map* $\phi : A \to R$ defined by $\phi(\sum_i \alpha_i v_i) = \sum_i \alpha_i$, to become a ring homomorphism and we will show that these conditions are indeed necessary and sufficient in Proposition 2.8.

It is well-known that for a finite group G, the group ring $R[G]$ is self-injective if and only if R is self-injective [3]. The next example illustrates how this result does not extend to invertible algebras.

Example 2.1. It is well known that a finite-dimensional commutative local algebra A over a field $R = F$, is self-injective if and only if A has a unique minimal ideal [7]. Consider the F_3-algebra $A = \frac{F_3[x,y]}{\langle x,y\rangle^2}$. It is easily checked that the basis $B = \{1, 1+x, 1+y\}$ is an invertible basis for A. Now $\langle x, y\rangle$ is the unique maximal ideal of A and so A is a local ring. Now $\langle x\rangle$, and $\langle y\rangle$ are both minimal ideals and therefore A does not have a unique minimal ideal. Therefore, A is not self-injective even though it is finite dimensional over F (which, as all fields, is self-injective.)

A key element of the proof in [3] of the above characterization of self-injective group rings is the fact that the R-homomorphism $\phi : R[G] \to R$ given by $\phi(\sum_{g\in G} \alpha_g g) = \sum_{g\in G} \alpha_g$ is a ring homomorphism. It seems reasonable to ask, for a basis $\mathcal{B}$ of an algebra A over a ring R, when the map $\phi : A \to R$ given by $\phi(\sum_{b\in\mathcal{B}} \alpha_b b) = \sum_{b\in\mathcal{B}} \alpha_b$ is a ring homomorphism. We introduce next a few definitions that will be essential components of the answer to that question provided by Proposition 2.8 below. Furthermore, these definitions and Proposition 2.8 will be instrumental in providing a partial converse to Proposition 2.2 in [10].

Definition 2.2. Let A be an R-algebra with basis $\mathcal{B}$. We say that R *commutes linearly* with $\mathcal{B}$ if for all $v \in \mathcal{B}$ and $\beta \in R$, if $v\beta = \sum_{v_k\in\mathcal{B}} \delta_k v_k$ then $\sum \delta_k = \beta$.

Definition 2.3. Let A be an R-algebra with basis $\mathcal{B}$. We call $\mathcal{B}$ *linearly closed under products* if for all $v, w \in \mathcal{B}$, if $vw = \sum_{v_i\in\mathcal{B}} \alpha_i v_i$ then $\sum_{v_i\in\mathcal{B}} \alpha_i = 1$.

Definition 2.4. Let A be an R-algebra with invertible basis $\mathcal{B}$. We say $\mathcal{B}$ is *linearly closed under inverses* if for all $v \in \mathcal{B}$, if $v^{-1} = \sum_{v_i\in\mathcal{B}} \alpha_i v_i$ then $\sum_{v_i\in\mathcal{B}} \alpha_i = 1$.

Clearly Definition 2.3 is satisfied by group rings. However, in [10], an example of an invertible algebra that is not a group ring is given, namely $\dfrac{F_2[x,y]}{\langle x,y\rangle^2}$. An invertible basis of this algebra also illustrates that Definition 2.3 does not just pertain to group rings.

Example 2.5. Consider $A = \dfrac{F_2[x,y]}{\langle x,y\rangle^2}$. As stated in [10] we know A is not a group ring. However, A is an invertible algebra with invertible basis $\mathcal{B} = \{1, 1+x, 1+y\}$. The product of $1+x$ and $1+y$ is

$$(1+x)(1+y) = 1+x+y = 1(1) + 1(1+x) + 1(1+y).$$

The sum of the coefficients is 1. The other combinations are trivial. Therefore, $\mathcal{B}$ satisfies Definition 2.3.

Proposition 2.6. *Let A be an invertible R-algebra with invertible basis $\mathcal{B}$. Assume $\mathcal{B}$ is linearly closed under products and scalarly closed under products. Then $\mathcal{B}$ is a group.*

Proof. Let $v_i, v_j \in \mathcal{B}$. Since $\mathcal{B}$ is scalarly closed under products we have $v_i v_j = \alpha v_k$. But since $\mathcal{B}$ is linearly closed under products we must have $\alpha = 1$. Therefore, $\mathcal{B}$ is closed under products and by Proposition 2.12 in [10], $\mathcal{B}$ is a group. □

An obvious question is are there other examples of algebras with bases that are linearly closed under products and inverses, yet are not group rings. The previous example inspired a consideration of algebras of the form $\dfrac{R[x_1, x_2, \ldots, x_n]}{\langle x_1, x_2, \ldots, x_n\rangle^m}$. The following proposition will show there are numerous examples of algebras that are linearly closed under products and inverses yet not group rings.

Proposition 2.7. *Let $A = \dfrac{R[x_1, x_2, \ldots, x_n]}{\langle x_1, x_2, \ldots, x_n\rangle^m}$ where R is any ring, $n \geq 1$ and $m \geq 2$. Assume $\{x_i | i = 1, \ldots n\}$ is a commutative set. Then A has an invertible basis that is linearly closed under products and inverses, namely, $\mathcal{B} = \{1\} \bigcup \{1 + x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}\}$ where $0 \leq r_i < m$ for all i and $1 \leq \sum_{i=1}^{n} r_i < m$.*

Proof. Let $1 + x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}, 1 + x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n} \in \mathcal{B}$. Then

$$\begin{aligned} v &= (1 + x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n})(1 + x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}) \\ &= 1 + x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n} + x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n} + x_1^{s_1+t_1} x_2^{s_2+t_2} \cdots x_n^{s_n+t_n}. \end{aligned}$$

First suppose $\sum_{i-1}^{n}(s_i + t_i) < m$. Then $1 + x_1^{s_1+t_1} x_2^{s_2+t_2} \cdots x_n^{s_n+t_n} \in \mathcal{B}$. Writing v as a linear combination of elements from $\mathcal{B}$ we have

$$v = -2(1) + (1 + x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}) + (1 + x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}) + (1 + x_1^{s_1+t_1} x_2^{s_2+t_2} \cdots x_n^{s_n+t_n}).$$

The sum of the coefficients of the basis elements is $-2+1+1+1 = 1$. So in this case $\mathcal{B}$ is linearly closed under products.

Now assume $\sum_{i-1}^{n}(s_i + t_i) \geq m$. Then $x_1^{s_1+t_1} x_2^{s_2+t_2} \cdots x_n^{s_n+t_n} = 0$. Using this information we write v as a linear combination of elements from $\mathcal{B}$ to obtain

$$v = -1(1) + (1 + x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}) + (1 + x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}).$$

Again we notice the sum of the coefficients is 1 and $\mathcal{B}$ is linearly closed under products.

Given an arbitrary element $w = 1 + x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n} \in \mathcal{B}$ we see the inverse is

$$w^{-1} = 1 - (x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}) + (x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n})^2 - \cdots + (x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n})^k$$

where k is the smallest integer such that $(k+1)\sum_{i=1}^{n} s_i \geq m$. If k is odd then writing w^{-1} as a linear combination of elements from $\mathcal{B}$ we have

$$w^{-1} = 2(1) - (1 + x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}) + (1 + (x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n})^2) - \ldots + (1 - (x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n})^k),$$

and the coefficients of w^{-1} sum up to 1.

On the other hand, if k is even

$$w^{-1} = 1(1) - (1 + x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}) + (1 + (x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n})^2) - \ldots + (1 + (x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n})^k).$$

Now there are an even number of terms of w^{-1} not including 1. As the signs alternate we see they all cancel out and therefore when we add of the coefficients we have 1. Combining these two results we have $\mathcal{B}$ is linearly closed under inverses. □

Note that if the ring R is self-injective then the R-algebra A in the previous proposition is not a group ring. This holds as A is local and does not have a unique minimal ideal, thus is not self-injective.

Proposition 2.8. *Let A be a left R-algebra with basis $\mathcal{B} = \{v_i | i \in I\}$ and $\phi : A \to R$ the R-homomorphism given by $\phi(\sum_i \alpha_i v_i) = \sum_i \alpha_i$. Then the following are equivalent:*

(i) ϕ is a ring homomorphism.

(ii) $\mathcal{B}$ is linearly closed under products and R commutes linearly with $\mathcal{B}$.

(iii) For every $v, w \in \mathcal{B}$ and $\beta \in R$, if $v\beta w = \sum_{v \in \mathcal{B}} \beta_v v$ then $\sum_{v \in \mathcal{B}} \beta_v = \beta$.

Proof. (1) ⇒ (2): First suppose ϕ is a ring homomorphism. Let $v, w \in \mathcal{B}$ and write $vw = \sum_i \alpha_i v_i$. Now $\phi(v)\phi(w) = 1 \cdot 1 = 1$ and $\phi(vw) = \sum_i \alpha_i$. Since ϕ is a ring homomorphism we have $\phi(v)\phi(w) = \phi(vw)$ which gives $\sum_i \alpha_i = 1$.
If $v\beta = \sum_j \gamma_j v_j$, then $\phi(v\beta) = \sum_j \gamma_j$. Write $1 = \sum_k \delta_k v_k$. So $\beta = \beta \sum_k \delta_k v_k$. Since ϕ is a ring homomorphism we have $\phi(1) = 1$, but also

$$\phi(1) = \phi(\sum_k \delta_k v_k) = \sum_k \delta_k$$

giving $\sum_k \delta_k = 1$. Now

$$\phi(\beta) = \phi(\beta \sum_k \delta_k v_k) = \phi(\sum_k \beta\delta_k v_k) = \sum_k \beta\delta_k = \beta \sum_k \delta_k = \beta \cdot 1 = \beta.$$

So $\phi(v\beta) = \phi(v)\phi(\beta) = 1 \cdot \beta = \beta$. Hence $\beta = \sum_j \gamma_j$.

(2) ⇒ (3): Let $v, w \in \mathcal{B}$ and $\beta \in R$, if $v\beta w = \sum_i \beta_i v_i$ then $\phi(v\beta w) = \sum_i \beta_i$. Alternatively, $(v\beta)w = \sum_i \alpha_i(v_i w)$, where $\sum_i \alpha_i = \beta$. It follows that

$$(v\beta)w = \sum_i \alpha_i \sum_j \delta_{ij} v_j$$

where for all i, $\sum_j \delta_{ij} = 1$. Thus,

$$\phi(v\beta w) = \sum_i \alpha_i \sum_j \delta_{ij} = \sum_i \alpha_i = \beta.$$

Therefore, $\sum_i \beta_i = \beta$.

(3) ⇒ (1) It suffices to show that ϕ is multiplicative. Let $r, s \in A$. So $r = \sum_i \alpha_i v_i$ and $s = \sum_j \beta_j v_j$. Then

$$\phi(rs) = \phi(\sum_i \alpha_i v_i \sum_j \beta_j v_j) = \phi(\sum_{i,j} \alpha_i (v_i \beta_j) v_j).$$

Now, for every i, j, write $v_i \beta_j v_j = \sum_k \delta_{ijk} v_k$. By (3), $\sum_k \delta_{ijk} = \beta_j$. So,

$$\phi(rs) = \phi(\sum_{i,j,k} \alpha_i \delta_{ijk} v_k) = \sum_i \alpha_i \sum_j \sum_k \delta_{ijk} \sum_i \alpha_i \sum_j \beta_j.$$

It is easy to see that this is also the value of $\phi(r)\phi(s)$. □

Notice that the invertible basis in Example 2.1 satisfies condition (2) of Proposition 2.8.

Corollary 2.9. *Let $R^t G$ be a proper twisted group ring. Then the map $\phi : R^t G \to R$ given by $\phi(\sum_i \alpha_i \overline{g_i}) = \sum_i \alpha_i$ is not a ring homomorphism.*

Proof. Notice R^tG has an invertible basis, namely $\overline{G} = \{\overline{g} | g \in G\}$, that is scalarly closed under products. If ϕ were a ring homomorphism, then by Proposition 2.8 $\overline{G}$ would be linearly closed under products, and by Proposition 2.6 $\overline{G}$ would be a group. This contradicts the fact that R^tG is a proper twisted group ring. □

Corollary 2.10. *Let RG be a proper skew group ring. Then the map $\phi : RG \to R$ given by $\phi(\sum_i \alpha_i g_i) = \sum_i \alpha_i$ is not a ring homomorphism.*

Proof. RG has an invertible basis, namely G, that scalarly commutes with R. If ϕ were a ring homomorphism, then by Proposition 2.8 R would commute linearly with G, thus R commutes with G, a contradition. □

Proposition 2.8 enables a straightforward verification that proper field extensions do not have a basis that is linearly closed under products, an extension of Corollary 1.23.

Proposition 2.11. *Let $F \subset E$ be a proper field extension and suppose $\mathcal{B}$ is a basis, then $\mathcal{B}$ is not linearly closed under products.*

Proof. Clearly F linearly commutes with $\mathcal{B}$. Suppose $\mathcal{B}$ is linearly closed under products. Then by 2.8 we have $\phi : E \to F$, the F-homomorphism given by $\phi(\sum_i \alpha_i v_i) = \sum_i \alpha_i$ is a ring homomorphism. Then $ker(\phi) = E$ or $ker(\phi) = 0$. However, $e_1 - e_2 \in ker(\phi)$ so $ker(\phi) \neq 0$. Also $e_1 \notin ker(\phi)$ and so $ker(\phi) \neq E$. Thus $\mathcal{B}$ is not linearly closed under products. □

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Received: March 14, 2014.

Accepted: April 20, 2014.